Introductory material and appendix
to the new and first English translation of

Luca Pacioli's 1498
ON THE DIVINE PROPORTION

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Compiled April 2005
for the LaRouche Youth Movement

INTRODUCTION
by Dr. Jonathan Tennenbaum

You have in your hands one of the most famous scientific books in all history. It was written just a little over 500 years ago. Concerning its ostensible subject -- the geometrical proportion commonly known today as "the Golden Section" -- a vast literature has arisen, and quite a bit of controversy, too.

Despite Pacioli's well-documented influence on *De Divina Proportione's* illustrator, Leonardo da Vinci, this is the first time this book has been translated into English. Now that the book has found its way into your hands, dear Reader, a new adventure starts. Will you soon put the book away again, discouraged perhaps by the unfamiliar style and references, by the prolixity of its contents, or the many questions it seems to leave unanswered? Or will you happily find in it a pathway which, if followed with perseverance, might lead you to the most astonishing and sublime discoveries which the human mind has so far produced?

Anticipating the difficulties for a modern reader, and reviewing the drawbacks of the secondary literature the reader might turn to, the author of these lines- who has spent many years pondering over these matters- feels called upon to lend some modest advice and assistance. Modern secondary sources will, as a rule, not even mention the most essential thing, namely the *universal physical principle* expressed by the "Divine Proportion" and the five regular ("Platonic") solids, which form the subject matter of Pacioli's book. It is precisely the breakthrough in understanding of the physical significance of the regular solids, achieved by Cusa, Pacioli, Leonardo and their collaborators, that formed the starting-point for Johannes Kepler's founding of modern physics. Moreover, that very same issue of physics has also been the focus of a long campaign of mystification, disinformation and outright intellectual thuggery in the course of history. That is why we find the Divine Proportion and the regular solids are often dismissed today as mere mathematical curiosities, devoid of serious scientific importance; or even worse, relegated to the domain of esoterics and kabbalism.

Pacioli lives in a very different world than ours today, as even his opening paragraphs make clear. Pacioli is not at all academic, not stuffy; he is informal, lively, playful, humorous, exuberant. Imagine, in your mind's eye, the workshops of Leonardo da Vinci, the bustling activity of construction of cathedrals and cities, the astonishing works of art and poetry, the discovery of the New World, the rapid ups and downs of political and military fortunes. We feel not just the distance of 500 years, but the spiritual and intellectual poverty of our age. We notice, also, that Pacioli and his collaborators are spiritually much closer to the Greek classical culture, to its science, philosophy and art, than we are to either. And they were closer to the idea of universal history, to the idea of mankind as a continuous historical sweep, than we are today in our asocial and anti-historical age.

You can see this idea in the painting "The School of Athens" of Raphael. It is as if we have just arrived late to a discussion. There and in *De Divina Proportione*, Pythagoras, Plato, Euclid, Dante, Nicolaus of Cusa and others have already spoken. Though many of our most important interlocutors have passed away, you can deliberate with them, after a manner, by getting in on the dialogue, which has been going on for 2500 years.

So we should catch up, and also benefit from the latter reflections of Johannes Kepler, whose launching of modern astronomy and mathematical physics represents the culmination of the discussion. The discussion reaches a happy sort of crescendo with Kepler's demonstration of the "Divine Proportion" in the solar system -- more about that below – and then heads off in new directions.[1]

Pacioli's is a pedagogical work, using what appears to be elementary geometry, as a stepping-stone to bring the reader to the most profound scientific questions. This method, developed most masterfully by Nicolaus of Cusa in his *De Docta Ignorantia*, was a favorite method of the Renaissance.

To understand what Pacioli is talking about, you must read this book with paper and pencil readily at hand, doing the constructions, making models. Younger readers nowadays will demand "Give me the bottom line, save me the work. Why do I need to actually DO the constructions? Yes, dear friend, you must. For only then – together with a whole lot of *thinking* about what you have done -- will you be able to fully appreciate the earth-shaking significance of those constructions.

So work through the constructions, and also look at how Pacioli discusses them, not simply as geometrical exercises, but as the "footprints", so to speak, of a profound *physical principle*.

Turning Euclid on his head

At first glance, Pacioli appears to do little more than to rework certain results of the ancient Greek geometers, which are compiled in Euclid's famous "*Elements of Geometry"* written about 1800 years earlier. Pacioli focuses on Euclid's discussion of the geometrical proportion Euclid called "the extreme and mean ratio", which Pacioli renames the "Divine Proportion", its relation to the regular pentagon and Euclid's famous final chapter, Chapter 13, dealing with the construction of the regular, or "Platonic" solids. But there is a very big difference in spirit and content, between the two. Pacioli, in effect, "turns Euclid upside down", in a manner typical of the revolutionary quality that permitted the Renaissance to leap far beyond a mere scholarly transmission of ancient Greek science.

Euclid's "*Elements*" in the form it came down to us, is a "mixed bag". On the one hand, Euclid was studied down through the centuries as a unique compendium of Greek mathematics, relating a considerable part, although by no means all, of the results obtained up to his time (c. 300 BC). However, on the other hand, the "*Elements"* adopted a synthetic, deductive form of exposition, which has little or nothing to do with how Greek science had actually developed up to Euclid's time.[2]

We have no reason to think that the original author of the "*Elements"* intended to obscure or impede the

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[1] Kepler's initial demonstration of the "Divine Proportion" in the solar system, is contained in his 1596 "Mysterium Cosmographicum". The reader should follow the train of Kepler's development, from there, to his revolutionary, 1609 "Astronomia Nova", and finally to the 1619 "Harmonice Mundi". In a manner unequaled in the history of science, either before or after, Kepler makes his own thinking, his entire process of discovery, directly accessible to the reader. If you want to learn what fundamental scientific discovery is all about, then make Kepler your personal friend!

[2] The axiomatic method has a very specific pedagogical value, in drawing attention to the assumptions underlying any given body of formal scientific knowledge, and the way in which those assumptions circumscribe "hereditarily" every conclusion (or "theorem") contained in that body of knowledge. At the same time, one obtains a preliminary notion, albeit a negative one, of the "revolutionary" implications of a creative discovery, invalidating one or more of the axioms of such a body of knowledge. But, the axiomatic-deductive method is unable to account for how such creative scientific discovery actually occurs. For this and related reasons, the over-emphasis on axiomatic-deductive presentations of scientific knowledge, suppresses the actual process by which science progresses, the which is its actual content.

development of science. The problem is, that the majority of the most crucial writings of Greek science, which had originally existed alongside Euclid's compendium, were subsequently lost, destroyed, or at best preserved in fragmentary form, and the living continuity of Greek science was broken off by the "dark age" ushered in by the Roman Empire. "Euclid's *Elements"* subsequently was promoted as the primary source and reference for the study of Greek mathematics, and the model for elementary education in geometry well into the Twentieth Century.

That "elementary geometry" became virtually synonymous with Euclid's "*Elements"* in the teaching of mathematics helped strengthen the illusion, related to the overemphasis on the deductive method, that science, as well as the Universe itself, develops "from the bottom up" rather than "from the top down". So for example, compare Euclid's idea that the ultimate unit or "element" of geometry is the "point", and that (for example) the line is to be thought of as a mere set of points, with the *opposite* standpoint, which is inherent in the notion of constructive geometry and the method of Plato. From the latter standpoint, which proceeds from the *higher* existence to the lower one, the point loses its self evident "elementary" nature and instead is regarded as a singularity generated by the intersection of two lines; lines in turn, are generated as the intersections of surfaces, and surfaces as the boundaries or transitional regions between solids.

Similarly, Euclid's "*Elements"*, begin with *plane geometry*, and proceeds only in the final chapters to solid (three dimensional) geometry. Whereas, in reality the first and most "elementary" geometry is certainly not *plane* geometry at all, but rather *spherical* geometry -- the form of geometry associated with astronomy -- as Man's oldest science.

Such circumstances explain why Pacioli and his fellow thinkers of the Renaissance may express gratitude to Euclid, while at the same time struggling to rediscover the actual method and "soul" of Greek mathematics, which could at best be "read between the lines" of Euclid and other fragmentary surviving texts, of which the surviving dialogues of Plato were the single most decisive source.

The physical significance of the regular solids and the so-called "Golden Section", as developed by the Pythagorean and Platonic schools of ancient Greece, the which Pacioli and Leonardo Da Vinci understood better than practically every scientist living today, provides a crucial case in point. Related to this, the Renaissance revived the notion of spherical geometry and emphasized construction and generation of *physical models* rather than mere contemplation of "geometrical objects". The renaissance of spherical geometry emphasized in Plato's Timaeus, occurred in connection with the revival of Eratosthenes' work on the spherical form of the Earth and of the heliocentric conception of the solar system, all strongly connected with the person of Nicolaus of Cusa.

Pacioli's summary and its earth-shaking implications

Pacioli took a decisive step toward recovering the "lost knowledge" of Greek science, by insisting on the *physical* significance of the regular solids -- "without this knowledge presupposed, it is not possible to understand or employ any of man's sciences", he writes --, and by emphasizing the existence of a *single, unifying principle* behind them, he expresses the conception of the Divinity residing in the "Divine Proportion".

I now summarize Pacioli's main conclusions -- at least so as far as the geometrical aspect goes per se -- not exactly in the order and way he did, but with a view to helping the reader to catch on to the earth-shaking implications of this book:

First, that only *five* regular solids can be constructed in ordinary (visual) space: the tetrahedron, cube, octahedron, icosahedron and duodecahedron. This is not a mathematical deduction, but an *experimental physical fact* that anyone can verify directly, independent of any formal axioms or postulates. As Pacioli wrote, "In Nature it is not possible to form a sixth (regular solid)".

Second, that each regular solid determines a set of *geometrical proportions* and relationships which, taken together, and in view of the uniqueness of the five solids, deserve to be regarded as *characteristics or "constants" of visual space*. Chief among these are the values of the *ratios of the radii of the inscribed and circumscribing spheres* for each of the solids.[3]

Third, that the five regular solids can be derived, as embedded figures, from a single one: the duodecahedron, which, according to Pacioli, "sustains the existence of all the others" and governs the manifold harmonies and interrelations among all five.

Fourth, that the construction of the duodecahedron, inscribed in a given sphere, is impossible without forming a certain specific *proportion*, which Pacioli identifies as the *Divine Proportion* and which, among other things, manifests itself as the ratio of the diagonals to the sides of the pentagons, which form the 12 faces of the duodecahedron.[4]

Fifth, that an infinite variety of other geometrical bodies ("dependents") can be generated, which, although infinite in number, all derive their properties from the five regular solids and from the Divine Proportion.

Sixth, that, in virtue of these and other "powers and effects", the Divine Proportion lies beyond the domain of those proportions that can be expressed exactly (in finite terms) by means of *whole numbers*. Its value thus corresponds to what has come to be called an "irrational number". Pacioli says: "It is

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[33] The said ratios are defined as follows.

For any given solid, let r be the radius of the sphere inscribed inside the solid. This sphere is the largest sphere which can fit within the solid; it touches the solid at the midpoints of its faces.

Let R be the radius of the sphere circumscribing the solid. This sphere is the smallest sphere into which the solid can be fit; it touches the solid at its vertices.

Although r and R depend on the size of the solid, the ratio R/r does not. The value of the ratio is characteristic of the type of solid, and the five values corresponding to the five types of regular solids, constitute "constants of space". Expressed in algebraic and approximate numerical form, those values are as follows:

tetrahedron	$R/r = 3$	
octahedron and cube	$R/r = \sqrt{(3)}$	= (approx.) 1.73205
duodecahedron and icosahedron	$R/r = \sqrt{(3)} \times \sqrt{(5 - 2\sqrt{(5)})}$	= (approx.) 1.25841

[4] Golden Mean relationship between the sides and the diagonals of a pentagon

$$(a+b)/a = a/b$$

impossible to exaggerate… the privilege and preeminence due to our proportion, because of its infinite power... This gift is conferred on our proportion by the immutable nature of its higher principles."

Seventh, that the five regular solids ultimately derive from the *sphere*, from the regular divisions of a spherical surface; and that the sphere, in turn, is generated by the action of *rotation* applied to a circle (which is itself the product of rotation).

I assure the reader, these propositions, taken together, are among the most "pregnant" in all of science. They are worth many, many hours of thinking and rethinking, working through Pacioli's original text and constructions, reading what Plato and Kepler had to say about the same topics, coming back to the whole matter again and again later, maybe after months or years, to gain new insights, making these matters part of your life, as I have done.

I promised "earth-shaking implications". Follow me, and I shall reveal them!

<u>Where did space get its shape?</u>

To begin with, ask yourself, looking at the five regular solids and the above seven propositions: "Where does all this come from?" After all, the commonplace notion of *pure space*, has nothing about it, which would lead us to suspect the necessary existence of such *very specific* objects, constants and harmonic relationships. Space, after all, is just a "blah", isn't it? -- a *formless void* extended indefinitely in three directions. In and of itself, that concept is totally sterile; and has no obvious relationship to the existence of the five regular solids, the "Divine Proportion" and the other very specific "constants of space".

So, to the extent the five solids exist, as a matter of experimental *fact* and have those surprising and very specific properties described by Pacioli, their *cause* must lie in a reality *outside* the sterile, abstract concept of "pure three-dimensional space".

That is the *first, earth-shaking conclusion*.

But we can already hear loud objections. "Wait, didn't Euclid prove, that the existence of exactly five regular solids is a *logical consequence* of the axioms of Euclidean geometry? If so, why do we need to assume a 'reality outside the abstract concept of empty space?'"

I answer: "The simple circumstance, that a certain, experimentally demonstrated proposition can be logically derived from certain axioms, is merely a *phenomenon*, not a *cause* or explanation for anything. It does not tell us *why* the proposition is true, nor whether the axioms are actually valid as stated. For, Euclid did not create or ordain *space* with his definitions and postulates!"

That suggests posing the following, "dangerous" question:

If space is not the product of a set of definitions and axioms, then *what is it* that gives visual space the characteristics it has?

Some people will no doubt be surprised, that such as question even be posed. Our ordinary, three-dimensional space seems so simple and self-evident, that it would hardly occur to them to ask for an explanation. The very suggestion, that there might be something more fundamental, lurking "behind" space and causing it to have certain properties, seems to reverse the order of so-called common sense: --

trying to explain something apparently *simple* on the basis of something necessarily more complicated.

On the other hand, if space is really so "simple", and if there is nothing "behind" it, then how are we to account for such unexpected phenomena, as the existence of no more, nor less than five regular solids; and for all the other remarkable and unexpected relationships, that Pacioli develops in his book? Could it be, that there is something *defective* in our own conception of space, as something simple and self-evident?

The example of "Higher Arithmetic"

Let me suggest a kind of "laboratory experiment", which might help us to clarify the nature of the problem. Instead of the "space", let's take something that appears even more simple, more concrete and more self-evident: the *whole numbers*.

What, indeed, could be simpler, more elementary, than the series of whole numbers 1, 2, 3, 4, etc.? We start with the unit 1, and keep adding 1, again and again in succession. The whole number series is the paradigm of a completely {linear} series -- an endless "blah" with no other content, than adding 1. One would hardly expect anything of interest to come from such a "blah".

And yet, how totally different the reality is! Exactly the domain of the whole numbers, turns out to be full of the most beautiful and unexpected harmonic relationships, to be found anywhere in experimental science. These extraordinary phenomena make up the subject-matter of what Carl Friedrich Gauss called the "Higher Arithmetic".[5]

For example, already in ancient times the Pythagoreans recognized the existence of a variety of distinct *species* among the whole numbers: odd and even numbers, perfect numbers, square numbers, prime and composite numbers etc. They noticed, that numbers that are *close* to each other in the linear series, turn out often to belong to *completely different "species"*.

Investigating the species-characteristics of the numbers in this experimental-physical way, we uncover a highly complex structure, *totally different* from the simple linear ordering of the "whole number series". Within that structure a central role is played by the famous species of *prime numbers*, into products of which all other numbers can be uniquely decomposed, like chemical compounds from the elements. Yet, the prime numbers are distributed in a *curiously irregular* fashion relative to what initially appeared as the self-evident, supposedly "natural" ordering of the whole numbers as a linear series.

Could it be, that our conception of the whole numbers as a linear series is fundamentally *defective*? Could it be, that the *cause* of the existence and properties of the primes and other species of numbers lies *outside* the bounds of our conception of the whole numbers as a "blah", generated by nothing but repeated additions of 1 to itself?

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[5] See Gauss' foreword to his famous 1801 "Disquisitiones Arithmeticae", in which he distinguishes between "elementary arithmetic", which deals with such things as the decimal system, the methods of addition, multiplication, subtraction and division etc. on the one side, and what he calls "the higher arithmetic", which concerns the discovery of what are ultimately physical principles, underlying various phenomena manifested by the whole numbers and their relationships. Gauss writes, in recounting the origin of the "Disquisitiones": "I chanced upon a remarkable arithmetic truth...and, since I considered it not only as very beautiful, but also suspected, that it was connected with other remarkable properties, I devoted my entire energy, to discover the *principles*, upon which it was based..."

Indeed! Not only the anomalous distribution of the prime numbers, but above all the extraordinary harmonic relationships among the number species, investigated by Gauss for the first time in a really systematic fashion, point exactly to the existence of a body of "higher principles", beyond the deductive domain of ordinary arithmetic and algebra. For exactly that reason Gauss himself chose the term "Higher Arithmetic" for the branch of science he had founded.

A simple example of the mentioned, harmonic relationships: Observe, that certain prime numbers can be expressed as the sum of two square numbers -- such as 2 = 1 + 1, 5 = 4 + 1, 13 = 4 + 9, 17 = 16 + 1, 29 = 4 + 25, 37 = 36 + 1, 41 = 16 + 25 etc. -- whereas other prime numbers can not. Among the latter are 7, 11, 19, 23, 31, 43 etc. It turns out, that these two species of prime numbers are distinguished by a very simple harmonic criterion: each prime number, that leaves a remainder of 1 when divided by 4 (i.e. a prime of the form 4n+1), is the sum of two square numbers; the remaining prime numbers (with the exception of 2), namely those leaving a remainder of 3 when divided by 4, are not sums of two squares. [6]

The reader can easily confirm this phenomenon in many particular cases, by direct numerical experimentation. To discover the systematic reasons behind it, however, was the work of centuries. Gauss showed, in fact, that this and related phenomena involving relationships between squares and prime numbers, become fully intelligible only from the standpoint of the so-called *complex domain.* [7]

For readers unfamiliar with this topic, the essential point is that the existence of Gauss' complex domain, can in no way be derived by simple logical reasoning from the ordinary concept of whole number, but embodies the discovery of a *new principle*. The extent of resistance to recognition of that principle, prior to Gauss is exemplified by the case of Leonard Euler, the skillful algebraicist who insisted, contrary to accumulated evidence, that the complex numbers were mere "imaginary" inventions, and could not be considered physically real. Among the implications of Gauss' discoveries is the realization, that in a certain sense the complex number domain is more "elementary", more primary than the (seemingly simpler!) domain of the real numbers (including the whole numbers), and that the latter is derived from the former.

As a matter of fact, Gauss' discovery of the significance of the complex domain is intimately related to

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[6] In effect, Gauss discovers a kind of "periodic system" underlying the properties of the prime numbers, analogous to the periodic system of chemical elements, later to be discovered by Mendeleyev, and (ultimately) reflecting a common *physical principle*.

[7] See Gauss' account of what he called "an infinite extension of the domain of higher arithmetic", via the introduction of complex numbers, by means of which many extraordinary phenomena, observed in the domain of the ordinary whole numbers, first become fully intelligible. This account can be found in the opening of Gauss' first paper on the "Theory of Biquadratic Residues", and in his famous first and second "Commentaries" on that paper.

The broader ontological significance of Gauss' notion of the "complex domain" has been set forth in the most comprehensive fashion by Lyndon LaRouche in a number of articles (see, Visualizing the Complex Domain," 21st Century Science & Technology}, Vol. 16, No. 3, Fall 2003, pp. 24-52.) The essence of the matter lies in the singular distinction of Man relative to all other species of living organisms: namely, that Man is able, by virtue of the creative powers of the individual human mind, to discover, in the form of *ideas,* principles of organization of the physical Universe, that are not directly accessible to sense perception. The creative activity of scientific discovery, and the realization of scientific discoveries in human social practice, lead to an increasing power of Man over the Universe; thereby demonstrating, that reality is located not in the domain of objects of sense perception, but in the domain of universal physical principles, which can be known to the human mind only as *ideas*. The latter domain constitutes the "complex domain" in the broadest sense exemplified most powerfully by the cited work of Gauss and his devastating refutation, in 1799, of the empiricist viewpoint of Euler and Lagrange.

the subject-matter of De Divina Proportione and the Renaissance rediscovery of the primacy of {circular and spherical geometry} relative to the "flat" plane and space geometry of Euclid. A key distinction of the complex domain, relative to the notion of simple linear extension underlying common notions of so-called "real numbers", is the central role of *rotation*, as exemplified by the famous representation of the complex number "i" or Ö(-1) as a rotation of 90 degrees. Moreover, as Gauss already understood, and his student Riemann developed more explicitly, the intrinsic geometry of the complex number domain is based on the sphere and the higher-order "covering surfaces", which result from the successive introduction of singularities.[8]

Again, the technical details are not of interest here, but rather the observation, that the introduction of *rotation* places the supposedly self-evident whole numbers in a very different light: As opposed to mathematical abstractions formed by adding "1" to itself, we now see them as intrinsically embedded in the geometry of *physical action*, in the form of completed (closed) cycles of rotation.

Geometry as a problem of physics

Our short excursion into "Higher Arithmetic" will hopefully suggest to the reader, the sense in which one might come to discover *physical principles* existing beyond and "behind" the apparently self-evident facade of ordinary visual space: principles determining the "form" of visual space itself -- as a kind of physical effect -- and giving rise to the unique existence of the five regular solids and the other beautiful harmonic phenomena that Pacioli brings together under the concept of "Divina Proportione". By this I do not mean to say that space is somehow determined *separately* from the physical phenomena occurring "within" it, quite the contrary! I am suggesting, that the "form" of visual space is determined by the very same universal principles that govern the Universe as a whole, including the phenomena that appear to

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[8] The creation of a "constructive geometry of the complex domain", by Riemann, following Gauss, addressed the following fundamental paradox of mathematical physics. The Universe, in the course of its development, constantly changes its characteristics of action. The attempt to represent such changes in formal mathematical terms leads inevitably to the appearance of *discontinuities*, at which the chosen form of representation breaks down, in a manner indicating a fallacy in one or another of its axiomatic assumptions. But while such a failure is fatal to an axiomatic-deductive mathematical system, the Universe continues merrily on with its development! What we have in the real Universe, in place of the discontinuities, are regions of ambiguity -- *singularities* -- at which the Universe, in a sense, changes its physical geometry. Thus, the higher, evolutionary form of action in the Universe has the form of the successive generation of singularities and their *integration* into a transformed geometry of action.

This sort of process is most powerfully demonstrated in the "microcosm" of human activity, in the development of Man's physical economy. Here the introduction of new physical principles into the practice of production, through creative scientific discoveries (singularities) and their realization (integration) in the form of new species of *technologies* causes changes in the characteristics of the Man's action upon the Universe.

The paradoxical task of representing such an evolutionary process in mathematical -- albeit not axiomatic-deductive -- form, was first tackled by Riemann on the basis of Gauss' work on the complex domain. Riemann laid forth his essential conception in his dissertation "On the Hypotheses which Lie at the Basis of Geometry", and elaborated it in a preliminary, but extraordinarily fruitful manner in his "Theory of Abelian Functions". In the latter paper, the act of generation and integration of singularities is represented by the geometrical construction of surfaces of higher order, starting from the sphere, by "cutting and pasting". In the simplest, illustrative case, we cut a sphere along one or more segments, forming slit-like openings. Then, we attach a second copy of the sphere, with corresponding cuts, to the first one by pasting together along the edges of the cuts. The result is a surface of higher order. In the case of two cuts, for example, this procedure transforms the sphere into a torus. A process, initially defined in terms of a spherical geometry, is extended to the higher-order surface according to a principle of least action. The combined act of generation of discontinuities (cuts), the construction of a higher-order surface and the "least action" continuation of a given process into the new geometry, Riemann referred to as "Dirichlet's Principle". It provides a preliminary mathematical image of a "unit cycle" of "negentropically" evolving physical action.

occur "inside" space.

This means, in particular, that if we want to make progress in discovering the laws of the Universe, we should avoid imposing such a priori assumptions upon the evidence, as that of a supposedly "absolute" Euclidean three-dimensional space existing separate and apart from physical processes.[9] Instead, we should follow Pacioli in considering the experimentally-determined characteristics of space, including the five regular solids, as a crucial sort of *physical evidence*, bearing upon the laws of the Universe.

These issues become really interesting, when we turn to the most hotly debated of all the questions concerning the Divine Proportion and the regular solids, namely their relationship to the organization of *living processes*. Although Luca Pacioli does not address this question at any length in De Divina Proportione -- his focus is mainly on geometry per se -- it was well-known to his readers, as well as a subject of investigation by his collaborator Leonardo Da Vinci and other scientists, engineers, architects and artists of the Renaissance. Furthermore, Plato's dialog "Timaeus", which characterizes the Universe as a whole as a living process, was well-known and much discussed at Pacioli's time.

Exactly this aspect was emphasized by Johannes Kepler in his famous World Harmony and other writings, which any reader should consult, who wants to understand the content of De Divina Proportione.

The Divine Proportion and Living Processes

Since the time of Pacioli, a vast and controversial literature has arisen concerning the occurrence of "Golden Section" proportions in the human body and other living organisms, as well as concerning its role in painting, sculpture and architecture. The critical reader will soon find, however, in looking through modern sources, that the whole matter has been left in a highly unsatisfactory and confused state.

Typical, is the controversy which arose in the middle-to-late 19th century around the work of Adolf Zeising.[10] Zeising carried out extensive measurements of the proportions of the human body in a large number of individual cases. On the basis of this statistical material, he claimed to have provided a *decisive* empirical verification, that the human body is proportioned according to the Golden Section -- not in the sense of precise agreement in each individual case, but as a *norm,* corresponding not only to the statistical average, but to a kind of biological ideal or "target", to which the development of each individual naturally tends. A norm, in other words, that would actually be reached, were it not for

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[9] The struggle to overcome "ivory tower" mathematical conceptions of space and time, exemplified by "Euclid's Elements", and to conceptualize instead a *physical space-time* free of aprioristic assumptions, has been a long and ongoing one. The issue was put forward forcefully by Leibniz in his famous correspondence with Clarke - Alexander, ed., {Leibniz-Clarke Correspondence,} Manchester University Press, 1956.
There Leibniz rejects Newton's notion of "absolute mathematical space and time", insisting that space and time are "purely relative" and derive from the actual ordering of physical events. Later, Abraham Kaestner, a leading member of the Leibnizian circles associated with Benjamin Franklin in Europe, developed an incisive critique of Euclidean geometry, that became a crucial starting-point for the investigations of his student Carl Gauss, and of Gauss' student Bernhard Riemann, on what Gauss called "anti-Euclidean geometry". The latter, in turn, inspired the work of Einstein, Goedel and others on relativistic physics.

[10] A. Zeising, 1854, "Neue Lehre von den Proportionen des menschlischen Koerpers aus einem bisher unbekannt gebliebenen, die ganze Natur und Kunst durchdringenden morphologischen Grundgesetz" ("New theory of the proportions of the human body, based on a fundamental morphological law, unknown until now, but pervading both Nature and Art.")

inevitable fluctuations and external influences. Zeising saw this as a special case of a "fundamental Morphological Law, pervading both Nature and Art". By introducing a special, but superfluous "morphological law", however, Zeising implicitly denied the most crucial conclusion of Pacioli and the classical Greek scientists, namely that the Divine Proportion expresses a characteristic of the "shaping" of visual space itself.

In the meantime, countless other empirical studies have been made, on the presence of the Golden Section and related proportions, in living organisms; from the form of Nautilus shells to the role of the Fibonacci numbers (closely related to the Golden Section) in phylotaxis in plants, the pentagonal form of many flowers and sea animals, and even the manifestation of Golden Mean proportions in the DNA molecule. All of this is well-known and can be found in countless books, articles and internet sites.

These kinds of empirical studies, however, could hardly answer the criticism of skeptics who point out, for example, that no amount of mere measurement could ever establish, that it is the precise, irrational ratio of the Golden Section -- and not some other ratio, that might by chance be close to it -- that is really "present" in a given natural object. Similarly, one might imagine reasons for the pentagonal form of flowers, or for regularities in the distribution of leaves and branches on plants, that do not require posing a "universal principle".

Skeptics also reject the often-cited evidence of the spiral shells produced by certain sea organisms, which express a principle of self-similar ("exponential") growth while at the same time displaying proportionalities in striking agreement with the geometrical series generated by the Golden Mean. The typical objection, is that the mathematically perfect spiral shells in question, represent a rare exception among the myriad forms produced by living organisms. The majority of organisms display neither a spiral form, nor do they grow in a simple exponential manner.

The confusion, reflected by these and similar objections, gives us all the more reason, to go back to the actual history of the discussion and to what Pacioli and the greatest of his predecessors and successors actually said, concerning the relationship of the Golden Mean and the Principle of Life.

The One and the Many

It was familiar to Pacioli's readers, as well as a subject of extensive investigation by his collaborator Leonardo Da Vinci and other scientists, architects and artists of the Renaissance, that the visible forms of macroscopic living organisms, such as plants and animals, display a characteristic sort of *harmonic order and proportion*, not ordinarily found in the realm of inanimate objects on the Earth, except among fossils and other immediate products of living processes. From at least the time of the Pythagorean School onward, Greek thinkers maintained, that the harmonic proportionings, characteristic of the visible forms of living organisms, must be intimately related, on the one hand, to the *unity* or *oneness* of living beings -- the circumstance, that each living organism constitutes a microcosm, a *One embracing a Many*, like the Universe itself --, and, on the other hand, to the sense of *The Beautiful*, which the contemplation of living forms normally awakens in us. Conversely, the distortion or violation of natural harmonic proportions was associated with the notion of ugliness and perversity, sickness and death, the latter occurring as a final dissolution of the oneness and integrity of a living organism.

The inner relationship of harmonic proportion, unity in multiplicity, life and beauty, as reflecting a common principle in the Universe, lay at the center of Greek science and art. Thus, the great artist Polycletus, is said to have recounted, in his theoretical work of sculpture called the "Canon", how he measured the human body and found that its parts stand in definite proportions, which he expressed in numbers. He regarded those numbers, however, only as a technical means toward discovering what he conceived as the "ideal proportions" of the human body, and which he brought into direct relationship with the notion of perfection and "the Good". This notion of perfection, however, embraced not only form, but body, spirit, thinking and action taken as a whole.

From this standpoint we can better understand the immense importance the Greeks, and later the Renaissance thinkers, attributed to the study of *geometrical proportion* and *number*. This was no abstract mathematical pastime, but a means through which to ascend from visible things to the Principle of Life itself, and of the Good, as *physical principles* embedded in the Universe. This requires going beyond our previous summary of Pacioli's conclusions, to examine the proposition, that the "Divine Proportion" may signify very much more than merely a *byproduct* of the regular solids.

Firstly, as the Pythagoreans taught, and Nicolaus of Cusa emphasized anew, the very concept of "proportion" presupposes a One that subsumes a Many. For, the *comparison* between two or more magnitudes, out of which a proportion is formed, cannot occur unless those magnitudes are juxtaposed within a single notion, a *relationship* between them. The same is true of geometrical *form*, and even of *number*, which both presuppose a "something" that exists over and above the individual, discrete parts making up the geometrical figure or number, and "binds them together" into a unity. Denying that "something" would destroy all numbers, form and proportion, as for example in the case of a "pure empiricist", who, when shown a triangular array of three dots were to declare: "I don't see three. I just see dot, dot, dot"!

Secondly, the Greeks distinguished, among the ways or *modes* by which a Many might be subsumed by a One -- whether by number, proportion, harmony, in organic Nature or in art -- between different *species*, *degrees* and *qualities*. A harmony, for example, might be regarded as the more "powerful", the more perfect the *unity* by which the Many is bound together; and the more "rich" the *multitude* and diversity of the Many so subsumed. The greatest or most powerful harmony, then, -- and most certainly the most beautiful -- would be one that subsumes the greatest possible diversity within the greatest possible unity.[11]

Now observe, how exploring these concepts by means of elementary geometrical constructions, leads us *inevitably* to the Golden Mean or Divine Proportion!

The simplest case concerns the division of a straight-line segment. If we divide the segment in half, we have a proportion which is "perfect" but also perfectly *trivial*, since there is no real difference or differentiation between the resulting segments forming the "Many". Whereas when the line is divided according to the "Golden Section", and only in that case, the resulting segments are not only different, but actually *incommensurable*; while at the same time a single proportion binds them together with each other and the original whole.

Moreover, as Pacioli emphasized, the single act of division of a line segment according to the "Golden Section", *sets into motion*, so to speak, the generation of an entire *infinite series* of different lengths by simple addition and subtraction of the resulting intervals. That entire infinite "Many", is thus embraced,

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[11] Beethoven's late works, particularly his Late Quartets, attest to a single-minded striving to achieve exactly this -- the most perfect unity of the greatest possible diversity. Beethoven's success, in this respect, is unequalled in all artistic composition to date.

in the *simplest possible manner*, by the power of a single proportion.

Shall we not agree with Pacioli, that these and related *"excellences"* of the Divine Proportion, justify our regarding it as the simplest and most elementary reflection or *projection* into the visual, geometrical realm, of a general physical principle -- a principle underlying the efficient existence and generation of "wholes" subsuming a multitude of different states or elements, in the Universe? Such a principle would have to be inseparably connected with the principle of *life*, embodied in *living organisms*, which constitute the most perfect and at the same time the most internally diverse "wholes" on the scale of normal observation. Most striking, the infinite series generated by the Golden Section displays, in the simplest conceivable way, the character of *self-similar ("exponential") growth* which we recognize as a fundamental tendency of living processes.

These general arguments concerning the relationship between the Divine Proportion and living processes, are admittedly quite suggestive; but can we regard them as *conclusive*? In fact Nicolaus of Cusa, Pacioli and their predecessors going back to Pythagoras and Plato, were in no way satisfied with mere "philosophical speculations", but drew upon the *entirety* of the physical evidence available at the time. The most relevant demonstration of this is Pacioli's close collaboration with Leonardo da Vinci, perhaps the greatest experimental scientist and observer of Nature, to have lived up to this time. It was exactly Leonardo, who, as scientist, inventor and artist, investigated the relationship of visible phenomena to the underlying laws of the Universe, in the most systematic and all-embracing way.

From Leonardo to Vernadsky

Apart from his extensive studies of perspective, of optics, of atmospheric and geological phenomena, of hydrology and hydrodynamics, of statics and mechanics, it was *living processes*, and above all *Man*, that constituted the chief focus of Leonardo's work. Although very little has come down to us in the way of explicit statements by Leonardo concerning the "Divine Proportion", his very participation in Pacioli's project attests to the importance he attributed to it; and to his judgement, that the Divine Proportion was no mere mathematical curiosity, but a matter of fundamental importance for every domain of his work.

Neither Leonardo nor Pacioli thought about that relationship in the literal, formal way that came to predominate among fanatical 19th and 20th century "believers" in the Golden Section. Those who meticulously measure Leonardo's paintings, in hopes of demonstrating the "secret" of the Golden Section, should beware: Leonardo was painting the human soul, not a mathematical proportion! Leonardo understood the Divine Proportion, as a principle shaping the entire Universe, and not as a symbolic number showing up magically in this or that object.

Indeed, it is only when we turn from the mere morphology of individual organisms, to Leonardo's extensive studies of inanimate and organic Nature *as a whole*, that the deeper significance of the Divine Proportion comes into focus. Those investigations ranged from the composition of rocks and sedimentary layers, the dynamics of the atmosphere and the flow of rivers, the anatomy and function of living organisms, all the way to what we today call the study of "ecosystems" . What has come down to us from Leonardo on these matters, attests to a revolutionary conception of Nature as an *organized totality*, including an extraordinary understanding of *geological evolution* and the role of living processes, and in particular of Man, in *transforming the Earth* – conceptions that anticipate by 450 years the founding of the science of the Biosphere by the great Russian scientist Vladimir Vernadsky! [12]

In fact, it is when we look back at Leonardo's work from the standpoint of Vernadsky as a continuer of Leonardo, that we can address the significance of the Divine Proportion in the most efficient and powerful way, opening a new approach to the age-old questions: What is life? What is the difference between living and nonliving processes? Is life on Earth an accident, or a lawful product of the development of our Universe?

Vernadsky's breakthrough in investigating the principle of life, came by shifting the focus of the investigation from the level of isolated individual organisms, to "living matter" as a whole -- the aggregate of *all* living organisms existing on the Earth at one time. And posing the question: "What is the *impact* of living matter upon its environment (the biosphere) in the course of geological time?" Thus, Vernadsky substituted in place of the perilously abstract query: "What is life?", a question that is experimentally far more accessible, namely: "What do living processes on Earth *do*?", as seen from the standpoint of the space-time scale of the geological development of the Earth. Thus Vernadsky was able to bring the entire, vast array of accumulated geophysical, geochemical, biological, paleontological and related data to bear, on what would otherwise have remained a matter of mere speculation. Thereby, Vernadsky emulated Leonardo's approach, conceptualizing a principle of Nature on the basis of the most sweeping synthesis of all available empirical evidence.

Vernadsky's conclusions, which were no less than revolutionary, can be summarized, *in part*, as follows:

1. In the course of evolution, the aggregate "free energy" of the living matter in the biosphere -- its ability to *do work* in transforming the environment -- has constantly *increased*.

2. As a result of that increase in free energy, living matter has become the *most powerful geological force* in the biosphere, totally transforming the physical-chemical and geological characteristics of the entire region of the Earth's crust, oceans and atmosphere populated by living organisms (i.e. the biosphere). Virtually the entirety of our present environment, including the composition of the atmosphere, the climate and geological characteristics of the Earth's surface, was created in its present form as a result of the action of living processes. In the course of geological time, what Vernadsky called the "biogenic flow of matter and energy" -- i.e., the throughput generated by the activity of living organisms -- has constantly grown, both in scale and intensity, to play an increasingly dominant role within the overall throughput of the biosphere as a whole.

3. The capacity for this specific sort of *"anti-entropic"* development, characterized by a continual increase and intensification of free energy, is *absolutely unique* to living matter, and is not found in the nonliving domain of the biosphere. While living matter has undergone a constant process of intensification and development in the course of geological time, the behavior of nonliving matter on the Earth has remained virtually constant over billions of years, *except* insofar as it has been modified under the action of living matter. While isolated nonliving processes on the Earth seem to tend toward equilibrium and an increase in entropy, living processes display the *opposite* tendency, developing further and further away from equilibrium.

On this basis, Vernadsky concluded that an *absolute, unbridgeable gap* exists between nonliving and living matter in the biosphere. Life, as expressed concretely in the geological, Earth-transforming activity

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[12] See Dino de Paoli, Leonardo da Vinci's Geology and the Simultaneity of Time," {21st Century Science and Technology,} Summer 2002 Issue, Vol. 15, No. 2.

of living organisms on the Earth, must therefore embody a distinct *physical principle* or set of principles, that are absolutely different from, and in a certain sense superior to, the principles dominating the apparent behavior of nonliving matter in the biosphere.

Does the "shape" of visual space reflect the principle of life?

Now "read" the significance of the Divine Proportion from the standpoint of Vernadsky's conception of evolution as a lawful process! Living matter in the biosphere constitutes a process of growth, not -- or not primarily -- in visible spatial dimensions per se, but (to a first approximation) in the intensity of the generation of free energy per unit area of the Earth's surface. That flux-density manifests an exponential tendency for growth, which (unlike an individual organism, which stops growing after a relative short period of time) has continued over billions of years! That development of living matter in the biosphere, occurred not in a linear way, but through a succession of distinct states of the global "ecosystem", each with its specific array of biological species.

To the extent the growth of the biosphere, in the sense just described, is the expression of a *principle* in the Universe, that principle constitutes a "One" subsuming the "Many" evolutionary stages. The principle itself remains *invariant* throughout the process; it is everywhere *similar to itself*. Seeking the simplest geometrical image of such a process, we arrive once more at the self-similar spiral based on the "Divine Proportion"! That implied relationship, between the "Divine Proportion" and a principle of development of living matter as a whole in the biosphere, is something much more solid and fundamental, than a mere morphological characteristic of a specific living organism, like a sea shell or the proportions of the human body.

Going further, we arrive at even more Earth-shaking conclusions! To the extent the development of living matter on the Earth expresses a distinct physical principle, operating in the Universe, life and its characteristics obviously cannot be regarded as some sort of a "lucky accident", as some biologists today seem to believe. Vernadsky emphasizes, in this context, the work of Louis Pasteur and Pierre Curie on the characteristic "molecular dissymmetry" of living processes, which Pasteur explicitly regarded as the "local" expression of a property of the Universe as a whole.[13] Vernadsky also spoke of life on Earth as a "cosmic phenomenon", inseparable, for example, from the organization of the solar system.

The notion, that living processes on Earth express a general property of the Universe, is of course very old. It was a central feature, for example, of Plato's Timaeus, in which the Cosmos is characterized as a "living being". In the same dialogue, the Pythagorean Timaeus associates the *duodecahedron* -- the most "universal" of the regular solids, and whose characteristic proportion is the Golden Section -- with the geometry of the Heavens.

Aha! Remember what we discussed earlier, about the "shaping" of visual space itself, by the same universal physical principles, that govern all processes in the Universe. Could it be, that the singular role of the Golden Section or Divine Proportion, relative to the five regular solids as characteristics of the "shaping of visual space", is a "product" and expression of a universal principle of life, embedded in the Universe as a whole?
This question raises a wonderful paradox. If we admit that there is a principle of life -- so profoundly

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[13] See Vernadsky, "On the Fundamental Energetic-Material Distinction of Living Matter", {21st Century Science and Technology,} Summer 2001, Vol. 14, No. 2.

embedded in the Universe, that it co-determines the very shape of visual space --, then that principle would necessarily *predate* the emergence of living organisms themselves, at least on the Earth. But, what could have been the meaning and effect of such a "pre-existing" principle of life in the Universe, at a hypothetical time *prior* to the emergence of the "first" actual living organism? Could it not be, that that principle already found expression in a "prebiotic" process of evolution of nominally nonliving matter, including the early evolution of the solar system and the Earth, which *in a directed way* created the pre-conditions for the later emergence of living organisms? This was the view of Vernadsky and his successors.

Furthermore, if the principle of life is really a universal principle, then it must necessarily, and constantly, leave some traces or effects of its action on each and every process in the Universe, including in ostensibly "dead" matter!

This thought suggests still another paradox. If the Universe as a whole is governed by a principle of life, and is even itself "living" in some sense, then why does there exist any "dead" matter at all in it? And whence comes the "unbridgeable gap" between living and nonliving matter, which Vernadsky demonstrates to exist in the biosphere?

Not accidentally, the interrelations among the five regular solids, described by Pacioli and later emphasized anew by Kepler, provide a most suggestive clue toward unravelling these paradoxes.

Going back to the geometrical investigations of Pacioli, the key thing to observe, is that the five regular solids naturally fall into two distinct, hierarchically-ordered groups:

The first, higher-order group, consists of the duodecahedron and icosahedron, which are "duals" or "twins" to each other, in the sense that each arises from the other by joining the midpoints of the faces.

The second, lower-order group, consists of the cube, octahedron and tetrahedron. Their relationship to the first group is that they can easily be derived from the duodecahedron (or icosahedron) as simple embedded figures. For example, we create a cube, by merely joining selected vertices diagonally across the pentagonal faces of the duodecahedron. By similarly simple means, the octahedron and tetrahedron derive from the cube.

However -- and this is *crucial* -- it is *not* possible to reverse this process, and to go from the lower to the higher-order solids in the same, straightforward way. For example, as Pacioli indicated, it is impossible to build the duodecahedron from the corresponding cube, without utilizing the Golden Section proportion, or some equivalent intermediate step. While the duodecahedron "automatically" gives birth to the embedded cube, it is virtually impossible to discover how to go from a cube to the corresponding duodecahedron, without "working backwards" from the duodecahedron and its Golden Section proportions, as if already pre-existing.

Doesn't the relationship between the first and second species of regular solids, cohere with that between living and nonliving processes in the Universe?

Kepler drew exactly this conclusion, in his delightful piece entitled "The Six-Pointed Snowflake", which laid the foundation for the modern science of *crystallography*. Kepler notes, there, that the cubic, octahedral and tetrahedral geometries, and geometries derived from them, are characteristic of the *crystalline forms* of nonliving matter in the small. These geometries have the property, that they can be generated by a kind of "mechanical necessity" alone, as in the example of the hexagonal or cubic crystal lattices that arise from the "close packing" of spheres. By contrast, five-fold, pentagonal forms, with their

Golden Mean proportions, are not generally encountered in inorganic Nature; nor do they arise naturally by the same sort of "mechanical necessity" as the crystallographic forms. Accordingly, Kepler associated the first group of regular solids with the domain of living processes, the second with non-living matter.

What should we conclude from this wonderful correspondence between the two groups of regular solids, and the two domains of the Universe -- the living and the nonliving? Nonliving matter is a byproduct of a "living Universe", but does not otherwise embody the principle of life itself, in the same, direct way, as living organisms. It is possible for living matter to generate dead matter, but there is no known way to generate "life" from dead matter, without the intervention of a living process. As a plant can organize and integrate originally nonliving materials, such as water and minerals, into its living tissue, so living processes exercise a kind of "dominion" over nonliving processes. But no combination of the "mechanical" principles per se, of the sort that dominate the ostensible behavior of nonliving matter could ever generate a living process.

The human mind

The ideas, we have laid forth here, hardly resolve all questions and paradoxes concerning the nature of "life". Yet, a study of the "Golden Ages" of science over the last 2400 years, demonstrates that our indicated reading of the significance of the Divine Proportion and the regular solids -- following Plato, Pacioli, Nicolaus of Cusa, Kepler et al, --, has served over the ages as an extraordinarily fruitful "higher hypothesis", leading the human mind again and again to fundamental scientific discoveries.

The human mind! We haven't talked about it much so far, although it is implicitly the whole subject of Pacioli's book! After all, only human Reason is capable of "seeing", in a mere geometrical proportion, what sense perception could never discover: an expression of the *Divine!*

Again, Vernadsky's work provides perhaps the simplest pathway to the point at hand. Vernadsky observed, that the emergence of Man introduced a new active principle into the biosphere: human Reason, which is *not* subsumed by the principles of living matter *per se*. Over the course of time, Man's productive activity, in such forms as agriculture, mining, industry, infrastructure etc., has emerged as a new and ever more dominant "geological force" within the biosphere. In fact, as Vernadsky demonstrated, the effect of human agriculture and related activity, has been to greatly *accelerate* the anti-entropic growth of "free energy" in the biosphere, while transforming the structure and intensity of its throughput of matter and energy in a manner unprecedented in the preceding evolutionary history of the Earth. Vernadsky spoke of the transition of the Biosphere into a "Noosphere",[14] in which the creative mental activity of human beings has become the dominant "vector" of development.

Just as Vernadsky demonstrated an "unbridgeable gap", separating the characteristics of living matter from those of nonliving matter in the biosphere, and reflecting the existence of a distinct principle of life, so human Reason reflects still another, still higher principle in the Universe.

The most rigorous, decisive demonstration of a second, "unbridgeable gap" between Man and all other

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14 See the author's introduction to Vernadsky's "On the Fundamental Energetic-Material Distinction of Living Matter" in the same issue of {21st Century Science and Technology} Summer 2001, Vol. 14, No. 2, as well as his conference presentation Vernadsky and the Science of Life" {Schiller Institute, Bad Schwalbach, Germany, May 4-6, 2001- Panel V} forms of living matter in the biosphere, has been elaborated by Lyndon LaRouche.[15] The kernel of the proof lies in the implications of the fact, that only the human species, among all living species, has demonstrated the ability to progressively increase its potential population density, in a deliberate and sustained manner. Indeed, as a result of successive improvements in the technological and related organization of human activity, the population potential of the human species on this planet, has increased by a factor of 1000 or more in the course of documented history and prehistory!

Those improvements, exemplified most clearly by the development of science and technology, involve successive, sweeping changes in human behavior, having no equivalent in the relatively fixed, genetically limited range of behavior of other living species. That unique power of Man, a power expressed in scientific and technological revolutions, lies in the faculty called "Reason".

But, what is this faculty of Reason, really? How does it work? We come back to Pacioli, and the reason, why he insisted on the term "*Divine*", for what others merely called a "golden" proportion.

What, indeed, is the quality that Pacioli demands from YOU, as readers of his book? He demands, that you not merely read letters and words, but grasp his *meaning* -- an *idea*, which lies beyond mere words and images, which cannot be seen or grasped by the senses, but can be generated inside the creative processes of your mind. The "Divine Proportion" is an *idea* of exactly that type.

The paradoxical existence of exactly five regular solids in visual space, and of their harmonic interdependence, points to the existence of principles lying outside and beyond visual space per se, but which determine the characteristics of visual space and the form of appearance of phenomena within it. These principles are not visible objects, but constitute objects of a different sort -- *thought-objects*, objects of the human mental processes. Such thought-objects include the principle of life, and the principle of reason. They are physically real, as Vernadsky demonstrated; they generate enormous *effects* in the domain of visible phenomena, yet they are not themselves objects of the senses.

The ability to discover and apply these and other "thought-objects", in the form of a growing array of physical principles integrated into human productive activity, is the basis of scientific and technological development, and thereby of Man's growing dominion over the biosphere and, by implication, over the Universe as a whole. Insofar as the Universe responds to such accomplishments of human Reason, by granting Mankind an ever-increasing power over the forces of Nature, human Reason, so defined, must be in correspondence with the principle of creation of the Universe itself.

Hence, Pacioli's insistence on the "Divine". And it is that quality, Dear Reader, that I hope, by this modest introduction, to have helped you to discover in yourself.

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[15] "V.I. Vernadsky and the Transformation of the Biosphere," a lecture by Lyndon H. LaRouche, Jr. on June 28, 2001 to the Lebedev Institute of Physics of the Russian Academy of Sciences, and the dialogue with Russian scientists which followed it. Executive Intelligence Review, v. 28, No. 28, July 27, 2001.

Luca Pacioli's *"Platonic Space"*
by John P. Scialdone

From the beginning of mankind's existence, the cognitive power of the human species as a whole has allowed a continuous, general growth of its population, eventually reaching a few hundred millions of living inhabitants on the surface of this planet; but at that level it stayed for millennia, up through the New Dark Age of the fourteenth century. This changed in the fifteenth century with the Golden Renaissance. Luca Pacioli's life and works, including the text before us, were at the center of this great shift that occurred in mankind's existence, -- a revolution, based on the idea of universal progress inherent in the uniquely *agapic* and creative nature of man.[16] Around this Christian-Platonic conception of man, was constructed the institution of the sovereign nation-state, which afforded every individual in society the opportunity, and implicitly the obligation, to participate in the larger affairs of society, through the free exchange of ideas, and the continuous development of a literate, vernacular language-culture.

With the break down of the Latin-based Charlemagnan Empire, following the death of Frederich II Hohenstauffen in 1250, the basis for the function of language-culture in the realization of the role of the individual in society, located in the individual's sovereign power of *agapic*[17] reason, was established by Dante Alighieri (1265-1321), in his ***De Vulgari Eloquentia*** (On the Eloquence of the Vernacular), ***La Vita Nuova*** (The NewLife), and other writings.[18] In ***De Monarchia Mundi*** (On World Monarchy), he advanced the concept of sovereign nation-states within a world monarchy.[19]

The bridge from the efforts of Dante, as advanced by Petrarca, Boccaccio and others, through the New Dark Age and into the Renaissance, was one of the most sublime figures in all history, Jeanne d'Arc (1412-1431). She opened the door from medieval feudalism to the world of the modern nation, as she elevated a disgusting, sluggard ruler to represent the nation of France, establishing the principle of

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[16] Jonathan Tennenbaum, Introduction to the present text. Jonathan Tennenbaum, *The Economics of the Noösphere*, Executive Intelligence Review Book, 2001: Appendix: Problems of Biogeochemistry II, Introduction p. 265-274. Lyndon LaRouche, "Conflict Is Not the Natural Condition Among Men and Nations," *Executive Intelligence Review*, Vol. 30, Number 1, January 10, 2003, pp.26-34.

[17] Gr. - *agape*; It. - **carità**; Christian-Platonic Love; love of God for humanity, as in Paul's Corinthians I:13 (charity or love); or love of truth, as in the ***Republic*** of Plato. See also Johannes Brahms' ***Vier ernste Gesaenge*** (Four Serious Songs), not only the fourth song, but the four taken as a whole.

[18] Similarly, the great Islamic renaissance was inseparable from the development of the Arabic language itself. See Muriel Mirak Weissbach, "The Power of Great Poetry to Shape the Character and Build the Nation: Dante, Humboldt, and Helen Keller," *Fidelio* magazine, Vol. V, No. 2, Summer, 1996. See also Muriel Mirak Weissbach, "Andalusia, Gateway to the Golden Renaissance," *Fidelio* magazine, Vol. X, No. 3, Fall, 2001.

[19] Indicative of the relationship between the education of society for the sake of love of truth and of mankind, and the effort to form a new form of sovereign nation-state, is the way that Book I of Dante Alighieri's ***De Monarchia*** opens: "For all men whom the Higher Nature has endowed with a love of truth, this above all seems to be a matter of concern; that, just as they have been enriched by the efforts of their forebears, so they too may work for future generations, in order that posterity may be enriched by their efforts. For the man who is steeped in the teachings which form our common heritage, yet has no interest in contributing something to the community, is failing in his duty...." (Dante, ***Monarchy***, tr. Prue Shaw; Cambridge University Press 1996.)
Or as Pacioli states in chapter I of the present text: "[T]hat person be worthy of praise before God and man, who, having received some particular gift, communicates it voluntarily to others: because he awakes in them **carità** . . . ."

sovereignty of a people over their nation.[20] The genius of this teenage maiden, shattered the corrupt feudal order of the Holy Roman Empire, and made possible the development of the institution of the sovereign nation-state by Nicholas of Cusa (1401-1464) around the ecumenical Council of Florence of 1439.[21] Cusa not only developed the constitutional outline of the modern self-governing nation-state with his ***Concordantia Catholica*** (Catholic Concordance), but established the foundation of modern science

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[20] For a concise discussion and for other important sources of Jeanne d'Arc's historic mission, see Lyndon LaRouche, *The Historical Jeanne d'Arc*, Executive Intelligence Review, Nov. 17, 2000, Vol. 27, No. 45, p.68; Irene Beaudry, *The Military Genius of Jeanne d'Arc, and the Concept of Victory*, ibid. p.64. See also the transcripts of the Trial of Jeanne d'Arc of 1431, tr. W.P. Barrett from the Latin and French documents, Gotham House, Inc. 1932. In the same Gotham House volume, is an essay and dramatis personae by Pierre Champion, of the "trial judges and other persons involved in the Maid's career, trial and death," which gives some interesting comparisons of testimony by some who were involved in the trial of condemnation, who also testified at the trial of Rehabilitation of 1455, usually with much changed stories.

[21] The comprehensive solution to the mission of creating a republican form of sovereign nation-state, was finally supplied by Cardinal Nicholas of Cusa, in his ***De Concordantia Catholica***, written for the Council of Basel in 1434. Cusa defined the institutional forms for Constitutional, representative government, including separation of powers, and the responsibility to promote the general welfare of all the people and finding in the concern for posterity, the true meaning of the pursuit of happiness -- the sovereignty responsibility of the state deriving from the sovereignty, and therefore the equity and freedom, of each individual, who is *imago Dei*. He writes,

"... since all are by nature free, every governance, whether it consists in a written law, or in living law in the person of a prince ... can only come from the agreement and consent of the subjects. For, if men are by nature equal in power and equally free, the true, properly ordered authority of one common ruler, who is their equal in power, can only be constituted by the election and consent of the others, and law is also established by consent."

Later we would see these ideas further developed by Gottfried Wilhelm Leibniz in ***On Natural Law***, in 1690, by Cotton Mather in ***Essays To Do Good*** in 1710, by Emmerich de Vattel in ***The Law of Nations***, 1758, and finally, of course, by Benjamin Franklin himself.

In 1438, Nicholas of Cusa and his allies forced a break with the conciliar movement within the Latin Church to establish an ecumenical council to reunite the Eastern and Western Churches as a whole. This was the great Council of Florence (1439-1441), which was the launch point of the Renaissance, and the watershed of modern history. It brought together over a period of years, some of the greatest minds living in all Christendom at that time, along with the most extensive collection of ancient Greek, Arabic and other texts ever assembled, since the burning of Aratosthenes' Great Library in Alexandria, Egypt in 48 B.C., and Al-Khwarizmi's House of Wisdom of the Baghdad Caliphs of the eighth and ninth centuries. Held appropriately at the Cathedral of Santa Maria del Fiore – begun in 1296, the largest church then yet built in Italy -- under the *duomo* completed by Brunelleschi in 1436, Gemistho Plethon (1356-1450) delivered there the famous lectures that ended the hegemony of Aristotle in European philosophy, and established Plato as the philosopher whose Socratic method was coherent with and necessary for true Christian thought, as the Plato-Aristotle debate had also been decisive earlier during periods of the Islamic Renaissance.

This council sought the unification of the Eastern and Western churches, around the disputed principle of the *filioque*, the idea that the Holy Spirit, or *Logos*, proceeds equally from the Father and from the Son. If the *Logos* flowed from Christ as from God, and Christ was also man, then this potential existed in the free will of the individual to do good, defining the sovereignty of the individual, and the ability to have an unmediated relationship with the Creator. Travelling to Constantinople to study the Church texts, Cusa successfully demonstrated that this disputed idea, which had been incorporated into the Creed of the Council of Nicaea in 325, was indeed always present in the Church writings of the Greek and other Orthodox Churches. Cusa saw man as the living image of God, *imago viva dei*, and being capable of participating in God, *capax dei*, through improvement of that talent which uniquely sets him apart from the beast -- agapic and creative reason.

with ***De Docta Ignorantia*** (On Learned Ignorance) and his later writings.[22] With Cusa and the Council of Florence, the modern world begins.[23]

The Renaissance returned science to its classical roots in ancient Greek-Egyptian knowledge, through the mediation of the Islamic-Andalusian renaissance[24] (roughly concurrent with, and collaborating with, the Charlemagnan Empire of the eighth through thirteenth centuries), intentionally overthrowing the Aristotelean fixed-axiomatic method of description of simple sense-perception, and reinstated the Platonic constructive-geometric method in science, of locating and solving the apparent contradictions or paradoxes which emerge from sense-perception, through a rigorous process of higher hypothesis, as described for example in Plato's ***Timaeus*** dialogue.[25] Thus, the progress of classical science was picked up once again, following its virtual destruction by Aristotelean thought, from the point of the Roman Empire's murder of Archimedes in 212 B.C., and its burning of the great Library in Alexandria, Egypt in 48 B.C.

Typical of such revolutionary science, are Leonardo Da Vinci's investigations of hydro-dynamics, optics, and curvilinear perspective; Johannes Kepler's investigation of the implications of the Platonic solids inscribing and circumscribing nested spheres, in the harmonic spatial organization of the Solar System; the negative curvature found in the use of the hanging chain, the catenary, by Filippo Brunelleschi in the design of the great dome over the Florence Cathedral; or the purpose of the bronze ball atop that dome, which the young Leonardo helped cast in the workshop of Andrea del Verrocchio.[26]

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[22] Jasper Hopkins, *Nicholas of Cusa On Learned Ignorance, A Translation and an Appraisal of De Docta Ignorantia*. The Arthur J. Banning Press, 1981. For a very large selection of predominantly first translations of Cusa's scientific/theological works, see William F. Wertz, Jr., ***Toward a New Council of Florence, 'On the Peace of Faith' and Other Works by Nicholas of Cusa***, Schiller Institute, Inc., 1993.

[23] See Helga Zepp-LaRouche, "A Contribution for Nicholas of Cusa's 600th Birthday: A Dialogue of Cultures," *Fidelio* magazine, Spring 2001, Schiller Institute.

[24] Op. cit., Muriel Mirak Weissbach, "Andalusia, Gateway to the Golden Renaissance."

[25] For an extended discussion of Pythagorian/Platonic vs. Aristotilean science, see the pedagogical series, ***Riemann for Anti-Dummies***, by Bruce Director at http://www.wlym.com/pages/pedagogicals.html

[26] Bruce Director, "The Long Life of the Catenary: From Brunelleschi to LaRouche", *Fidelio* magazine, Vol. XII, No. 1, Spring 2003, p.100. See also Claudio Rossi, "The Apollo Project of the Golden Renaissance - Brunelleschi's Dome," *21st Century Science and Technology*, Vol. 2, No. 4, July-August, 1989, p. 24.

The construction of the Cathedral of Santa Maria del Fiore, was a project that would shape Florentine culture and science over a span of 150 years. Begun in 1296, and completed just before the Council of Florence commenced in 1439, this project survived repeated visits of the plague which depopulated the city, but this "Apollo Project" of its time, kept the minds and skills of the city alive. The walls of the cathedral rose 140 feet above the ground level, upon which an octagonal drum rising an additional 30 feet would stand as a ringed base upon which the cupola was to sit. Vaulting would thus begin at 170 feet high, with no machines yet in existence to make work at such heights very practical, let alone possible. The cupola incorporated an important innovation developed earlier in the Islamic renaissance, by building them with a double shell, giving greater strength to the domes of their mosques. Spanning 143.5 feet at the drum, the dome had a radius of 10 feet at its apex upon which a lantern would stand. Due to the scarcity of wood for scaffolding, Filippo Brunelleschi's ingenious method of construction, which included the use of the hanging chain, allowed the dome to sustain its own weight as it went up, both giving an intrinsic strength to the structure, and simultaneously solving the problem of scarcity of wood, since no scaffolding for centering from the ground was needed.

We can see from this project alone, the Apollo Project of its day, the determination to rescue human existence, once, and for all, from its tortured condition of existence; and what exactly was the larger mission of Luca Pacioli, in his lifelong effort to revolutionize and promulgate the mathematical sciences, for the city building efforts needed to transform Italy into a nation-state.

Even the bronze ball supported atop the lantern, which the young Leonardo helped to cast in the workshop of Andrea del Verrocchio, represented a revolution in metallurgy and building techniques, giving strength to the entire structure.

Luca Pacioli was a follower of Nicholas of Cusa, and, just as he was a student and collaborator of Piero della Francesca, he was a teacher and collaborator of Leonardo da Vinci. He was a master of all forms of modern mathematics and geometry of his time; but his leading role was within the conspiracy to transform Italy into a sovereign nation-state, through making the most advanced conceptions available to the society as a whole, through his perpetual activity in teaching, and through his efforts to put these ideas forward in the vernacular language. This revolution in science and statecraft directly challenged all previous oligarchical forms of organization of human existence, and eventually led to the creation of the first true republic on the shores of North America, and the ensuing development of the modern world.[27]

It is a bit odd that this five hundred year old seminal work in mathematics has never heretofore appeared in the English language. Actually, it was not the original intention of either translator, to work on a published translation of ***On the Divine Proportion***. While Pacioli dedicated much of his life to putting the greatest ideas of science into the vernacular, his is not the Italian of Dante or Petrarca, or of Pacioli's literary contemporaries. In the course of *searching* for a definitive translation, the acquaintance of the publisher was made, who informed us, that he had already gone through five unsuccessful translators. Abraham Kaestner, the founder of "anti-Euclidean" geometry, and teacher of the great Carl Gauss, translated and transcribed fragments of the work for his great ***Geschichte der Mathematik*** (History of Mathematics) at the close of the eighteenth century, and commented,

"I don't understand everything that I've transcribed ... as indeed it happens to the greater number of scholars .... Pacioli's Latin is, as one sees by trying, not the clearest, and those for whom he wrote, understood what to us is dark, as we don't know all that he refers to. I will therefore do with my author [Pacioli] as is always done by the interpreters of authors, use what I understand, and leave the rest to future elucidation."[28]

And so, it happily falls to us to bring this great work finally to light, in the English language.

The text before us is from the original manuscript edition of 1498, although the first printed edition of ***On the Divine Proportion*** was published in 1509 in Venice, which included additional sections following the treatise itself. The title of the 1509 volume referred only to the text of ***On the Divine Proportion***, which is then followed by a second part on architecture, as Vitruvius based it on the proportions found in the human body, and a third on the construction of the alphabetical letters. A final part contains the translation into the vernacular of the ***Libellus de Quinque Corporibus Regolaribus*** of

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By setting society to accomplish such a great project, not with slave labor, but precisely by creating the most skilled labor force possible (and fracturing the feudal guild system in the process), the builders of Florence were able to bring forth from society as a whole the greatest achievements, laying the basis for the needed revolutionary changes in organization of human existence. We see the relationship between the creation of the institution of the sovereign nation-state, and the concomitant dirigistic economic policies already maturing, antedating the Renaissance itself. Following precedents such as the great cathedral building projects of the Charlemagnian period, this approach of "Great Projects," that is, the which require going beyond all previous frontiers of knowledge and accomplishment, has been a characteristic of successful republican government ever since.

[27] Lyndon LaRouche, *The Economics of the Noösphere*, Executive Intelligence Review Book, 2001: "The Gravity of Economic Intentions," chapter 4, *"The Sovereign Nation-State Economy"* pp. 246-250. See also H. Graham Lowry, *How the Nation Was Won, America's Untold Story 1630-1754*. Executive Intelligence Review Book, 1988.

[28] Abraham Gotthelf Kaestner, ***Geschichte der Mathematik***, tr. Margaret Scialdone. Ironically, this seminal work by Abraham Kaestner has also not yet appeared in English, although, already in the 1740's, Kaestner was engaged in translating into German the English language works of the scientific circles of Benjamin Franklin. Note also the extreme measures taken to prevent such circulation of the scientific and philosophical works of Franklin, Leibniz and others, between Europe and the North American English colonies. David Shavin, "Leibniz to Franklin On 'Happiness,'" *Fidelio* magazine, Vol. XII, No. 1, Spring 2003.

Piero della Francesca.[29]

Luca Pacioli opened his manuscript ***On the Divine Proportion*** with a dedication to the Duke of Milan, Ludovico II Sforza, describing a great gathering in the Duke's presence, in his illustrious city of Milan: "an assembly dedicated to scientific debate, composed of people of all ranks, famous and most wise, both religious and secular, with whom your court continually abounds." He turns again to this great gathering at the conclusion of the book, alluding to the prominent participation of his own Franciscan Order, and others; and, although he states in both cases that it occurred in the current year, 1498, it is not clear whether this gathering actually took place, or whether it is a metaphorical description of the great scientific ferment occurring over time at the Academy of Leonardo Da Vinci at the Castle of the Duke. He leaves us in a quandary by identifying the presence of individuals not living at that time, including his great scientific predecessor, Nicholas of Cusa himself.[30] Raffaello Sanzio gives us a hint to solve this riddle, with his great mural in the Stanza della Segnatura of the Vatican, ***The School of Athens***, describing just such a scene of great scientific debate, with Plato in motion (with the face of Leonardo) carrying his ***Timaeus***, and Aristotle in an awkward, immobile stance with his ***Ethics,*** entering a hall filled with groups of historic personalities from different times in history, engaged in various discussions.[31] Interestingly, along the floor, below Raphael's murals in that Vatican Room, there is an ingenious wood inlay, showing, in perspective, a cabinet, which appears to be in the wall of the room, with the Platonic solids placed on shelves, partly concealed behind the half-open cabinet door.

Similarly, in his extended tribute to his friend Leonardo, Pacioli describes Leonardo's casting of a massive equestrian statue. It is generally acknowledged that the project was under way; however, either it was not completed before the invading armies of France entered Milan in 1499, or it was, and was promptly melted down for cannons and ammunition and the like, with Ludovico's defeat, both Pacioli and Leonardo having already fled the city.[32]

Before moving to the divine proportion itself, Pacioli discusses the origins of science *per se*, and the role of science in building and defending cities, a subject of great interest to Ludovico Sforza at that time [ch. II]:

". . . the defense of great and small republics, otherwise called the military arts, is not possible without

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[29] See Richard Sanders' "The New Renaissance Is Here," immediately following this preface. Far from the shameful and absurd charge by Giorgio Vasari, that Pacioli had plagiarized della Francesca's work, Pierro della Francesca and the prodigious Luca Pacioli were both engaged in the reworking of many source texts, including, of course, Book XIII of Euclid's ***Elements***, (the Campanus Latin translation from the Arabic, containing extensive annotations on the "Platonic" Solids), which itself was a reworking largely of the work of Theaetetus, student of Plato's Academy, where the work of the Pythagoreans was being reworked and advanced, as well as earlier Egyptian science centered at the Temple of Ammon at Cyrenaica. In this respect, the Renaissance was based on reviving, reworking and advancing Greek/Egyptian and Arabic/Hiindu sources.

One is reminded of the response of Johannes Brahms, to the criticism of a resemblance in one of his early works, to a passage of Mendelssohn: "Some booby has already been telling me something of the kind. (So was hab' ich schon von einem Rindvieh gehört)... Such things are always discovered by the donkeys." (Florence May, ***The Life of Brahms***, Paganiniana Pub., Inc., Neptune, NJ, 1981, repub. of 1905 ed., Vol. I, pps. 137-138.)

[30] It is precisely this paradox, which leads to mistaken efforts to claim that the Nicholas of Cusa that Pacioli refers to, could not have been Cardinal Nicholas of Cusa, because the Franciscan Friar Pacioli would have referred to him as a Cardinal. Indeed, in chapter II, Pacioli refers to Saint Augustine as Aurelius Augustine, even in the context of referencing Augustine's ***The City of God***.

[31] Lyndon LaRouche, *The Economics of the Noösphere*, Ibid., "Shrunken Heads in America Today," pp.49-54. See also Gerry Therrien, unpublished, *On Raphael's "The School Of Athens,"* 2003, Montreal, Canada.

[32] A commemorative casting by sculptor Nina Akamu, was recently completed at the Tallix Art Foundry, sponsored by Leonardo da Vinci's Horse, Inc (Sovereign Building, Courtyard Level, 609 Hamilton St., Allentown, PA 18101), and was placed at the Cultural Park in Milan, Italy on September 10, 1999. See www.leonardoshorse.org.

being able to exercise with merit and advantage the knowledge of Geometry, Arithmetic and Proportion. And never could it be said that an army assigned to carry out a siege or defense, were provided with everything necessary, if in that army there were not to be found engineers, and if there were not ordered new machines of war, as we said just now about Archimedes, the great geometer of Syracuse."

In the third chapter, entitled "What the name, mathematics and the mathematical disciplines, means and implies," Pacioli proposes to solve an old issue within science, taking the side of his friend Leonardo; namely,

"This term, mathematics, is Greek, derived from *μαθηματικός*, which in our language simply means something which can be an object of study, and for our purposes, by the mathematical sciences and disciplines is understood Arithmetic, Geometry, [Astronomy], Music, Perspective, Architecture, Cosmography, and any others dependent upon these. Nonetheless, scholars commonly acknowledge only four, that is Arithmetic, Geometry, Astronomy and Music as primary, and call the others subordinate.... But in our [Luca and Leonardo? - tr.] judgement ... we must confine it to either three, or five of them: that is, Arithmetic, Geometry and Astronomy, and exclude Music for the very reasons that other scholars omit Perspective; or five, by adding Perspective to the four said disciplines, for just as good reasons as others gave for adding Music to the three we indicated."

In Book II of ***On Learned Ignorance***, Cusa states, "In creating the world, God used arithmetic, geometry, music and likewise astronomy."[33] Was Cusa, the father of the Renaissance, mistaken? Not at all! In the period following the Council of Florence, the work of those pioneers, with whom Pacioli collaborated, such as Leon Battista Alberti and Piero della Francesca, set the stage for the revolution in curvilinear perspective by Leonardo. The results are seen in the famous perspective drawings done by Leonardo to illustrate the solids at the end of the first 1498 manuscript copies of ***On the Divine Proportion***. Pacioli asserts that Perspective had now indeed become a science, as Leonardo also attests in the notes for his projected book on Painting, "the mother of perspective."[34] This was one of the great advances of the Renaissance, as applied to painting, engineering, machine building and military fortifications and weapons design. Also indicative of the maturation of perspective as a science, is the work one of Pacioli's greatest students, Albrecht Dürer, in his "Der Zeichner" series of etchings.

With these introductory remarks, Pacioli is ready to proceed to the subject at hand, and tells us why he is the first individual to give the name Divine to this proportion. He states: "A fitting title for our tract, it seems to me, should be 'On the Divine Proportion,' because many of the qualities which I find in our proportion, are also those which we attribute to God." Describing four such divine attributes, he then identifies a fifth, which gives intelligence to an otherwise chaotic universe, -- the "Quintessence", which Plato also described in the ***Timaeus***, as giving order to the four primary elements, earth, water, air and fire. As Pacioli stated it in chapter V: "It is possible to arrogate the fifth quality, not unworthily to the aforementioned: that is, just as God confers Being to the Heavenly Virtue, by another name called Fifth Essence," according the quality of life to the divine proportion. Of this we'll say more below.

So, with Plato's ***Timaeus*** as his guide, Pacioli developed the implications of the Divine Proportion and the five regular Platonic solids, using the Latin edition of Euclid's ***Elements***, which included books XIV and XV, translated from the Arabic with annotations and commentaries by Johannes Campanus of Novara, chaplain to Pope Urban IV (1261-1264). The first printed edition appeared in 1482. Pacioli referenced the Campanus edition more than 175 times in the course of the book, and would later defend it, with its commentaries and annotations, against other new translations of Euclid, such as that of Bartolomeo Zamberti in 1505, exemplified by Pacioli's insistence also on retaining some of the Arabic

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[33] Op. cit., Jasper Hopkins, *Nicholas of Cusa On Learned Ignorance*, II: par.175, 176.
[34] DaVinci, *Leonardo On Painting*, Yale University Press, 1989. See footnote in Chapter III of the present text.

terminology over use of Latin equivalents. The Campanus edition was preferred by Johannes Kepler long after other editions appeared.[35]

On the Divine Proportion is largely a reworking of the controversial book XIII (books XIII-XV in the Campanus edition), also dealing with books X-XII, from the standpoint of Plato's ***Timaeus***. Pacioli stated that the Divine Proportion and the Platonic solids are at the core of a higher science, which could help to advance the existence of mankind, "because of its infinite power, since, without its knowledge, a great many things could never come to light, whether in philosophy or in any other science." While Pacioli utilized the axiomatic system of proofs used by Euclid, it is these last of Euclid's books which require none of the axiomatic suppositions contained in ***The Elements*** whatsoever! And this is precisely the significance of the subject of Pacioli's book, namely the principles of spherical geometry. As he states in chapter LV:

"It turns out that only these five (solids) are called regular, – not, however, to exclude the sphere, being above all others most regular, all the others deriving from it, as from the cause of the most sublime causes. And in the sphere, there is not any dissimilarity, but uniformity throughout, and in every place it has its beginning and end, its left and right..."

As Nicholas of Cusa implies in his ***On Learned Ignorance,*** the set of interdependent formal axioms and postulates of the so-called Euclidean geometry, starting, for example, from *a priori* definitions of points and lines, were preceded by the constructive geometry of Pythagoras and Plato, a geometry based on circular action as such, -- the tradition revived by Cusa, and continued by Pacioli, da Vinci and Kepler. Indicative of the importance of Books X - XIII of Euclid's ***Elements*** vs. the first nine books, in Book XI, Proposition 3, Euclid puts forward a constructive basis for the existence of the line, as the intersection of two surfaces, a higher order of dimensionality determining the lower order, in place of the flawed definition in Book I.[36] Similarly, Cusa describes a perfect line as circular. Think of a line as the perimeter of a circle or sphere of infinite radius:

"When an infinite line is considered as contracted in such a way that, as contracted, it cannot be more perfect and more capable, it is circular; for in a circle the beginning coincides with the end. Therefore, the most nearly perfect motion is circular; and the most nearly perfect corporeal shape is therefore spherical. Hence for the sake of the perfection, the entire motion of a part is toward the whole."[37]

Now, let us carry out an experiment.[38] With the help of a sphere of some kind, such as a ball or globe, to assist us in the following discussion, let us suppose, that we would take no prior supposition, other than *action itself, in a universe which would be the best of all possible worlds*.[39] An uncompleted action being useless, we would look for the best completed action possible, which Nicholas of Cusa tells us, would contain the maximum area with the minimum perimeter – or, as Gottfried Leibniz would tell us later, the most work with the least action. The unique result that would fulfill such an action is the circle.

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[35] Johannes Kepler, *Mysterium Cosmigraphicum*, Abaris Books, Norwalk, Ct. 1999: for example, pp.149, 151.
[36] Bruce Director, "The Making of a Straight Line," *The New Federalist* newspaper, August 4, 2003, Vol. XVII, No. 22, p. 8.
[37] Op. cit., Jasper Hopkins, *Nicholas of Cusa*, Book II: par.163.
[38] Ideal for this exercize, would be a glass or plastic sphere, on which dry erase markers could be used. A taut spherical balloon would also do. See the 12 part pedagogical exercise on spherical geometry, December 1998 - March 1999, by Jonathan Tennenbaum and Bruce Director: *How to Purge Your Mind of 'Artificial Intelligence,* http://www.wlym.com/pedagogicals/purge.html.
[39] Cusa, op. cit., Book I, Chapter One, p. 49: "We see that by the gift of God there is present in all things a natural desire to exist in the best manner in which the condition of each thing's nature permits this."

Next, acting on the circle, the simplest and most perfect rotation of that circle, namely, around an axis defined by two points maximally distant from one another (a diameter), would give us a sphere, containing the most volume with the least surface (which we can observe from something as simple as soap bubble experiments). And those two points, which moved the least, we can call poles. And the locus of points which had the greatest velocity, the largest possible circle on the surface of the sphere, we can call an equator or great circle. If we took any two other points (not maximally distant), there would be no maximal or minimal singularity generated.

Consider the fact that the surface of a sphere is completely uniform, and appears to be undifferentiated; yet, if we but put it into motion, we find it to be bounded by rigorous divisions!

So, next let us rotate our sphere at two new poles located anywhere on our equator, and no matter which two maximally distant points we chose, we would find that we have created a new equator passing through our two original poles. This then would generate a new singularity at the points where the two equators intersect. (We have now also discovered the right angle, the same event that Pacioli describes in the book thus: "And before Plato there was Pythagoras, a most acute contemplator of nature, imbued with geometry's gentle sweetness, who, in order to celebrate the discovery of the right angle, as we read in the story of him told by Vitruvius, ordered a very great feast and jubilee with 100 oxen sacrificed to the gods....") Now, let us rotate the sphere again, around the axis defined by these two points, creating a third equator, which passes through all four of the first two sets of poles we used before! If we examine for a moment our division of the sphere thus far, we discover that we have eight regular divisions, which gives us the spherical octahedron. Continuing the process, let us now take two points at the midpoints of two opposing edges of the spherical octahedron and rotate, creating a new great circle intersecting both of the other two circles of the octahedron. We can repeat this five more times, using the midpoints of the other ten edges of the octahedron for the points of rotation. We have generated the spherical hexahedron or cube, and the tetrahedron as well.

As stated above, Plato was the first to draw out the implications, in his ***Timaeus***, of the fact that only five such unique, regular (equilateral and equiangular) divisions are possible, -- no more! -- a paradox which identifies a fundamental principle of the boundedness of visible space. And when we will have constructed all of them, we will find the proportionality which governs this bounding process to be our divine proportion. We will leave it to Luca Pacioli to demonstrate how to generate the icosahedron and the dodecahedron, with the method used by Euclid in his Book XIII (incorporating the work of Theaetetus, a student of Plato), from the diameter of the sphere, and how each of the five solids in turn generates others, along with the other non-regular derivations from them. He will show us, however, that the only regular solid which can generate all of the other four, including the icosahedron, is the dodecahedron. But the key to this is, again, our divine proportion.

If we look ahead a bit, we find Kepler pondering these same considerations concerning the boundedness of space, as he writes his ***Mysterium Cosmographicum*** (The Secret of the Universe) in 1596, and ***Harmonice Mundi*** (Harmony of the Worlds) 1619.[40] Kepler also deals with the controversy around these last books of Euclid's ***Elements***, including the tenth, which Pacioli tells us are the core of Geometry. We hear Kepler chastising Petrus Ramus (1515-1572), and Lazarus Schoener for following Ramus when he stated that

"... [Schoener] could see absolutely no use for the five regular solids in the world, until he perused my little book which I entitled ***The Secret of the Universe***, in which I prove that the number and distance of

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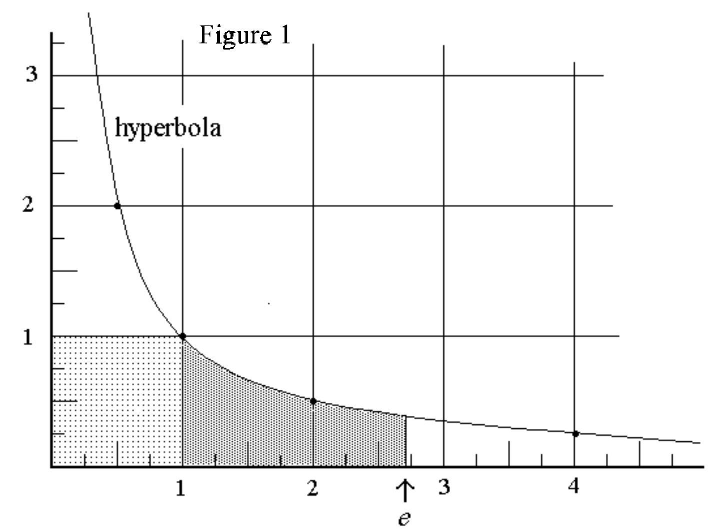
[40] Johannes Kepler, *Mysterium Cosmigraphicum*, Abaris Books, Norwalk, Ct. 1999. Johannes Kepler, *Harmonice Mundi*, American Philosophical Society, 1997. Looking further ahead to our own time, see *21st Century Science and Technology*, Fall 2004, Vol. 17, No. 3, and other related references cited therein, for a series on the Model of the Nucleus developed by Dr. Robert J. Moon and his collaborators, again considering the implications of the five regular solids.

the planets were taken from the five regular solids.... [Ramus] did not believe [Proclus] when he asserted, which was quite true, that the ultimate aim of Euclid's book, to which absolutely all the proportions of its books were related, was the five regular solids. Hence there arose in Ramus a very confident conviction that the five solids must be removed from the aim of the books of the ***Elements*** of Euclid.

"With the aim of the work removed, as if the form were removed from a building, there was left a formless heap of propositions in Euclid."[41]

And on the tenth book on irrational quantities, also central to Pacioli's work, Ramus complains of its obscurity, purposeless, and describes it as the "superstition of the Pythagoreans[42]." Kepler responds, "Let your part be to carp at what you do not understand: for me, a hunter for the cause of things, no other paths to them had opened but in the tenth Book of Euclid."[43]

Although Book X is on the different types of irrational quantities, the divine proportion is given special treatment in Book XIII. Throughout the book, Pacioli is careful to distinguish which type of irrational he is dealing with in any given situation: whether the number is irrational in length only, but not in its square, as, for example, with the $\sqrt{2}$ or $\sqrt{5}$, whose squares or second power are rational; or whether they are irrational not only in length, but also in their second power, or any higher power as well, due to the fact that they are composite irrational numbers, as in the case of the Divine Proportion, which Euclid calls an *apotome*, of which, as Pacioli points out in chapter VIII, Euclid defines six types in Book X. Along with the divine proportion, there are other irrational numbers, but which are of a higher order still, not being composite numbers. For example, *pi*, the ratio between the circumference and the diameter of a circle; or *e*, the natural logarithm, which can be found according to the length at the base of an hyperbola, which gives the second unit of area under the curve, after the square of area one. [Figure 1] These are irrational in all powers, as Cusa identifies in ***On Learned Ignorance***, for the specific reason that they require a comparison between two different species of line, the curved and the straight, which species as a whole are incommensurable, defining these quantities as what we call today "transcendental."[44] Thus did Cusa correct Archimedes on the issue of the quadrature of the circle, and in so doing, established the foundation for modern science. So, although our divine proportion remains irrational in any power, it is not a transcendental, since it can be generated through strictly rectilinear comparisons -- most simply of the relationships within the 1,2,$\sqrt{5}$ right triangle, giving us its exact, but irrational, value of $(\sqrt{5}+1)/2$ or, alternatively, $2/(\sqrt{5}-1)$. Or, as Pacioli describes as the "ninth effect, surpassing all the others," any two



[41] Ibid., p.11.
[42] Ibid., Kepler quotes Ramus, p.10.
[43] Ibid., p.11.
[44] Lyndon LaRouche, *The Economics of the Noösphere*, op. cit.: "The Tragedy of U. S. Education Today," p. 49. Cusa, op. cit. Book I, Chapter Three, paragraph 10, pg. 52: "For Truth is not something more or something less but is something indivisible. Whatever is not truth cannot measure truth precisely. (By comparison, a noncircle [cannot measure] a circle, whose being is something indivisible.)...."

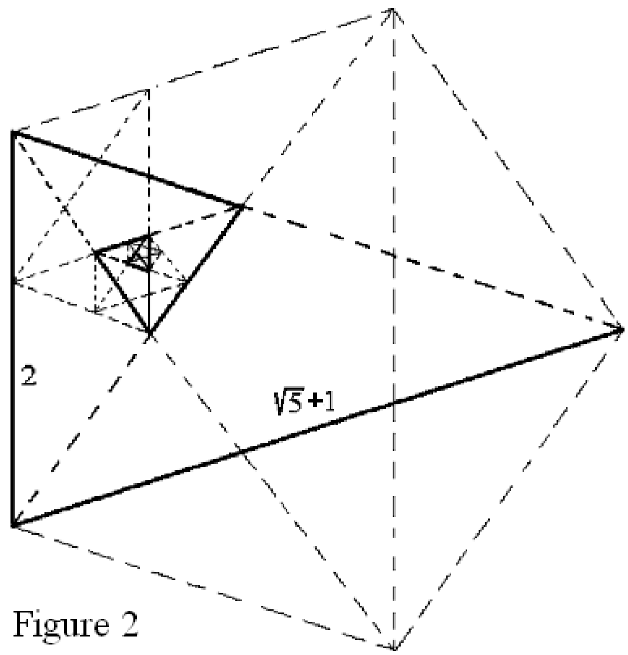


Figure 2

intersecting diagonals of a pentagon formed in a circle, will divide each other according to our proportion.

Many of these irrational and other attributes of the divine proportion are characterized by Pacioli in chapters VII - XXII, which illustrate thirteen wonderful "effects" of the divine proportion, recreating in the vernacular the propositions which he identifies from Euclid's Book XIII, to which he refers the reader who desires to know the geometric demonstration of the laws or the principles involved. But the primary attribute, which Pacioli identifies as its fourth "ineffable" effect, which makes the divine proportion the foundation of this controversial thirteenth book, is its characteristic of self-similar growth.[45] We see this, for example, in the way that logarithmic spiral action is embedded within the five point symmetry of the divine proportion [Figure 2]. This growth function occurs characteristically in living processes, as Kepler emphasizes the distinction between "five-ness" in living processes vs. six-ness in non-living, in his paper, "On the Six Cornered Snowflake" (1611).[46]

These characteristics which Pacioli emphasizes, make the divine proportion alone, as we noted above respecting the dodecahedron, the universal metric of commensurability of all the regular solids contained within a given sphere. Kepler developed the implications of the role of the regular solids in defining the general organization of Euclidean, or better, "Platonic" space, within our solar system, -- his "Secret of the Universe."[47]

Frior Luca explains that he shows us only thirteen effects and no more, in reverence to Christ and the twelve apostles, as also the Duke had done in commissioning "our great Leonardo with his faithful brush" to depict in a mural, the thirteen reunited in *The Last Supper*, at the church of Santa Maria delle Grazie in Milan. From there he proceeds directly into the extended and delightful discussion of the regular solids, followed by their dependents, and then a discussion of different types of pyramids and columns.

Pacioli identifies in chapter LXVI yet another irony; that, even though you cannot give an exact value for the volume of either a cone or "round pyramid", or a round column, due to the transcendental value of *pi* used in the calculation of the area of the base, nevertheless, of a cone and a column of equal height and equal circular base, it is the case that the volume of the cone is *exactly* 1/3 the area of the column.

Finally, in the conclusion of his work, Pacioli draws out for his patron, the Duke of Milan, the implications of his new science for society – the true source of happiness in the ongoing discovery of the lawful ordering of the universe, in furtherance of the continuous progress of mankind.

If this preface has stirred up some of the centuries old assumptions, and hopefully provoked new questions, about the great turning point in history in which Luca Pacioli lived and played a unique role,

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[45] Lyndon LaRouche, *Science of Christian Economy and Other Prison Writings*, Schiller Institute, Inc., Washington, D.C., 1991: "In Defense of Common Sense," Chapter 7 on the Golden Section, p.42.
[46] Johannes Kepler, *The Six-Cornered Snowflake*, 1611; tr. L. L. Whyte, Oxford Univ. Press, 1966.
[47] *Mysterium Cosmographicum*, op. cit.

then you're probably anxious to work through this seminal work by the master mathematician of the Renaissance. Pacioli will bring you many delightful surprises and discoveries, if you read this, not as a text book, in anticipation of the exam, but as a workbook of experiments. It is useful to have material handy to do the drawings and constructions for yourself as you read. It is not necessary to have Euclid's ***Elements*** to refer to while reading this, although exploring them from Pacioli's standpoint, makes them much more valuable books to read. Indeed, Luca Pacioli has given us here a new pedagogy for geometry, which starts at the *beginning*, with the Divine Proportion, and Plato's ***Timaeus***."

It is appropriate to commemorate the quinticentennial of the period of the collaboration between Luca Pacioli and Leonardo da Vinci, as a watershed period of the Renaissance as a whole, with this the maiden voyage of the first complete English translation of ***On The Divine Proportion***. Let this be an impetus to recover and rejoin a living history of science, with the increased use of original texts in our schools, colleges and homes, a new Renaissance, in place of sterilized text books, which purport to give us the science, without the method by means of which nor the purpose for which it was created.

The New Renaissance is Here
by Richard Sanders

A new spirit haunts the sterile world of academia and the dank haunts of the would-be monopolizers of knowledge: a new spirit of Renaissance; and the first English language translation of ***De Divina Proportione*** — after a half-millenium of inaccessibility — is one of its visible results.

Never has the need of a new Renaissance been so clear: how could its offspring, the United States of America — home of such moral and intellectual giants as Benjamin Franklin and Abraham Lincoln — come to the point of tolerating the national disgrace of the recent White House occupants.

Up to the Golden Renaissance, 95% of the population was constrained to live like cattle. The great thinkers like Plato, Augustine, Dante, knew all men and women were made in the image of the Creator, — but that technology, the applied science whose power enabled one man and a machine to do the work of a hundred, and to free all mankind from drudgery, did not yet exist. Only when Brunelleschi and Pacioli's student and collaborator, Leonardo da Vinci, invented the science of invention — which promised to spread to the entire world, including the newly discovered Americas — did mankind stand on the edge of the final defeat of slavery and serfdom.

But, after the defeat of the League of Cambrai, the sly old doges of Venice and other manipulators killed some major thinkers, like Thomas More, and waited for the youth movement to age and die, without being replaced. A few who slipped out of control, like Rafael Sanzio, did not get old enough to reproduce themselves. And at the same time, the great financial powers associated with the Venetians, and the Anglo-Dutch, as well as old landowners, unleashed the Reformation-Counter-reformation disaster that plunged Europe into internecine religious wars until 1648, just like the dogs of war being unleashed in our own time.

Happily, now there is a new Renaissance afoot, finally, a youth movement inspired and led by Lyndon LaRouche, which will never be stopped. Why? It is not just the energy of young people, nor perhaps their impatience, that is key. It is their growing sense of their own mastery of the work of the greatest thinkers of world history, starting "*in media res*" from the work of the incomparable Carl Gauss, and working back through Plato, Pacioli, Kepler et. al., that is making them unstoppable. It is these young men and women, most in their early 20's, who are just mastering Gauss's fundamental theorem, — not simply learning to manipulate it, as in the universities, but knowing it from the inside — who will lead the new renaissance and brush the old cynics aside.

The Golden Renaissance was characterized by the rebirth of Platonic thought, necessarily accompanied by the translation of Plato, Archimedes and others into the vernaculars spoken by the common folk of the day. "Necessarily," for the same reason that the success of the American Revolution depended to a large degree on the fact that the Americans had twice the literacy rate of their would-be masters. Hence the great translating projects of Cosimo de Medici, to name one of the most famous, and Federico da Montefeltro, the patron of Luca Pacioli and of Raphael Sanzio's father.

But the Renaissance geniuses did more than just translate the Greeks: they focussed the truth and beauty of Greek thought, more laser-like, onto agape, the love for mankind, as expressed by Plato and by St. Paul in his letters, and expressed as statecraft by Joan of Arc, Louis XI of France, Henry VII of England, and Machiavelli. Luca Pacioli wrote the Divine Proportion, certainly as much more than a "translation" of

Euclid's ***Elements*** — he looked at Euclid through the lens of the **Timaeus**, showing us that all this time we had been looking at geometry through the wrong end of the telescope, and thus once again the study of geometry was able to become a powerful force, for the good of the soul; and thus the Divine Proportion is also a treatise on statecraft, as emphasized in Chapters 1, 2, and 57 and by the polemics of Chapter 69, which might appear to be extravagant praise of the Duke of Milan, but actually defines the highest standards of the general welfare for any ruler or representative to live up to.

Fra Luca Pacioli – A Biographical Sketch

Pacioli was born around 1445, six years before Nicholas of Cusa's colleague Guillaume Cardinal d'Estouteville would convene the second phase of The Trial of Nullification, the judicial process which would lead to the full exoneration of Jeanne d'Arc in 1456. His birthplace, Borgo San Sepolcro, a narrow strip of Tuscany carved away by the course of the river Tiber, lies between the two provinces of Umbria and the Marches. Today it is richly populated, with an important agricultural and industrial base. At the time of Luca's birth that small Tuscan town had already become an important center of commerce.

In 1472 Pacioli joined the Franciscans from whom he had received his early education. By far the greatest aesthetic-educational influence upon Luca Pacioli during his youth was his relationship with his 'paesano' Piero della Francesca. We do not know at precisely what age Luca became Piero's disciple, however the Maestro, who was a generation older than Luca, was in and out of their home town of Borgo San Sepolcro from 1450 through 1463 were he was largely occupied with the painting of the fresco of the Resurrection, strikingly beautiful not only for its skillful execution but especially for its use of metaphor and geometrical spatial relationships; during this same time frame Piero also worked on the visual-narrative the Legend of the True Cross frescos at nearby San Francesco, Arezza. From the time Luca was five years old, Piero della Francesca would have been a very visible presence.

Luca and Piero's relationship would grow from one of Maestro and student, to the most dedicated of friendships. Piero became blind and turned his attention to writing in his later years. Luca assisted Piero's work on the writing of ***Libellus de Quinque Corporibus Regularibus*** [On the Five Regular Solids]; Piero later insisted that Luca translate and rework it, from the Latin into the venacular, to make it more accessible for the general population. Pacioli did so, and included it as an appendix to the 1509 printed version of ***De Divina Proportione***, Pacioli thus fulfilling the Maestro's wish.

The most important patron of Piero was the beloved political and military leader, and patron of learning, Federigo da Montefeltro, to whose son Pacioli dedicated his ***Summa de Arithmetica Geometria Proportioni et Proportionalità***. Federigo, besides being a great patron of the arts and letters, was a brilliant strategist involved in many complex military and diplomatic matters; as a sign of the tremendous geopolitical instabilities of the fifteenth century, it is noteworthy to mention that in 1445, at only 23 years old, Federigo was excommunicated; the order for this was not lifted until 1450 - by the new Pope Nicholas V, who has become known to us as the humanist pope, the founder of the Vatican library, and a friend of Nicholas of Cusa's.

Piero and Luca would often, over the years, journey by foot from Borgo San Sepolcro over the rugged and scenic terrain of the Appenines mountain chain, to the palace of Federigo da Montefeltro in Urbino, where together they made use of his extensive library, considered at that time to be the greatest in all of Europe, containing over four thousand manuscripts, a great proportion of which were rare first manuscripts and translations. It was at this library that the two did their research and experiments for Piero's ***De Prospectiva Pingendi*** and ***Libellus de Quinque Corporibus Regularibus***.

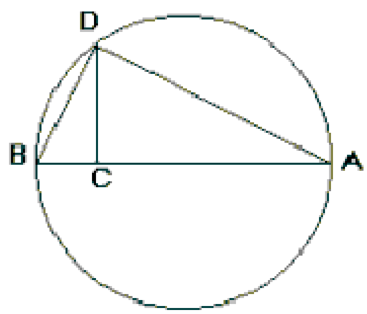
One high point of Luca Pacioli's life relates to the most famous portrait ever made. When the French conquered Milan, Pacioli and Leonardo da Vinci fled Milan and soon found themselves together in Florence, which was the hotbed of the Renaissance, and had already been, for over half a century a center of ferment for the project which led into the discovery of the Americas. The Florentine astronomer Paolo dal Pozzo Toscanelli and his friend Nicholas of Cusa, had been conspiring to do this, probably since they were fellow students in Padova in 1428. Hence the intellectual ferment, voyages of exploration, discussion, finances, concretized in such personalities as Amerigo Vespucci, Giovanni da Verrazzano, and Cusa's friend Ferdinand Martins, the Canon of Lisbon, who made sure that Toscanelli's map showing "how to go east by going west", got into Columbus' hands.

One generation later, Leonardo and Machiavelli are conspiring to build a canal from Florence to the Mediterranean in order to make Florence into a seaport, with a direct connection to the New World! That brings us to the famous smile of the Mona Lisa. At this very time, the Giocondo family was closely associated with the Vespuccis and Machiavelli, and this is the time that Leonardo painted the "Mon(n)a Lisa," which is the portrait of Signora Lisa Giocondo, or familiarly to Italians, "La Gioconda." Perhaps she was smiling because she knew the identity of the landscape behind her; perhaps because as she was sitting for Leonardo, she realized that seldom in history had so many geniuses been found in one place planning and fighting for such a beautiful future for mankind; or because she knew that when Columbus reached the new world, that there the plans and projects of Machiavelli, Pacioli, Leonardo, Cusa, Toscanelli and Brunelleschi might come to fruition.

It is to help finish this work, to contribute to this new Renaissance, started in the "New World," that the translators make you the present of this labor of love.

APPENDIX

Chapter 29

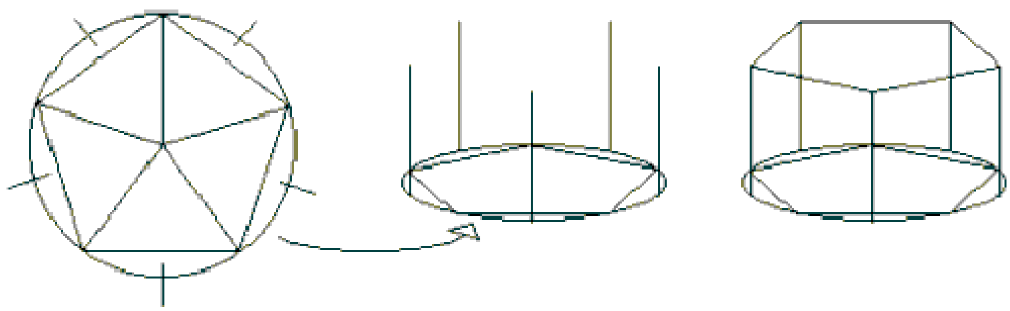


AB is the diameter of the sphere in which the icosahedron is inscribed.

AC = 4xBC

Use the length BD as the radius of a new circle, (which will girdle the sphere above and below the equator), in which a pentagon is inscribed, and connect the vertices to the center L of the circle.

Inscribe a decagon in the same circle.
(Not fully drawn in here.)



Raise posts ("cateti") from the five vertices of the decagon which are midway between the vertices of the pentagon, which posts are equal in height to the length of the radius of that circle, which is length BD in the original circle.

Connect the tops of these posts with five horizontal bars ("corausti"), which are equal in length to the sides of the first pentagon, and which form another pentagon equal in perimeter.

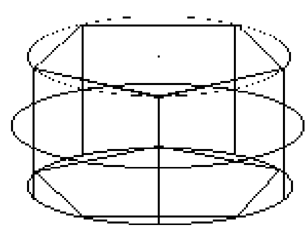
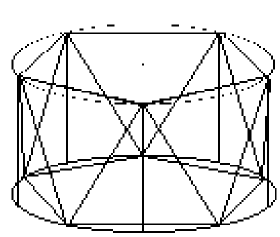
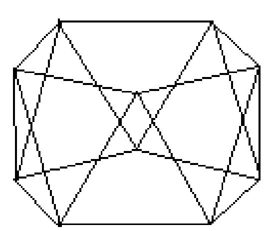


Figure on right includes the original circle inserted at the equator of the sphere.

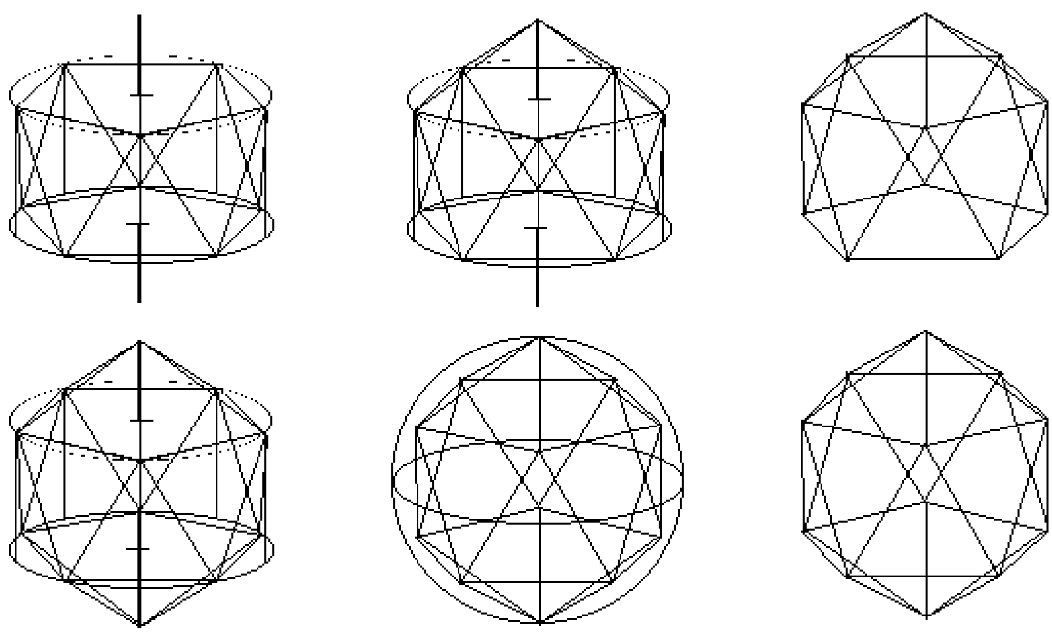
From the top of each post, let fall, two by two, hypotenuses made between the post and a side of the decagon (not shown, the attaching point of the decagon would be at a vertex of the lower pentagon). An easier way to see it is simply to connect the vertices of the upper and lower pentagons. The value of each hypotenus is equal to a side of the pentagons. Figure on far right shows current construction lines removed.





And these 10 hypotenuses together with the ten sides of the two pentagons, create 10 equilateral triangles, within the latitudes of the sphere defined by the two smaller circles, and do not touch the larger circle at the equator at any point. The triangles thus formed are equilateral for the following reason: Knowing that the radius of the circle is equal to each one of the posts raised (equal to length BD), each post is also equal to the side of the hexagon inscribed in the same circle. And since the square of the hypotenuse is equal to the sum of the square of the side of the decagon and the square of the post (which equals the radius of the circle); and since the square of the side of the pentagon inscribed in the same circle is also equal to the sum of the square of the radius and the square of the side of the decagon, the hypotenuses are equal to the sides of the pentagon, and therefore, the triangles sre equilateral.

From the center of the lower circle, raise another post of equal length (equal to DB) up to the center of the upper circle. And extend that central post upward and downward, from each end, an additional length equal to the side of the decagon. From the highest point of this post, extend downward five lines, equal to the length of the side of the pentagon, to each of the vertices of the upper pentagon. Do the same from the bottom to the vertices of the lower pentagon.



ON THE DIVINE PROPORTION

De Divina Proportione

BY

LUCA PACIOLI

-1498-

TRANSLATED FROM THE ORIGINAL BY

RICHARD SANDERS

AND

JOHN PASQUALE SCIALDONE

Luca Pacioli

painted by Jacopo de Barbari, 1495

Translators' Notes

1. All footnotes are by the translators, and brackets are used to indicate included clarifying words or phrases in the text.

2. Words which are left in the original, are in **bold**.

3. Latin, Greek and Arabic words are in *italics*.

4. Book titles are in ***bold italics***.

5. All figures are reproduced from the original, sometimes exactly and sometimes redrawn for clarity, unless otherwise indicated.

6. The edition of Euclid's ***Elements*** to which Pacioli is constantly referring, was of the 1482, fifteen-book Latin edition translated around 1250, from the Arabic, with annotations and commentaries, by Campano da Novara (Johannes Campanus), to whom Pacioli also repeatedly refers (e.g. chapters 6 and 21). Some differences in the enumeration of propositions, etc. between Pacioli's references and the common Dover edition (translated from Heiberg's Greek text published between 1883 and 1888) have been indicated in brackets, or in the footnotes when some explanation is required, for the convenience of the reader.

On The Divine Proportion

by

Luca Pacioli

1498

First English Translation

by

Richard Sanders

John P. Scialdone

Introduction by Jonathan Tennenbaum

April, 2005

for the

LaRouche Youth Movement

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To the very excellent Prince Ludovico Maria Sforza Anglo, Duke of Milan, ornament in times of peace as in times of war, this epistle of Brother Luca de Borgo San Sepolcro, of the Minor Orders, Professor of Sacred Theology:

On the Divine Proportion

CHAPTER I

Today, Eminent Duke, the 9th day of February, the year of our Lord 1498, we are gathered in the impregnable citadel of your illustrious city of Milan, the most worthy place of your usual residence, in the presence of your majesty, and an assembly [dedicated to] praiseworthy scientific debate, composed of people of all ranks, famous and most wise, both religious and secular, with whom your magnificent court constantly abounds. Among whom, besides most reverend Lord Bishops, Protonotaries, and Abbots, there are present from our sacred seraphic Order[1], the reverend father and sublime theologian, Maestro Gometius; the most worthy preacher of the Sacred Scripture, friar Dominico, surnamed Ponzone; the reverend father

[1] The Franciscans

Maestro Francesco Busti, presently acting regent in our worthy convent of Milan. And among the laymen present, first my own illustrious patron Signor Galeazo Sforza, the powerful Vicar of Sanseverino and a general officer in the service of your Excellency, a captain of arms second to none today, and a diligent practitioner of our disciplines. And among the most perspicacious, powerful, and outstanding orators, supreme in medicine and astronomy: Ambroso Rosa, a most lucid and acute investigator of the works of Serapione and Avicenna, and of the heavenly bodies, and interpreter of things to come; the very learned Aluisi Marliano, capable of treating every illness; and most diligent observer of all aspects of medicine, Gabriel Pirovano -- and much admired and venerated by all the aforesaid scholars, Nicolò Cusano[2], along with the expert in the same profession, Andrea Novarese, and other outstanding and often-consulted doctors of both civil and ecclesiastical law; and from your illustrious Magistracy, counselors, secretaries, chancellors.

Among the company of most discerning architects, engineers and assiduous inventors of new things, Leonardo da Vinci, our Florentine compatriot, who, with every cast, sculpture and painting, confirms his surname[3]. As with the admirable and stupendous equestrian statue, -- dedicated to the sacred memory of your glorious father --, whose height from the top of the head to the level ground is 12 **braccia**[4] i.e., 36 times as much as the line AB given here in the margin [see Figure 1], and in all its enormous mass amounts to about 200,000 pounds, where each pound is made up of 12 common ounces which has nothing to envy from the work of Phidias and Praxiteles in Monte Cavallo; and of like excellence, the beautiful painting of the ardent desire of our Salvation[5], painted by his hand in the worthy and sacred refectory for receiving both bodily and spiritual nourishment, in the sacred temple of the Graces[6] -- to whom today, Apelle, Miro, Policretus, and others, it is clear, must give the palm. And not satiated with all these, he expects to bring his invaluable work on locomotion, on percussion, weights, and all kinds of forces, that is, accidental weights, to a worthy conclusion - -- having already with all diligence finished his valuable book on painting and human motion.. . And [also present] his,

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[2] Nicholas of Cusa 1401-1464, seminal figure of the 1439 Council of Florence, and founder of modern science with the writing of ***De Docta Ignorantia*** (1440) and other works. Cusa's work was so pivotal to the Renaissance, that he and his ideas were, as they are today, immortal; and hence his place at the gathering.

[3] **Vincere**, to win, conquer or excel

[4] singular, **Braccio**, a unit of measurement, lit. "arm", not standardized throughout Italy, but equal in Florence to approximately 58 cm. ***Leonardo on Painting***, Yale Univ. Press1989, p.311. This would make the statue approximately 7 meters high. For more about the statue, contact Leonardo da Vinci's Horse, Inc. at leonardoshorse.org, Allentown, PA.

[5] The Last Supper.

[6] Santa Maria delle Grazie, in Milan

so to speak, brother, Giacomo Andrea da Ferrara, a scrupulous disciple of Vitruvius, but not for that in any way diminished in remarkable military endeavors.

Your Grace has said with mellifluous and gilded words, that person to be worthy of praise before God and man, who, having received some particular gift, communicates it voluntarily to others: because he awakens in them **carità**[7], and merits praise and honor in making his, the sacred maxim: Quod et sine figmento didici et sine invidia libenter comunico.[8] The sense of these most sweet words is engraved in my mind as firmly as anything ever written in marble. And even though before this, almost by natural instinct, I acted this way towards others, and especially with respect to those gifts with which, among others, the most High in his immense bounty, was pleased to endow me, that is, the most noble and most necessary mathematical sciences; yet exhausted from my laborious efforts, both day and night, corporeal and spiritual -- the which will be easily understood by anyone who will read with care our great work compiling the mathematical disciplines[9], a work dedicated to your magnanimous relative, Guido Ubaldo, the Duke of Urbino, and especially what is adduced in the fifth chapter of that book -- I was already at a sunny place with others, recounting the years. But, greatly stimulated by those words, I took heart again, and undertook to cross the barren hill[10] and prepared this compendium, to be, as if a sauce to all our other works that deal with this same subject, and to give the highest and most delightful flavor to all the aforesaid sciences and mathematical disciplines; and again for use by your Excellence and your reverent subjects, and to gild and ornament your most worthy library with its multitude of volumes on every subject, to put at your disposal this brief compendium and most useful treatise, entitled **De Divina Proportione**. This treatise, along with the drawings that will present the solids in their material form and appearance, will not awaken less admiration in those who will look into it, than all the other volumes with all the excellent things to be found in your library, since the said forms have been up till now hidden from our contemporaries. In it, we will speak of subjects noble and sublime, which are truly the crucible and the cement of all our exquisite sciences and disciplines: and from that derive all the other speculative operations, scientific, practical and mechanical; and without this knowledge presupposed, it is not possible to understand or employ any of man's

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[7] The Italian synonym for the Greek, *agape*, as translated, for example, in the King James 1 Corinthians, 13 as "charity," viz.: "Though I speak with the tongues of men and of angels, and have not *agape*, I am become as sounding brass and a tinkling cymbal.... And now abideth faith, hope and *agape*, these three; but the greatest of these is *agape*." See also Executive Intelligence Review, Vol. 28 no. 21, "*Faith, Hope and Agape*."

[8] What I truly learned, I freely communicate without envy.

[9] The ***Summa de Arithmetica, Geometria, Proportione et Proportionalità*** **[Elements of Arithmetic, Geometry, Proportions and Proportionality]**, first published in Venice, November 20, 1494, by Paganinus de Pagannini.

[10] Dante's ***Inferno***, I, 28-29.

sciences, as will be demonstrated. And hence the Duke your Excellence, with acute intelligence, will exhort your familiars and other reverent subjects, to go over that which is a delight and the highest pleasure, as well as most fruitful, since they are not old wives' tales, nor other ridiculous and false tricks, nor again mendacious or poetic inventions of little credibility, which only as the wind, nourish the ear. Now it happens that, according to the philosopher, false things are useful, for the cognition of truths which follow them, they being useful precisely as the not-right, the one opposite to the other; and indeed true things for the most part will become useful and fruitful to us because we arrive at them by these things not true. But of true things, as affirmed by Aristotle and Averroes, our mathematical truths are the most true and of the first degree of certainty, and all the other truths of nature follow them. Whence for introduction and argument for what follows here, let this suffice. With constant humility and due reverence, I commend myself continuously in the highest degree, to your Excellency the Duke, to whom I wish the greatest happiness. *Quae felicissime ad vota valeat.*

CHAPTER II

Preface to the present treatise called De la Divina Proportione

Propter admirari ceperunt philosophari.[11] And so, excellent Duke, the authoritative proposition of the "master of those who know," would have it that the origin of knowledge lies in the sense of sight, just as the same is affirmed elsewhere, saying *Quod nihil est in intellectu quin prius fuerit in sensu*, that is to say, that there is nothing in the intellect which was not presented to us first in some way through the senses. And for the sages of old, sight was considered the most noble of the senses, whence, not without merit, it is still commonly said by people today, that the eye is the first door through which the intellect understands and enjoys.

As disclosed there: the priests of Egypt, seeing an eclipse of the moon, stood there in admiration, and searching for the reason, by true science they found that this happened naturally by the interposition of the earth between the sun and the moon, of which they were left pleased. And from that time on, their successors, becoming more and more imaginative by the light of the five intellectual windows, have filled, for our utility, innumerable volumes with their profound science, since just as from one thought another is sparked, so there were born from that thought, many others afterwards. Reflecting upon this myself, I decided to take up the pen to compose this very useful compendium, and at the same time to physically construct for the common utility, with my own hands, the solids each according to its particular form, and to offer them, together with the present treatise, to your Excellence, the Duke. And I do not doubt

[11] Because they admired, they began to philosophize.

that from their unusual aspect, as if a thing come to our times from heaven[12], your gracious and perspicacious intellect will take great pleasure, and especially, when, with the aid of the light mentioned above, with no less vigor of investigation than the ancient Egyptians vis a vis the said eclipse, you will be able, aided and abetted by the present treatise, to find the cause of such forms and their most sweet harmony. Whence I am certain of the following: that if, in the past, the mathematical sciences gave abundantly to those who had some aptitude for these sciences and disciplines, in the future this science should show itself even more generous; and the more since Your Excellence will exhort his dear familiars and respectful subjects, and others of good will, to acquire these sciences with all diligent care, it being the case that mathematics is the foundation and the stairway for arriving at the knowledge of every other science, for it is in the first degree of certainty, as this is affirmed by the philosopher, speaking thus: *Mathematicae enim scientiae sunt in primo gradu certitudinis et naturales sequuntur eas***.** That is to say, the sciences and mathematical disciplines are in the first degree of certainty, and all the natural sciences follow them, and without a knowledge of them, it is impossible to understand well any other. And, it is written again in the ***Sapientia***: *Quod omnia consistunt in numero, pondere et mensura*, that is, that everything that unfolds[13] in the lower and the upper universe, must be subjected to number, weight and measure. And in these three things, Aurelius Augustine in the ***De Civitate Dei*** says, that the Great Artificer is most to be praised because, in these, *Fecit stare ea quae non erant*[14]. In whose loving exhortation, I include many people ignorant of the sweet utility of that fruit, who ought to awaken from their torpor and their mental sleep, and with all study and care give themselves up to inquiring into those arts; and let this be the occasion for a secular renewal of our time, and with greater reality and alacrity, let each person come to perfection. And besides the fame and worthy commendation to his Excellence the Duke, there will be no small increase of virtue in your excellent dominion, in your dear familiars and beloved subjects, always ready to defend it from all, no less than was the noble and ingenious geometer and most distinguished architect, Archimedes for his own fatherland, who, as it is written, with his new and various inventions of machines, for a long time kept the city of Syracuse safe and sound against the successive assaults of the Romans, until, through Marco Marcello, Rome openly sought to raze

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[12] See Dante's sonnet to Beatrice in La Vita Nuova, XXVI:

Tanto gentile e tanto onesta pare	So kind and so virtuous seems
La donna mia	My lady ....
Ella sen va, sentendosi laudare,	She goes along, hearing herself praised,
Benignamente d'umiltà vestuta;	Benignly clothed in humility;
E par che sia una cosa venuta	And she seems to be something come
Da cielo in terra, a miracol mostrare….	From heaven to Earth, to show a miracle….

[13] Pacioli uses the same word "squaterna", translated here as "unfolds," as Dante, "squaderna" the beautiful concept of the unfolding of the universe being bound with love into one volume, when Dante looked into the eye of God and was consumed by the vision. Paradiso, 33, ll. 85-87.

[14] St. Augustine, ***The City of God***: He brought into being, things that were not.

it from the face of the earth. And from daily experience, it is not hidden from your Excellence the Duke -- since it happens that already for many years the very dear memory of your father has been the authority, teacher and living example of this to all of Italy and to both transalpine and cisalpine Gaul -- that the defense of great and small republics, otherwise called the military arts, is not possible without being able to exercise with merit and advantage the knowledge of Geometry, Arithmetic and Proportion. And never could it be said that an army assigned to carry out a siege or defense, were provided with everything necessary, if in that army there were not to be found engineers, and if there were not ordered new machines of war, as we said just now about Archimedes, the great geometer of Syracuse.

If one looks carefully at artillery in general, take whatever he will, such as bastions, and other redoubts, mortars, armor, pitfalls, mangonels, flame throwers, ballistics, catapults, different kinds of rams, with all other innumerable ingenious machines and instruments, they will always be found to be fabricated and formed with the strength of numbers, measure and their proportions[15]. What else are fortresses, towers, moats, walls, ante-walls, ditches, bridges, fortified battle towers, embrasures, mantles, and other strong points in the boroughs, cities and castles, than all geometry and proportion, all balanced and put into order with levels and plumb bobs? For no other reason were the ancient Romans so victorious, as written by Vegetio Frontino and other outstanding authors, if not by the great care and diligent preparations of their engineers and other admirals of the land and the sea, who could not possibly manage without the mathematical disciplines, that is Arithmetic, Geometry, and Proportion. All these things are fully and clearly manifest in the ancient histories of Livy, Dionysius, Pliny and others; such as the work of Ruberto Valtorri, a very expert mind from Rimini, who brings all this out in a worthy book entitled ***De Instrumentis Bellicis***, dedicated to the illustrious Signor Sigismondo Pandolpho. And your Excellency's late lamented neighbor and close relative, Federigo Feltrense, very illustrious Duke of Urbino, had these said machines and instruments, as Valtorri places them ad litteram in his book, as well as many others by other authors, arranged in order around the base of his noble and admired palace in Urbino, in a frieze of living and beautiful stone, done at the hands of most worthy stone workers and sculptors. Just as one reads among other things, in the ***Commentaries*** of Julius Caesar, of his artful bridge[16], and how, to this day, in the noble city of Todi in Umbria, in our sacred convent of the church of San Fortunato, one can still see depicted the work of your dear departed father, who had a

[15] See also Nicholas of Cusa, ***De Docta Ignorantia*** (*On Learned Ignorance*) Banning Press 1981, tr. Jasper Hopkins, in Book II, Ch 13: "In creating the world, God used arithmetic, music and geometry, and likewise astronomy.... Through arithmetic God united things. Through geometry He shaped them, in order that they would thereby attain firmness, stability and motility in accordance with their condition. Through music He proportioned things....

"And so, God, who created all things in number, weight, and measure, arranged the elements in an admirable order. (Number pertains to arithmetic, weight to music, measure to geometry.) ... And when Eternal Wisdom ordained the elements, He used an inexpressible proportion..."

[16] Which he had thrown over the Rhone river.

great number of huge ropes hung down in such a way, that they could string a bridge over the Tiber, which consequently allowed him to skillfully gain his famous victory.

Nor yet by other means, did our most subtle Duns Scotus attain to the grand speculations of sacred Theology, if not by knowledge of the mathematical disciplines, as is apparent in all his sacred works, especially if one looks carefully at the question of the second book of his ***Sententie***, where he inquires if an angel might have his own and determined place of existence; in which he demonstrates his clear understanding of the entire sublime volume of our most perspicacious philosopher from Megara, Euclid. Similarly, it is for no other reason, that all the texts of the prince of those who know physics, metaphysics, posterior [Analytics] and others, present themselves as difficult, for lack of knowledge of the aforesaid disciplines. Not for any other reason is there such a paucity of good astronomers, if not by poor training in Arithmetic, Geometry, Proportions, and Proportionality. And 9 out of 10 of them, base their judgements on tables, notebooks and other things calculated by Ptolemy, Albumasar, Alì, Alfragano, Geber, Alphonso, Bianchino, Prodocimo and others; and copied out by incautious scriveners, so the sources may be stained and spoiled. Consequently, in trusting these poor editions, they end up with very great and obvious errors, causing no small damage and prejudice to those who have faith in them.

Moreover, the supreme subtlety of all municipal law, consists, as it was several times expounded to me by experts, in making decisions relating to floods and breakwaters to prevent inundation from floods, such as in the treatise composed by the most distinguished one among them, the excellent Bartholo da Saxoferrato, called the ***Tiberina***, in whose prologue, he very much extolled geometry and arithmetic. Likewise, one of our brothers named Guido, professor of Sacred Theology, affirms this in a treatise on the give and take of the river Tiber which floods from time to time in those parts, especially from Perugia towards Deruta. His work was always supported with rectilinear and curvilinear geometrical figures throughout, citing our perspicacious philosopher Euclid, and concluded with great subtlety.

I speak not of sweet and soft musical harmony; nor of the supreme beauty and intellectual consolation of perspective, and how architecture is situated with respect to the description of the maritime and terrestrial world, and the knowledge of the courses of, and the aspects of the heavens, because, that the aforementioned mathematical disciplines are indispensable to these all, is clear from what we have said so far. Rather than risk boredom, I leave to the reader to consider a whole host of practical and speculative sciences with all the mechanical arts necessary to human affairs, which cannot be acquired nor carried out properly without the sufferance of mathematics. And so, one should not be surprised to find so few good mathematicians in our times, for the cause of this is the scarcity of good teachers, along with gluttony, slothfulness, and in part the feebleness of today's spirit, whence, even among wise men, the common proverb is deftly proving itself: *Aurum probatur igni et ingenium mathematicis.* That is, the purity of gold is tested by fire, and the rigor of the mind by the mathematical disciplines. This aphorism means that the mind having a bent for mathematics, is most apt for all the sciences, it being the case that mathematics is of great abstraction and subtlety, since it always has to consider things that are not perceived by the senses. And truly, it is the mathematicians, as the Tuscan proverb says, who will split a hair in mid-air.

By the same token, the ancient and divine philosopher Plato, correctly denied entrance to his renowned Academy, to all who were inexpert in geometry, when above the main door, he had placed the following words, in large clear-cut letters, *viz., Nemo huc geometriae expers ingrediatur,* that is, who was not a good geometer should not enter. And he did this because in geometry, every other hidden science is to be found. And before Plato there was Pythagoras, a most acute contemplator of nature, imbued with geometry's gentle sweetness, who, in order to celebrate the discovery of the right angle, as we read in the story of him told by Vitruvius, ordered a very great feast and jubilee with 100 oxen sacrificed to the gods, as we will describe below.

And let this much suffice for now concerning mathematicians and their commendation. Their number has already begun to swell in this your famous city, thanks to the sponsoring by your Excellence the Duke, of newly introduced public lectures, with profit to the outstanding audiences. As I [the lecturer] am so graced in these sciences by the Most High, explicating clearly and with all diligence -- in their judgment -- the sublime volume of the aforesaid Euclid, on the sciences of Arithmetic and Geometry, Proportions and Proportionality; and bringing his 10 books to a most worthy end, for their greater utility and more ample intelligibility always interposing to his theory our practice, while I dedicate the remaining time to finishing the present treatise.

CHAPTER III

What the name, mathematics and the mathematical disciplines, means and implies.

This term, mathematics, excellent Duke, is Greek, derived from mathmatikoz*,* which in our language simply means something which can be an object of study; and for our purposes, by the mathematical sciences and disciplines are understood Arithmetic, Geometry, Astrology, Music, Perspective, Architecture, Cosmography, and any others dependent upon these. Nonetheless, scholars commonly acknowledge only the first four, that is Arithmetic, Geometry, Astronomy and Music, and call the others subordinate, that is dependent upon these four. And this is said by Plato and Aristotle, and Isodore in his ***Etymology***, and Severinus Boethius in his ***Arithmetica***. But in our judgment, imbecilic and low as it might be, we must confine it to either three, or five: that is Arithmetic, Geometry and Astronomy, and exclude Music from them for the very reasons that other scholars omit Perspective from the five primary disciplines;

or five, by adding Perspective to the said four disciplines, for just as good reasons as the others gave for adding Music to the three we indicated.

If these scholars say that music satisfies the hearing, one of the senses given us by nature, I say that perspective satisfies the sight, which is much more worthy, as it is the first door of the intellect; if they say that music applies to the sonorous number, and to the implied measure of its extension in time, I say for my part that perspective observes the natural numbers according to all its definitions, such as the measure of the visual line. If the former is recreation for the soul through harmony, then the latter does this by the necessary distance and the very delectable variety of many colors; if the former respects its harmonic proportions, then the latter, respects the arithmetic and geometric proportions[17]. In brief, excellent Duke, for many years and until the present moment, this question has tormented my spirit; nothing makes clear to me why four should be better than three or five, even if we should say that so many scholars could not err; yet by what they say, my ignorance is not remedied. Who is it that sees a graceful figure [painted] with its necessary lineaments well disposed, where the only thing missing seems to be the breath, that does not judge it a thing divine rather than human? Painting imitates nature as much as could possibly be done. The which appears clearly to our eyes, if we will look at that marvelous representation of the ardent desire of our redemption,[18] in that painting it is not possible to imagine the apostles more alive to the sound of the voice of the infallible truth when he says: *"Unus vestrum me traditurus est",* one of you shall betray me; where with acts and gestures to one another, with living and painful astonishment, it seems that they be speaking: so justly did Leonardo depict this moment for us with his skillful hand.

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[17] See Leonardo Da Vinci, ***Leonardo On Painting*** - Yale University Press 1989. Leonardo puts forward a similar argument: "Those sciences are termed mathematical which, passing through the senses, are certain to the highest degree, and these are only two in number. The first is arithmetic and the second geometry, one dealing with discontinuous quantity and the other with continuous quantity. From these is born perspective, devoted to all the functions of the eye and to its delight with various speculations. From these three, arithmetic, geometry and perspective - and if one of them is missing nothing can be accomplished - astronomy arises by means of the visual rays. With number and measure it calculates the distances and dimensions of the heavenly bodies, as well as the terrestrial ones. Next comes music, which is born of continuous and discrete quantities and which is dedicated to the ear, a sense less noble than the eye. Through the ear, music sends the various harmonies of diverse instruments to the *sensus communis*. Next follows smell, which satisfies the *sensus communis* with various odors, but although these odors give rise to fragrance, a harmony similar to music, nonetheless it is not in man's power to make a science out of it. The same applies to taste and touch." (p. 15) He argues that painting, "the mother of perspective," would not be considered a science simply because, since "writers had no access to definitions of the science of painting, they were not able to describe its rank and constituent elements." (p. 14) "The science of painting includes all the colors of surfaces and the shapes of the enclosed bodies, and their closeness and distance, with their due degree of diminution according to their degrees of remoteness. And this science is the mother of perspective, that is to say, visual rays...." (p. 16) And in the subsection, "*How music is to be called the younger sister of painting*": Music is not to be regarded as other than the sister of painting, inasmuch as she is dependent on hearing, second sense behind that of sight. She composes harmony from the conjunction of her proportional parts, which make their effect instantaneously, being constrained to arise and die in one or more harmonic intervals. These intervals may be said to circumscribe the proportionality of the component parts of which such harmony is composed - no differently from the contours of the limbs from which human beauty is generated." (p. 34)

[18] Leonardo's ***Last Supper***

This is similar to what Pliny says about Zeuxis in ***De Picturis,*** when he is challenged by Parrhasius, to each do an exercise with a paintbrush; and the former used the paintbrush to depict a basket with bunches of grapes in it, and put it out for the public to see, and the birds came and threw themselves on it as if they were real grapes. The other made a painting depicting a veil. Parrhasius also put his work out to be seen by the public, and Zeuxis, believing that a veil was covering the work that was the response to his challenge, said, "Lift the veil and let everyone see your work, as I am doing with mine". And thus he was vanquished, because if he deceived the birds, irrational animals, the other deceived a rational animal, and a maestro.

Unless I am deceived by the great delight and love that I have for painting - even though I am not myself an artist - there is no gentle spirit which is not delighted by painting, both the rational and the irrational are charmed by it. Whence until I find a contrary proof, I will align myself with those who say there be three principal disciplines, and the others subordinate; or else five, if music is included, and in no way does it seem to me that perspective runs behind, it being no less worthy of praise. And I am certain, since it is not an article of the faith, that indulgence will be granted me. And so much for what is relative to the said name.

CHAPTER IV

About those things which the reader needs to make note of, in order to understand this book, and about the symbols used.

Next, to make things easier, note the following conventions we have adopted: whenever we cite the 1st of the first, and the 4th of the second, the 10th of the fifth, the 20th of the sixth, and so on, to the fifteenth, we always mean by the first notation, the number of the conclusion [proposition] and by the second, the number of the books of our philosopher Euclid, whom everyone imitates, as the archimandrite of this discipline. In other words, saying "by the fifth of the first" signifies, "by the 5th conclusion of his first book", and likewise of the other sections of his whole book on the Elements and first principles of Arithmetic and Geometry.[19] But when we cite the authority of another of his works, or of another author, we will name such a book and such an author.

Again, because of the many various symbols and abbreviations which are required of us even more than they are in other disciplines, whence in medicine they use their symbols for scruples,

[19] Henceforth, we will use Arabic numerals for the Proposition number, and Roman numerals for the Book number, of the ***Elements***. For example, "the 6th of the XIth" signifies the 6th Proposition of Book eleven. The definitions, propositions and books in the edition used by Pacioli, which was the 1482 fifteen-book printing, translated from the Arabic and annotated in the late 1200's by Johannes Campanus (Campano da Novara), has been correlated as much as possible with the modern Dover edition (translated into English from Heiberg's Greek text published in the 1880's.) The discrepancies are indicated in brackets [ ], or in footnotes, when some explanation is required.

ounces, drams and maniples, the silversmiths and jewelers use theirs for grains, dinars, and carats, the astrologers theirs for Jove, Mercury, Saturn, Sun, Moon and the others, and the merchants theirs for lire, soldi, groats, dinars, and likewise for brevity the others, and this only to avoid prolixity in writing and the time consumed in reading, or they would fill with ink much paper. Likewise in mathematics, by algebra, that is, the speculative practice, we use abbreviations for other things which denote the square and the cube of the unknown and other terms, such as those contained in our aforesaid work: a number of which we will use again in the present work, along with the following commonly used[20], namely:

Rx - that is, root, one or more as they be;

RxRx - that is, root of a root: one or more as indicated

m - that is, "less" [minus] in every quantity of whatever nature;

p - that is, "more" [plus] similarly in every quantity;

q^{ta} - that is, one or more quantities, designated by *a* or *e*[21];

po^{a} - that is, one or more powers, designated by *a* or *e*;

li^{a} - that is, one or more lines, designated by *a* or *e*;

Geo^{a} - that is, geometry; and geo^{ca} - that is, geometric;

arith^{ca} - that is, by arithmetic by *ca/ce*;

cti^{a} - that is, continuous, singular or plural by *a/e/i/o*;

propor^{e} - that is, proportion or proportions, by *e* or *i*;

N^{o} - that is, number or numbers, by *o* or *i*;

^{o} - that is, square or squares, by *a/e/i/o*;

dra~ or dre~ - that is, difference or differences;

p^{o} - that is, first or first ones, by *a/e/i/o*;

2^{o} - that is, second, second ones, by *a/e/i/o*;

m~.cato or m~.cata - that is, multiplied, by *a* or *o*;

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[20]The following symbols are used in the previous work, ***The Summa***, as well as the later 1509 printed edition of ***On the Divine Proportion***, which included other texts, but not in Pacioli's1498 manuscript, which is translated here.

[21] For all of the following cases, by *a* or *e* and so forth, Pacioli is simply giving abbreviations for the Italian singular and plural, masculine and feminine and their variations.

m~.care - that is, to multiply;

s. proport.[e] h. el m. e doi ex.[i] - that is, according to the proportion having the mean and two extremes.

Similarly, the following names, that is, multiplication, product, and rectangle, all signify the same thing. And also these, that is, the square of a quantity and the power of a quantity, are one same thing, because the power of a line is with respect to its square, by the last [the 46th][22] of the first [Book], and the most that a line can make is its square. And these things should be observed all the time in our process, so that there be no mistake in the sense of the words.

CHAPTER V

On the fitting title of the present tract or compendium

A fitting title for our tract, it seems to me, excellent Duke, should be "On the Divine Proportion," and this because many of the qualities which I find in our proportion, are also those which in this our very useful discourse, we attribute to God. From these we will take four qualities as sufficient for our proposition.

The first is that it makes one proportion only and not more, and it is possible to assign to it neither other nor different species: the which unity makes the supreme epithet of God, according to all of the theological and also philosophical schools. The second quality is that of the Sacred Trinity: that is, just as *in divinis* one selfsame substance exists among the three persons, Father, Son and Holy Spirit[23], so one selfsame proportion of this kind always finds itself suitable among the three terms. And never is it possible to find our proportion in either more or less, as we will show. The third quality is that, just as it is not possible to define God precisely, nor to understand him in words, so it is not possible for this, our proportion, ever to be determined intelligibly by number[24], nor to express it by any rational quantity, but its value is always hidden and secret and by mathematicians called irrational. The fourth quality is that, just as God can never change and is all in all, and all in every part, so our present proportion,

[22] See Translators' Notes, regarding reconciling Pacioli's references for the Propositions and Books of Euclid's ***Elements*** with that of the modern Dover edition.

[23] See Cusa {On Learned Ignorance}, Book I, chapters 7- 9.

[24] Cusa, op. cit., Bk. I, Ch 16 - par 44: "And in harmony with this, Rabbi Solomon states that all the wise agreed that the sciences do not apprehend the Creator. Only He himself apprehends what he is; our apprehension of Him is a defective apprehension of His apprehension." And Bk. II, Ch 13 - par 176: "And when Eternal Wisdom ordained the elements, He used an inexpressible proportion..."

always continuous and discrete in every quantity, being large or being small, is one selfsame and always invariable proportion; and in no way can it be changed, nor be otherwise apprehended by the intellect, as our proceedings will demonstrate.

It is possible to arrogate the fifth quality not unworthily to the aforementioned: that is, just as God confers Being to Heavenly Virtue, by another name called Fifth Essence, and by its mediation, to the other four simple bodies, that is, to the four elements earth, water, air and fire[25], and, through these, to every other thing in nature, so this our sacred proportion gives the formal being - according to the ancient Plato in his ***Timaeus*** **-** attributing to heaven the figure of the solid called dodecahedron[26], otherwise the solid of 12 pentagons, which, as will be shown below[27], is not possible to make without our proportion. And similarly, to each one of the other elements, he assigned its own, absolutely distinct form; that is, to fire, the pyramidal figure called tetrahedron, to earth the cubic figure called hexahedron, to air, the figure called octahedron, and to water that called icosahedron[28].

And these such forms and figures are called by scholars the regular solids, as we will discuss each of them individually below. And then, by means of these, we will speak of the infinite other solids called dependents. Nor is it possible to relate the 5 regular solids to one another, nor to understand their circumscriptions by the sphere, without our said proportion. All of which will appear below. Even if we might cite many other qualities, let these suffice to show that we have made a fitting choice of name for this compendium.

CHAPTER VI

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[25] Plato, ***Timaeus***: "When the ordering of the universe was undertaken, fire, water, earth, and air possessed only some vestiges of their characteristics and were generally in that condition which everything is likely to be in when God is absent from it, and this then being the case, God first of all gave them distinct form by means of kinds and numbers. We must always, above all else, bear witness to this, that God composed them to be beautiful and as perfect as possible while they were not before .... First of all, it is obvious to everyone that fire and earth and water and air are bodies; and that all bodies have depth; and that depth in turn is bounded by surface; and the rectilinear surface is composed of triangles .... We must now ask: which are the four most perfect bodies that can be constructed ...."

[26] ***Timaeus***: "There is yet a fifth construction, which God used to paint the zodiac of the universe."

[27] Chapter XXII

[28] ***Timaeus***: "Let us give earth the cubical shape, since of the four kinds the earth is the most immobile and most plastic of all bodies and that which has the most stable bases must necessarily be like her; also, of the triangles that we hypothesized in the beginning, the base of the [right] isosceles is by nature more secure than that of the others and, also, the equilateral quadrangle which was constructed from isosceles triangles is by necessity more stable both in parts and as a whole. Therefore we preserve the likely account when we assign this shape to earth.

"Of the rest, water is the most immobile, fire the most mobile, and air is the intermediate between the two; we shall assign the smallest body [tetrahedron] to fire, the largest [icosahedron] to water, and the intermediate [octahedron] to air; and the sharpest to fire, the next to air, and the third to water."

On the praise due to our proportion.

It is impossible to exaggerate, excellent Duke, the privilege and preeminence due our proportion, because of its infinite power, since, without the knowledge of this proportion, a great many things very worthy of admiration could never come to light, whether in philosophy or in any other science. This gift is conferred upon our proportion by the immutable nature of its higher principles, as was said by the great philosopher Campanus, our very famous mathematician, in his comment on the 10th of the XIVth, and especially seeing that our proportion is that which brings into accord with one another, in a certain irrational symphony, so many diverse solids, both in size and in the number of faces, as well as of figure and form, as will be understood in our process of posing the stupendous effects, which – for a line divided according to the said proportion -- are to be called not natural, but truly divine.

CHAPTER VII

On the first effect of a line divided according to our proportion

When a straight line be divided according to the proportion having the mean and two extremes -- for this was another name given by scholars to our admirable proportion -- if to its larger part is added the half of the entire line so proportionally divided, it will follow of necessity that the square of their sum always is the quintuple, that is, 5 times as much, of the square of the said half of the whole.

Before proceeding to other things, we must clarify how the said proportion is to be understood and interposed among the quantities, and what it is called in the works of the greatest scholars. That is why I say it is called *proportio habens medium et duo extremi,* that is, "the proportion having a mean and two extremes," to which every ternary must be subjected; no matter what the ternary might be, it will always consist of a mean and two extremes, since the mean without the extremes is not possible to conceive. It is thus that we were taught to divide a quantity in the 29th [30th] of the VIth, having first described in the 3rd Definition of the VIth how one must understand this division; even though in his IInd, by the 11th, he demonstrates how to divide the line with the same quality and force, without naming the proportion until the end of the Vth book. And by Campanus, it is adduced from the numbers in the 16th [18th] of the IXth. And this much on the subject of its name.

How its mean and its extremes are to be understood

Once we understand how our proportion came to be called by its particular name, we still have to clear up how the cited mean and extremes, in whatever quantity you will, must be understood and how they must be arranged, so that among them one might find the said divine proportion.

And for that, it is necessary to know, as is pointed out in the Vth, that among three terms of the same type, there are always of necessity, two dispositions or we would say proportions: that is, one between the first term and the second, the other, between the second and the third.

For example, let there be three quantities of the same type, whose proportions among them otherwise are not known. Let the first be A and let it's number be 9; the second B and its number 6; the third C and 4 [see Figure 2]. I say that among them there are two proportions, one between A and B, that is, of 9 to 6, all which congruent proportions we call in our work the **sexquialtera**, and is obtained when the larger term contains the smaller, one and one-half times. Since 9 contains 6 and still 3 more, which is the half of 6, so it is called **sexquialtera**. But since we do not here intend to discuss proportions generally, having extensively treated and clarified them, together with proportionalities, in our work cited above, I will not elaborate on them further here, but we should always take as presupposed, what is generally said of them, with their definitions and divisions. And we will speak here only of this one proportion, since we have not found it treated in such a very useful way, by anyone else before this.

Now turning to the purpose with which we began, of the three quantities, let the second, B, be to the third, C, as 6 is to 4, also a **sexquialtera** proportion. For now, we do not care whether the proportions are equal or unequal: but our intent is only to make clear how among the three terms of similar type, there are by necessity two proportions. Similarly, I say that our divine proportion observes the same conditions; that is, that always between its three terms, that is, the mean and two extremes, it invariably contains two proportions always of one same denomination. The which with other proportions, whether they be continuous or discontinuous, can occur in infinitely various ways, because sometimes between their three terms it will be double, sometimes triple*, et sic in ceteris* for all the common species. But there is no possible variation between the mean and the extremes of this our proportion, as will be shown.

Whence, I rightfully make this [invariance], the fourth property that our proportion shares with the Supreme Architect; and because it is counted among the other proportions, without any difference in species or of any other kind, in being subject to the conditions established by their definitions, in this we can compare it to Our Savior, who came not to dissolve the Law, but rather to fulfill it, and conversed with men, and made himself subject and obedient to Mary and Joseph. Thus this our proportion, sent by heaven, dwells with the other proportions in definition and conditions, and does not degrade them, rather, makes them more magnificent, maintaining the principle of unity among all quantities indifferently, itself never changing, as our Saint Severino says of the great God*, videlicet: Stabilisque manens dat cuncta moveri*[29]. From this we can know how to distinguish our proportion from others that might occur: that we will always

[29] That is, while remaining stable, he gives motion to everything.

find the three terms disposed in the same proportion as follows: namely, that the product of the smaller extreme and the sum of the smaller and mean, will be equal to the square of the mean, and consequently, by the 10th Definition of the Vth, this sum, of necessity, will be its larger extreme. And when 3 quantities of whatever type find themselves ordered thus, they are said to be according to the proportion having the mean and two extremes. And its larger extreme always equals the sum of the smaller and the mean, so that we are able to say that the said larger extreme is the entire quantity divided into those two such parts, that is, the smaller extreme and the mean in that way. The reason why we must note, that the given proportion cannot be rational, nor is it ever possible, if the larger extreme be rational, for either the smaller extreme or the mean to be denominated by any number, since it will always be irrational, will be made clear below. And in this, it agrees in the 3rd manner with God, as above[30].

CHAPTER VIII

How we are to understand the quantity divided according to the proportion having the mean and two extremes.

These things noted well, we must know that to divide a quantity according to the proportion having the mean and two extremes, means to make of that, two unequal parts, such that the product of the smaller and the whole undivided quantity, be equal to the square of the larger part, as declared by the 3rd Definition of the VIth, by our philosopher. And therefore, if there were ever a case when it were not said explicitly to divide the said quantity into the proportion having the mean and two extremes, but it said only that two parts should be made, so conditioned, that the product of the one part and the whole, should equal the square of the other part -- that person who understood these things, and were an expert in the art, must reduce the proposition to our given proportion, since it is not possible to interpret it otherwise.

For example, if one said: "make me of 10 two parts, such that, the one part multiplied by 10, makes as much as the other multiplied by itself", this case and others similar, operating according to the proofs given by us in the speculative practice called algebra and *almucabala*, by other name the method of that which was put forward in our previously offered work, the one part that is the lesser will work out to be 15 minus the square root of 125, and the other greater part makes the square root of 125 minus 5[31]. The which parts thus described are irrational, and

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[30] Chapter V.

[31] These operations, which Pacioli discusses in the ***Summa***, derive their name from the work of Muhammad ibn Musa al-Khwarizmi, (approx. 800 - 850 A.D.), called Al-jabr wa'l muqabala, written around 830, at the House of Wisdom in Baghdad. It describes a system for the solutions of

in the art are called residues; which our philosopher assigns, in the 79th of the Xth, to be of 6 species[32]. And the said parts are commonly expressed thus: the lesser, fifteen minus the square root of one hundred twenty five. And so stated means: taking the square root of 125 which makes a little more than 11, and that subtracted from 15, will leave a little more than 3, or we might say a little less than 4. And the greater is expressed: the square root of one hundred twenty five minus five. And this means: take the square root of 125, which is a little more than 11 as was stated, and subtract 5 from that, which would leave a little more than 6, or we can say a little less than 7, for the said greater part. But, by having fully demonstrated in our aforementioned work similar acts of multiplication, summation, subtraction and division of binomial residues and roots and all other rational and irrational quantities, whole numbers or all kinds of fractions, here we need not concern ourselves with this. Here it is a matter of stating the new, not reiterating what was said already.

And any quantity thus divided, will always have three terms ordered in continuous proportionality, one of which will be the entire quantity thus divided, that is, the larger extreme, as in the proposed case here, 10; and the other makes the greater part, that is, the mean, as is the square root of 125 minus 5; and the third, lesser part, makes 15 minus the square root of 125. And among them, the same proportion reigns; that is, the first is to the second as the second is to the third, and similarly, vice versa, the third to the second as the second to the first. And multiplication of the lesser, that is, 15 minus the square root of 125, by the larger, which is 10, makes as much as the multiplication of the mean by itself, that is, the square root of 125 minus 5, so that the one and the other product make 150 minus the square root of 12,500, just as our proportion requires. And by this reason, 10 is shown to be divided according to the proportion having the mean and two extremes: and its greater part is the square root of 125 minus 5, and the lesser part is 15 minus the square root of 125, the one and the other of which are of necessity irrational, as is proven by the 6th of the XIIIth, and again in the 11th of the IInd and the 16th [18th] of the IXth. And so much on the quantity thus divided.

CHAPTER IX

quadratic expressions, using geometric principles for completing the square. The word "algorithm" derives from the Latin form of al-Khwarizmi's name. Al- Kwarizmi also used the same values derived from dividing 10 according to the divine proportion, as a pedagogical example. Much of this was transferred to the Renaissance period through the writings and Latin translations from the Arabic of Campanus de Novara.

[32] The third set of Definitions in Book X, immediately preceding Proposition 85, defines the 6 species of *apotomes* or composite irrational values. Propositions 85 through 90 describe the 6 species individually. The term *apotome* is defined in the 73rd of the Xth, and more relevantly in the 6th of the XIIIth.

What is the square root of a number and of other quantities.

And because in our process it will often happen that square roots are spoken of, it appears to me necessary to succinctly explain its import. While throughout our work we discuss it in various ways, nonetheless let me here define the square root of a quantity to be itself a quantity, which, multiplied by itself, makes that quantity of which it is said to be the square root. And that such multiplication made times itself, is called the square of the stated square root. As we say the square root of 9 to be 3, and of 16 to be 4, and of 25 to be 5, and thus with others; and 9, 16 and 25 are called squares.

But along with this, we must know that there are some quantities which do not have square roots that can be named precisely with a number. As 10 does not have a number which multiplied by itself makes 10; and thus 11, 12, 13 and others similarly. Yet they exist and originate in two ways: one type of square root called discrete, or that is to say, rational, and is that for which it is possible to assign an exact number, as the square root of 9 is 3; and the other is called surd, and is that for which you cannot give an exact number, as we have shown of the square root of 10 and others. And by another name these are called irrational, in as much as all those quantities for which it is not possible to designate an exact number, in art, are called irrational; and those for which it is possible to give a number, are called rational. And let this suffice for our purpose on the square roots.

CHAPTER X

Continuation on the first proposed effect.

Those things noted well, let us return to the first proposed effect, which we make clear with conclusive examples. And, for its elucidation, we take up again the same case of 10 as we did above, without further troubling ourselves with other laborious quantities, since it always comes out the same for each as for this one. And, to more fully inform your Excellency, we will examine all the other effects by way of arithmetic, yet presupposing that the scientific proofs of our whole procedure, will be put forward in those places, where our philosopher Euclid deals with them geometrically, according to the requirements of the conclusions.

I say, then, that if 10 is divided according to our proportion, its greater part will be the square root of 125 minus 5, to which adding 5 for the stated result, that is, half of the entire 10, will make exactly the square root of 125, because that "minus 5" becomes restored and replaced with the "plus 5", half of 10. This combination, that is, the square root of 125 multiplied by itself, which makes, by its square, 125, is 5 times the square of the half of 10, which is 5, and its square 25. Whence 125 is exactly quintuple to the stated 25, the square of the stated half of 10, as was said. And this will be the result in every quantity no matter what kind, as the 1st of the XIIIth of our guide clearly demonstrates.

CHAPTER XI

On its second essential effect.

If there were a quantity divided into two parts, and onto the one part is added a [third] quantity, such that the square of their sum be quintuple of the square of the quantity added, it necessarily follows that the added quantity must be half of the initial whole that had been divided into the two said parts, and that to which it is added is the greater part [of the initial whole], and the [initial] whole line must be divided according to our proportion.

For example, taking 15 minus the square root of 125 and the square root of 125 minus 5 for the two parts that add up to one whole quantity, and to the one, that is, the square root of 125 minus 5, add 5 for a third quantity; the sum is the square root of 125 whose square is 125; and the square of the quantity added is 25. Whence 125 is quintuple to 25, which is the square of the quantity added. I say the square root of 25, that is, 5, is half of the initial quantity which was divided into those two such parts, and that to which it is added to be the greater part of the said initial quantity divided according to our proportion having the mean and two extremes, that is, 10. And this is the converse of the preceding effect, just as the 2nd of the XIIIth concludes geometrically.

CHAPTER XII

On its third singular effect.

If a quantity be divided according to our proportion, and if to its smaller part is added half of the larger part, then the square of the sum will always be quintuple of the square of half of the cited larger.

For example, let 10 be the quantity divided according to our divine proportion, such that the one part, that is, the larger, will be the square root of 125 minus 5, and the smaller part, 15 minus the square root of 125. I say, if to 15 minus the square root of 125, which is the smaller part, is added half of the square root of 125 minus 5, which is the larger part, the sum then of the smaller and the said half, multiplied by itself, will be 5 times as much as the square of half of the said larger part. And thus it turns out, since half of the square root of 125 minus 5, is the square root of 31 1/4 minus 2 1/2, added with 15 minus the square root of 125, which is the smaller part, makes 12 1/2 minus the square root of 31 1/4; whence 12 1/2 minus the square root of 31 1/4 multiplied by 12 1/2 minus the square root of 31 1/4, makes 187 1/2 minus the square root of 19,531 1/4. And this is the said square of the sum.

Then, half of the said larger part squared again, that is, the square root of 31 1/4 minus 2 1/2, multiplied by the square root of 31 1/4 minus 2 1/2, will make 37 1/2 minus the square root of 781 1/4, and this is called the square of half of the larger part, which is exactly 1/5 of the square of the sum. And consequently, the said square of the sum is the quintuple of the square of half of the given larger part of 10 thus divided. The which power, along with the others, is much to be admired, which is all proven geometrically by the 3rd of the XIIIth of our author.

CHAPTER XIII

On its fourth ineffable effect.

If a quantity be divided according to our divine proportion, and if to the entire said quantity is added its larger part, the said conjoined first quantity and the larger part of that, will then be parts of another quantity thus divided, and the larger part of this second quantity [thus divided, will always be the entire first quantity].

For example, let 10 be the quantity divided according to our unique proportion, such that its larger part will be the square root of 125 minus 5, and the smaller, 15 minus the square root of 125. Whence, if to 10, the first quantity, is added the larger part, the square root of 125 minus 5, it will make a second quantity, that is, the square root of 125 plus 5. And this second quantity, that is, the square root of 125 plus 5, I say to be similarly divided according to our proportion into the two said parts, that is, the square root of 125 minus 5, the larger part of the first quantity, and 10, which was the first quantity, and this 10 makes the larger part of the second quantity. And it turns out this way, because the product of the square root of 125 minus 5 -- which was the larger part of the first quantity and now makes the smaller of this second quantity -- by this entire second quantity, that is, by the square root of 125 plus 5, makes as much as the square of the mean or we would say the larger part of this second quantity, which is 10, the one and the other of which both make exactly 100, as is required by the said proportion. The which power is manifested to us geometrically again in the 4th [5th]of the XIIIth.

CHAPTER XIV

On its fifth marvelous effect.

If a quantity be divided according to our said proportion, the sum of the square of the smaller part with the square of the whole quantity, will always be triple the square of the larger part.

For example, let 10 be the quantity divided as we have said, such that one part makes 15 minus the square root of 125, that is, the smaller part, and the other the square root of 125 minus 5, that is, the larger part. I say that the square of 15 minus the square root of 125, added with the square of the entire quantity 10, their sum will be triple, that is, three times as much, the square of the larger part, that is, the square root of 125 minus 5. Whence the square of 15 minus the square root of 125, is 350 minus the square root of 112,500, and the square of 10 is 100, which, added with 350 minus the square root of 112,500, makes 450 minus the square root of 112,500, by the said relation. And the square of the square root of 125 minus 5, is 150 minus the square root of 12,500, which makes 1/3 of the said sum as it turns out, since 150 minus the square root of 12,500 multiplied by 3, will make exactly 450 minus the square root of 112,500. Accordingly, the said sum is triple the said square, just as we said. The which effect is geometrically proved by the 5th [4th] of the XIIIth.

CHAPTER XV

On its sixth unnamable effect.

It is never possible for any rational quantity to be divided according to our said proportion, without both parts being irrational, called residual.

For example, let 10 be the rational quantity which would be divided according to the proportion having the mean and two extremes. I say that each one of its parts must of necessity be residual. Whence the one part will be 15 minus the square root of 125, that is, the smaller part, and the other larger part makes the square root of 125 minus 5. Whence each part appears as residual, as thus they are called in the art, according to the 79th of the Xth. And this such effect we have by the 6th of the XIIIth.

CHAPTER XVI

On its seventh inestimable effect.

If the side of the equilateral hexagon be added to the side of the equilateral decagon, both of which are understood to be inscribed in the same circle, their sum will always be a quantity divided according to our said proportion, and its larger part will be the side of the hexagon.

For example, let the side of an equilateral hexagon in the designated circle, be the square root of 125 minus 5, and the side of the equilateral decagon in the same circle, be 15 minus the square root of 125; of which circle the diameter will be the square root of 500 minus 10. I say that the sum of the square root of 125 minus 5 with 15 minus the square root of 125, which makes 10, will be divided according to our proportion, and its larger part makes the square root of 125 minus 5, and the smaller part 15 minus the square root of 125, as 10 was said at other times to be divided. And this is manifested geometrically by the 9th of the XIIIth.

CHAPTER XVII

On the eighth effect, the converse of the preceding.

If a line be divided according to the proportion having the mean and two extremes, of that circle of which the larger part of the line makes the side of the [inscribed] hexagon, in the same circle, the smaller part will always make the side of the decagon.

For example, if the divided line were 10, its larger part which is the square root of 125 minus 5, will always be the side of the hexagon of a circle, of the which the diameter will be double the square root of 125 minus 5, that is, the square root of 500 minus 10. I say that, of that same circle, the smaller part 15 minus the square root of 125, makes the side of the equilateral decagon collocated in it. And Ptolemy makes much use of this converse in the 9th chapter of the first book of his ***Almagest***, to demonstrate the length of the chords of the arcs of the circle, as is likewise clearly demonstrated geometrically in the aforementioned 9th of the XIIIth.

CHAPTER XVIII

On its ninth effect, surpassing all the others.

If the equilateral pentagon be formed in the circle, and from two neighboring vertices, are subtended two straight lines as diagonals, of necessity, those lines will be divided by each other according to our proportion, and each one of their larger parts will always be equal to the side of the said pentagon.

For example, let the pentagon be ABCDE, and from the vertices C and A, let us draw the chord AC, which subtends the angle B; and from the vertices B and E, let us draw the other chord BE, which subtends the angle A [see Figure 3]. I say that these two lines AC and BE divide each other at the point F according to the proportion having the mean and two extremes, and the larger part of each one makes the side of the said pentagon, exactly. Whence, of the line AC, the larger part is CF, and the larger part of the line BE is EF; and each one of these is always equal to the side of the said pentagon. And the two said lines are otherwise called by the mathematicians, chords of the pentagonal angles.

So, if each of the given chords were 10, and they will be equal because their pentagon in the circle is equilateral, CF would be the square root of 125 minus 5, and AF 15 minus the square root of 125; and the part EF would be similarly the square root of 125 minus 5, and BF would be 15 minus the square root of 125, and the side of the pentagon would be similarly the square root of 125 minus 5. And all this in a beautiful way, is demonstrated geometrically by the 11th [8th] of the XIIIth. And by this such effect, we are able by the knowledge of the [length of the] side, to arrive at the knowledge [of the length] of all its chords and of all their parts. And thus, by the inverse, by the knowledge of the chords, we arrive at the knowledge of the side, and of the parts of the said chords, working arithmetically and geometrically, as we have instructed for dealing with them in our above-cited work, where we show how to easily manage binomials and other irrational lines, which our philosopher treats in his Xth Book. And he demonstrates this proportion of the lines [in the pentagon], in the 11th of the Second and in the 29th [30th] of the Sixth, so that easily one arrives in all ways at the knowledge of the one and of the other, which is something of very great utility in our scientific and speculative works.

CHAPTER XIX

On its tenth supreme effect.

If a quantity be divided according to the aforementioned proportion, all its possible effects, for it and for its parts, will as a rule be the same effects in number, species and type, for any whatsoever other quantity thus divided. For example, two lines being thus divided, that is, the one AB divided at C, and its larger part is AC; and the other DE, and its larger part is DF; and, as we speak of these two lines, thus we understand infinite others, which can be dealt with easily by way of arithmetic [see Figure 4]. Placing AB equal to 10, AC would be the square root of 125 minus 5, and the other 15 minus the square root of 125; and placing DE equal to 12, DF would be the square root of 180 minus 6, and the other would be 18 minus the square root of 180. I say that all that could ever occur to one of the said lines, compared, multiplied, divided and worked in all other ways, the same would always happen to the other: that is, the same proportion is made by each one to its larger part, and thus the same proportion is made by each one to its smaller part; and thus conversely, the same proportion by each part to its whole. And likewise, the [sum] of the one and each of its parts to the whole, and conversely, that sum to the said parts. And likewise, in division and subtraction. Whence the proportion of 10 to its larger part, the square root of 125 minus 5, is the same, as that of 12 to its larger part, the square root of 180 minus 6; and the proportion of the sum of 10 and the square root of 125 minus 5, to the square root of 125 minus 5, is the same as of the sum of 12 and the square root of 180 minus 6 to the square root of 180 minus 6.

And thus concisely to infinity, taken and inverted in whatsoever and howsoever many ways, by permutation, converse, addition, subtraction, inverted or not-inverted proportionalities, always will converge upon one same denomination and the same intensive effects, the which demonstrates without fail the great harmony in all quantities thus divided, as will come into view below in the regular and dependent solids. And all this is included through geometry in essence in the second of the XIVth.

CHAPTER XX

On the very excellent eleventh effect.

If one will divide the side of an equilateral hexagon according to our divine proportion, its larger part, of necessity, will always be the side of the decagon, circumscribed by the same circle as the hexagon.

For example, if the side of the hexagon were 10, divided in the said manner, its larger part will be the square root of 125 minus 5, which I say to be exactly the side of the decagon circumscribed by that same circle, of which the diameter will work out to be 20. And this is concluded by the 3rd of the XIVth. Whence, provably, given one side, the other side is easily found, and thus if we have the diameter of the circle, or its circumference, or its area, or any other of its parts, we can always find any one of the others by the one that we know, and likewise, conversely, we can find all the properties of the circle, proceeding from the hexagon, decagon, or also triangle, operating arithmetically and geometrically. And what a very useful thing this is, just as we said above on the 9th effect of the pentagon, *ideo, etc.*

CHAPTER XXI

On its twelfth almost incomprehensible effect.

If a quantity be divided according to our said proportion, the square root of the sum of the square of the entire quantity and the square of its larger part, will always be in proportion to the square root of the sum of the square of the said quantity and the square of its smaller part, as with the side of the cube to the side of the triangle of the solid of 20 faces[33].

For example, let 10 be the quantity divided according to the proportion having the mean and two extremes, of which the one part that is the larger will be, as has often been stated, the square root of 125 minus 5, and the smaller, 15 minus the square root of 125. Now, let us square, that is multiply by itself, the said quantity, that is 10. This will make 100. And also, let us square its larger part, that is the square root of 125 minus 5, which multiplied by itself, will make 150 minus the square root of 12,500; and again, let us square the smaller part, that is 15 minus the square root of 125, which multiplied by itself, makes 350 minus the square root of 112,500. Now, let us add to the square of the larger part, that is, to 150 minus the square root of 12,500, the square of the entire quantity, that is, of 10, which is 100, which will make 250 minus the square root of 12,500. And let us add the same square of the said quantity, that is, again 100, to the square of the smaller part, which we find to be 350 minus the square root of 112,500, to which adding 100 will make 450 minus the square root of 112,500. Now I say that the proportion of the square root of the one sum, that is, of 250 minus the square root of 12,500, made of the square of the said quantity and the larger part, to the square root of the other sum, made of the square of the said quantity and its smaller part, that is, of 450 minus the square root of 112,500, is exactly as the proportion of the side of the cube to the side of the triangle of the solid of 20 faces, when both of the said solids be circumscribed or enclosed by one same sphere.

The which square roots of the sums are called lines of power on the said sums: that is, the square root of (250 minus the square root of 12,500) means a quantity whose power or square is exactly the said sum, and thus the square root of (450 minus the square root of 112,500) means a quantity of which the power, or we would say the square, is exactly 450 minus the square root of 112,500. The which square roots by another name, by those versed in these things, are called universal square roots or connected square roots [34], as it appears in our aforementioned work in the 3rd tract of its 8th distinction, beginning on page 120 of the said volume. The which quantities are of very subtle investigation, and await speculative practice, as is made abundantly clear in the cited work. And it is not possible, eminent Prince, to give these a name using more proximate terminology; and this entire speculative effect is demonstrated geometrically in the 9th of the XIVth, with some other annotations put forward in that location by Campano.

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[33] The proportion being the edge of a cube to the edge of a triangle of an icosahedron, both inscribed in the same sphere.

[34] Radici legate.

CHAPTER XXII

On its thirteenth most worthy effect.

There is no little admiration for its 13th effect, since without its suffrage one might never form the pentagon, i.e., the figure of five equal sides, as cited above in the 9th effect, and as cited again below. Without that pentagon, as will be shown, it is not possible either to form or imagine the solid, most noble above all the other regulars, called dodecahedron, that is, the solid with twelve equilateral and equiangular pentagons, by another name called the solid with twelve pentagonal faces, whose form, as will be shown, the divine Plato attributed to the Quint Essence, that is, to Heaven, and rightly so[35]. Whence our philosopher in the IVth book by the 10th, instructs us in knowing how to make a triangle of the following kind: that is, that each one of its two angles which are in its base, are double the other. And he did this, since, wishing us to know how to form the equilateral and also equiangular pentagon, and to inscribe it and circumscribe it to the circle, that is, to form it precisely inside and outside the circle, this would not be possible, had he not taught us to know how to make the said triangle, as demonstrated by the 11th and 12th of the cited IVth.

And to make the said triangle, we must perforce divide a line according to our divine proportion, as he shows us by the cited 10th of the IVth; it happens that in that place Euclid does not tell us to divide the said line according to the said proportion and its conditions, since he has not yet told us what the proportion is, the which is reserved to the Vth [VIth] book, because it is not his custom to cite in his demonstrations things still to follow, which the reader does not yet know. But he uses only things proved already, and this order he keeps throughout his 15 books. And so, regarding the said triangle, he does not tell us to divide the said line according to the proportion having the mean and two extremes, but tells us, according to the 11th of the IInd, to make of it two parts, such that the square of the one [larger] be equal to the product of the other part and the entire said line. The which in essence means nothing other than dividing it according to our said proportion, as appears by the 3rd Definition of the VIth and by the 29th [30th Proposition] of the cited. And again above, we said the same thing on this subject, when we explained how to understand the mean and its extremes, with respect to the first of its effects adduced.

CHAPTER XXIII

[35] Timaeus, see note above, chapter V.

How, through reverence for our salvation, our discussion of the effects is ended.

It does not seem to me, excellent Duke, that I should go on any further about its infinite effects, because the paper would not suffice for the ink to express all of them. But we have selected only these 13 from among the others, in reverence of the group of twelve and its most holy Leader, Our Redeemer Jesus Christ, since having attributed to them the name divine, we should also end their study with the number of our Redemption, the 12 Articles [of the Creed], and the 12 Apostles with Our Savior. And I understand that your Excellence the Duke, has a singular devotion to that assembly and commissioned our aforementioned Leonardo, with his graceful paintbrush to paint them in the place cited above, the most sacred temple, Santa Maria delle Grazie in Milan. Nonetheless, in the following discussion, we will not omit to mention other effects of the divine proportion, as required, it being the case, as will be shown, that it would otherwise not be possible to form nor imagine the harmony and noble agreement among them, of all the regular solids, and their dependents; to which end we have put forward these already mentioned here, so that their implications might be made more clear.

CHAPTER XXIV

How the said effects concur with the composition of the regular solids.

Now, excellent Duke, the virtue and potential of our aforementioned proportion, with its singular greatest effects, as we have said above, manifests itself in the formation and composition of the solids, the regulars as much as the dependents. For the better understanding of which we will speak of them here in order; and first of the five essential ones, which by other name, are called regulars, and after that, of quite a few of their remarkable dependents.

But first, it is necessary to clarify why they are called regular solids; secondly, we must prove how in nature it is not possible to form a sixth. Whence, they are called regular, because they have equal sides, angles and faces, and the one is contained by the other exactly, as will be shown, and they correspond to the five elements in nature; that is, earth, water, air, fire and the Fifth Essence, that is the Celestial Virtue which sustains the existence of all the others. And just as these five elements are necessary and sufficient in nature, otherwise it would be to argue that God is superfluous to, or less than, natural necessity, which is absurd, as the philosopher affirms that God and Nature do not operate in vain, that is, do not lack necessity and do not exceed it, thus similarly the forms of these five solids, precisely those of which we will be speaking, are five *ad decorem universi*, and there cannot be more as we will demonstrate. And so, not unjustly, as will be shown below, the ancient Plato in his ***Timaeus***, attributed the figures of the said regulars to the five elements, as was said above in the 5th concordance of the divine name attributed to our proportion. And this much on their denomination.

CHAPTER XXV

How it is not possible for there to be more than five regular solids.

It now behooves us to show that there could not be more than five such solids in nature, that is, where all their faces are equal, solid and plane angles equal, and similarly their edges equal. And this will turn out to be so, since, in order to constitute each solid angle, it is necessary for at least 3 surface angles to converge, because it is not possible to complete a solid angle of only two surface angles. Whence, because the 3 angles of each equilateral hexagon are equal to 4 right angles; and also of the heptagon, that is a figure of seven sides, and generally of any figure of more equilateral and also equiangular sides, its 3 angles are always greater than 4 right angles, just as shown by the 32nd of the Ist, and each solid angle is less than 4 right angles, as the 21st of the XIth attests. So it is impossible that 3 angles of the hexagon and of the heptagon, and generally of any equiangular figure with more equilateral edges, might form a solid angle. And for that reason, it is manifest that no equilateral and equiangular solid figure can be formed of hexagonal surfaces, or [polygons] of more sides; because if the 3 angles of the equilateral and also equiangular hexagon are greater than a solid angle, it follows that 4 or more will exceed a said solid angle to a much greater extent.

But it is manifest that the 3 angles of the equilateral and equiangular pentagon, are less than 4 right angles, and 4 of them are more than 4 right angles. Whence, of the 3 angles of an equilateral and equiangular pentagon, it is possible to form the solid angle, but of 4 of its angles or more, it is not possible to form a solid angle. And so, only one solid of equilateral and equiangular pentagons is formed, the which is called by the philosophers *dodecahedron*, or otherwise solid of 12 pentagons, in which the angles of the pentagons three by three, form and contain all the solid angles of the said solid [see Plates XXVII and XXVIII].

The same reason pertains in quadrilateral figures of equal sides and angles, as said of the pentagons; if equilateral and equiangular, it will by definition be a square, because all of its angles will be right as is shown by the 32nd of the Ist. Whence, of the 3 angles adduced of such a surface figure, it is possible to form a solid angle, but of four of them or of more it is impossible. And for that reason, using such faces, since they are equilateral and equiangular quadrilaterals, it is possible to form of them one solid, which we call cube, which is a body contained by 6 square surfaces and has 12 edges and 8 solid angles [see Plates VII and VIII].

As for equilateral triangles, 6 angles are equal to 4 right angles as given in 32nd of the Ist; therefore, less than 6 are less than 4 right angles, and more than 6 are more than 4 right angles; and so, of 6 angles or more of such triangles, it is not possible to form a solid angle; but it is possible of 5 and 4 and 3. And thus it is, that 3 angles of the equilateral triangle contain a solid angle, so that equilateral triangles form the solid of 4 triangular faces of equal sides, called tetrahedron [see Plates I and II]. And when 4 such triangles converge, the solid of 8 faces is formed called *octahedron* [see Plates XV and XVI]; and if 5 equilateral triangles contain a solid angle, then the solid is formed called *icosahedron* of 20 triangular faces of equal sides [see Plates XXI and XXII]. Whence, for the reasons just given, we have made fully manifest why there are so many and such, and no more, regular solids.

CHAPTER XXVI

On the fabrication and formation of the five regular solids and on the proportion of each one to the sphere.

Having seen and understood what and exactly how many are the regular solids, it follows now to say how they would be formed, such that they be exactly circumscribed by a sphere, and also what would be the proportion and relation of their edges to the diameter of the sphere which would circumscribe them exactly, by means of which, one gets to know all of the solids. And so, first we will speak of the tetrahedron, that is, of the 4 equilateral triangular faces, and then we will speak of each one of them in order.

On the tetrahedron inscribed precisely in the sphere

I say therefore, this solid must be formed as follows: that is, first let us take the diameter of the sphere within which we intend to place it, which we set to be line AB, and let this line be divided at point C so that part AC is double part BC [see Figure 5]. And let us make over it a semicircle ADB, and let us draw the line CD perpendicular to line AB, and lines BD and DA. Next let us make the circle FGH on the center E, the semidiameter of which is equal to line CD [see Figure 6]. And let us make in this circle, an equilateral triangle, as we are taught in the 2nd of the IVth, and let this triangle be FGH, and from the center to its vertices, let us drawn the lines EF, EG, EH. Then on the center E let us raise the line EK, perpendicular to the surface of the circle FGH, as the 12th of the XIth instructs; let us set this perpendicular equal to the line AC, and let us drop from point K the hypotenuses KF, KG, KH[36]. The which things being precisely observed, I say the pyramid of 4 triangular faces of equal sides is finished; and it will be circumscribed exactly by the sphere that has the diameter AB. And I say, that the proportion between the diameter of the sphere and the side [edge] of the fabricated pyramid, is such that the square of the said diameter will be six fourths times[37] the square of the side of the said pyramid, that is, that the square of the diameter contains the square of the side of the pyramid

[36] Equal to AD.

[37] **sexquialtero**

one and a half times, that is as 3 to 2 and 6 to 4. And it means that, if the square of the said diameter were 6, the square of the side of the pyramid would be 4. And this has been found to be so in geometry.

CHAPTER XXVII

On the fabrication of the cube and its proportion to the sphere.

It follows to demonstrate how the cube be formed, and what will be the proportion of its edge to the diameter of the sphere which exactly circumscribes it. To that end I say that the said cube must be formed thus: that is, first, let us take the diameter of the sphere in which we intend to place it precisely, and let this be line AB, upon which we will make the semicircle ADB [see Figure 7]. And next, I will divide the diameter at point C, just as I did in the formation of the pyramid earlier, that is, in such a way, that the part AC will be double the part BC. And let us draw the line CD perpendicular to line AB, and let us draw also the lines DB and DA. Then let us make a square, and let each of its sides be equal to line BD; and let that square be EFGH, and let us erect upon its 4 vertices, 4 lines perpendicular to the surface of the said square, as the 12th of the XIth instructs, and let us make each of these perpendiculars equal to the line BD, and let these 4 perpendiculars be EK, FL, GM, HN [see Figure 8]. And these 4 perpendiculars will all be equidistant from one another, according to the 6th of the XIth. And the angles contained by those and by the sides of the square, are right angles by the definition of the line perpendicular to the surface. Next, let us join the end points of these perpendiculars by drawing the lines KL, LM, MN, NK. The which things with diligence carried out precisely, the cube which we are trying to form will be finished, contained by 6 square surfaces, which is proven by the 34th of the Ist. The 4 surfaces which circumscribe it, are those which are formed by the 4 perpendiculars, and are all four of them square.

As for the base being square, this is made clear by our exposition; and further, that the upper surface also is square, that is, KLMN, is likewise shown by the cited 34th of the Ist, and by the 10th of the XIth. And likewise, by the 4th of the said XIth, all the edges of the cube in question are shown to be orthogonal to their two opposite surfaces. And this cube will be circumscribed exactly by the sphere of the given diameter. Whence the said diameter will always be triple in power to the edge of the stated cube; that is, the square of the diameter in question will be three times the square of the edge of the cube. For example, if the diameter were the square root of 300, the edge of the cube would work out to be exactly 10. The knowledge of this is fitting for many necessary cases.

CHAPTER XXVIII

How the octahedron is formed in the sphere, precisely placed, and its proportion to the sphere.

In the third place it follows to fabricate the solid of 8 triangular faces called the octahedron, which is likewise precisely circumscribed by a given sphere, of which we know only the diameter. And it is to be made in this way. Take the diameter of the sphere which is the line AB, and divide it in half at point C, and over the entire length of the line, draw the semi-circle ADB, and let us draw CD perpendicular to line AB; and from there, let us join point D with the endpoints of the said diameter, that is, with A and with B [see Figure 9]. Then let us make a square, all of whose sides will be equal to the line BD, and let this square be EFGH [see Figure 10]. And in this square let us draw two diameters, of which one is EG and the other FH, which are divided by each other at point K, whence it is manifested by the 4th of the Ist that each one of these diameters is equal to line AB, which is taken as the diameter of the sphere, since angle D is a right angle, according to the first part of the 30th [31st] of the IIIrd. And also each one of the angles E, F, G, H is a right angle, by the definition of the square, and it is also clear that those two diameters EG and FH divide each other in half at point K, and this is apparent and easily deduced from the 5th and 32nd and 6th of the Ist. Now raise above K the line KL, perpendicular to the surface of the square, the which perpendicular is set equal to half of the diameter EG or else FH, and this leaves the hypotenuses LE, LF, LG, LH; and all these hypotenuses, for the reasons stated and presupposed in the penultimate [47th] of the Ist, replicated as many times as necessary, will be equal to one another, and also equal to the sides of the square.

Now so far, we have a pyramid of 4 triangular faces, with equal edges, situated upon the said square, the which pyramid makes half of the solid of 8 faces which we intend. Next, under the said square, we will make another pyramid similar to this, in this way: that is, we will extend the said line LK penetrating and going below the said square, down to point M, such that line KM, which is below the square, will be equal to line LK, which is above the said square. And then point M will be joined with all of the vertices of the square by drawing 4 other hypotenusal lines, and let these be ME, MF, MG, MH, and these also will turn out to be equal to one another, and also to the sides of the said square, by the penultimate of the Ist, and others cited above, as was proven for the other hypotenuses above the square. And thus the above mentioned things always observed with diligence, the solid of 8 triangular faces with equal sides, will be finished, and it will be precisely circumscribed by the sphere.

The proportion between the sphere and the said solid, is that the square of the diameter of the sphere is double the square of the edge of the said solid, exactly: that is, if the stated diameter were 8, [each] edge of the eight faces would be the square root of 32, whose powers are in double proportion to each other. That is, that the square of the diameter makes double the square of the edge of the said solid; and thus we have the construction of this solid, and its proportion with respect to the sphere.

CHAPTER XXIX

On the fabrication of the solid called the icosahedron and its creation.

To know how to make the solid of 20 equilateral triangular faces, which is precisely surrounded or circumscribable by a given sphere, with a rational diameter. And the edge of the said solid will be evidently an irrational line, that is, that which is called the lesser line [**linea menore**].

For example, let the diameter here of the given sphere also be AB, which we set to be rational, whether in length or only in power. And divide it at point C, such that AC be quadruple, that is to say, 4 times as much as CB, and make over it the semicircle ADB; and draw CD perpendicular to AB, and draw line DB [See Figure 11]. Next, with the length of DB as radius, using L as a center point, draw a circle EFGHK, in which we inscribe an equilateral pentagon with the same annotation, [vertices EFGHK] [See Figure 12]. Let us draw to the vertices, from the center L, the lines LE, LF, LG, LH, LK. And also, in the same circle, let us make an equilateral decagon, accordingly dividing all the arcs whose chords are the sides of the pentagon by equal parts; and from the mid-points [of the arcs] to the end points of all the sides of the inscribed pentagon, let us direct straight lines; and also, on all of the vertices of the said pentagon, let us erect a cathetus, or [vertically elevated] perpendicular if you wish, as the 12th of the XIth instructs, each one of which is again equal to line BD. Then, let us connect, or join the extremities of these 5 catheti with 5 **corausti** [elevated horizontals]; and the 5 catheti thus erected will be equidistant from one another by the 6th of the XIth; and, since they are equal, the 5 **corausti** which join their end points, will also be, according to the 33rd of the Ist, equal to the sides of the pentagon.

Now drop from the summit of each of the catheti, hypotenuses to the two adjacent vertices of the inscribed decagon, and the end points of these 10 hypotenuses, which descend from the five end points of the catheti to the five points which are each middle vertices of the inscribed decagon, connected, or conjoined, form another pentagon in the said circle, the which also will be equilateral, according to the 23rd [24th] of the IIIrd. And when you will have made this, you'll see that you will have made 10 triangles, the sides of which are: the 10 hypotenuses, the 5 **corausti** and the 5 sides of this [latter] inscribed pentagon. And that these triangles be equilateral, you will comprehend as follows. It being the case that the semidiameter of the circle described, as well as each one of the catheti erected, will be equal, by hypothesis, to line BD, so each one of the catheti, by the corollary of the 15th of the IVth, will be equal to the side of the equilateral hexagon made in the circle, whose [semi]diameter is equal to line BD; and, because, by the penultimate of the Ist, each one of the 10 hypotenuses is more in power, than the catheto, by the side of the decagon[38]*;* and since moreover, by the 10th of the XIIIth, the side of the pentagon, likewise, in power exceeds the catheto by the side of the decagon, then each one of these hypotenuses will be, by common sense, equal to the side of the pentagon.

As for the **corausti** it has already been shown that they are equal to the sides of the pentagon. Whence all of the sides of these 10 triangles are either the sides of the equilateral pentagon inscribed the second time in the circle, or else equal to them. The said triangles are therefore equilateral. Continuing above L - the center of the circle - erect another catheto equal to the first ones, which will be LM; and join its upper end point - which we call point M - with each end point of the first catheti, by 5 **corausti**, or [if you wish] with 5 **corausti**; and, by the 6th of the XIth, this central catheto, that is, which is erected at the center, will be equidistant to each one of the vertex catheti. And so, by the 33rd of the Ist, these 5 **corausti**, will be equal to the semidiameter of the circle, and, by the corollary of the 15th of the IVth, each one will be as [the length of] the side of the hexagon. Now then, from each end of the said central catheto, let us add a line equal to the side of the decagon; that is, MN is added from atop, upward, and, downward, below the circle, LP is added from the center of the circle. Then, from point N, let us drop five hypotenuses to the 5 upper vertices of the 10 triangles which are around the circumference, and from point P, five other hypotenuses to the other five lower vertices. And these 10 hypotenuses will be equal to one another, and to the sides of the inscribed pentagon, by the penultimate of the Ist and by the 10th of the XIIIth, just as we demonstrated earlier for the other 10.

You have, then, the solid of 20 triangular and equilateral faces, all the edges of which are equal to the sides of the pentagon; and its diameter makes the line NP. And, of these 20 triangles, 10 of them are on the circumference of the circle, 5 rise up concurring at point N, and the other 5 descend, concurring at point P. And, that the given sphere precisely circumscribes this solid called *icosahedron,* formed in this manner, will be manifested as follows. Since the line LM is equal to the side of the hexagon, and the line MN to the side of the decagon, which are both equilateral and circumscribed by the same circle EFG, all of line LN will be, according to the 9th of the XIIIth, divided according to the proportion having the mean and two extremes at point M, and its greater part will be the line LM[39].

Let us now divide the line LM in half at point Q, and PQ will be, by common sense, equal to QN, since PL was put equal to the side of the decagon just as MN; whence QN makes half of NP, just as QM makes half of ML. Accordingly, it is therefore the case, that the square of NQ will be, according to the 3rd of the XIIIth, quintuple to the square of QM: the square of PN will

[38] I.e., the square of each hypotenuse minus the square of each vertical perpendicular, is equal to the square of each side of the decagon.

[39] See chapters XVI and XVII above.

also be, according to the 15th of the Vth, quintuple to the square of LM, since, according to the 4th of the IInd, the square of PN makes quadruple the square of QN, and the square of LM is also quadruple to the square of QM, for the same reason. And the quadruple to the quadruple is like the simple to the simple, as the 15th of the Vth affirms. And the square of AB is quintuple to the square of BD, according to the second part of the corollary of the 8th of the VIth, and by the corollary of the 17th of the same [book], since AB is also quintuple to BC, since AC was quadruple to the same BC. Because, then, LM is by hypothesis equal to BD, AB will be, by common sense, equal to NP. Whence, if a semicircle is constructed on line NP, and taken around until it returns to the place whence it began to move, that sphere which will be made by its motion, will be -- by the definition of equal spheres -- equal to the proposed sphere.

And, because line LM is in the mean position proportional between LN and NM, and hence between LN and LP, and, it being the case that LM is equal to the semidiameter of the circle, each semidiameter of the circle will also be in the mean proportional between LN and LP. Whence, the semicircle described on PN will pass through all the points of the circumference of the circle EFG, and, therefore, also through all the vertices of the fabricated solid, which are in that circumference. And since, by the same reasoning, all the **corausti** -- which connect, or you could say join, the end points of the angular catheti with the end point of the central catheto -- are in the mean position proportional between PM and MN, since each one of them is equal to LM, it follows that the same semicircle also passes through the other vertices of the icosahedron figure thus fabricated.

This such solid is therefore inscribable or, that is, locatable, in the sphere whose diameter is AB[40]. And the side [edge] of this solid figure I say to be the lesser line, because it is manifested that line BD is rational in power, since the square is subquintuple, or that is, the fifth, of the square of line AB, which was set rational either in length or only in power. Whence the semidiameter -- and [all] the semidiameters -- of the circle EFG will also be rational in power, since its semidiameter is equal to BD. Accordingly, by the 12th [11th] of the XIIIth, the side of the equilateral pentagon to this inscribed circle, is the lesser line; and also, just as was shown in the course of this demonstration, the side of this figure is as long as the side of the pentagon. Therefore, the side of this figure of 20 equilateral triangular bases is the lesser line, just as was proposed.

CHAPTER XXX

On the very noble regular solid called the dodecahedron.

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[40] Here Pacioli refers the reader back to the initiating Figure 11, and continues the discussion working back and forth between the two figures 11 and 12.

To know how to make the solid of 12 equilateral and equiangular pentagonal faces, such that the proposed sphere surround, or if you wish, circumscribe it exactly; and the edge of the said solid will be manifestly irrational, which is called the residue.

Construct a cube in the sphere as the method given instructs[41], so that the assigned sphere will exactly circumscribe it, and let AB and AC be two surfaces of this cube, and let us imagine now AB as the upper surface of this, and the surface AC be one of the lateral surfaces, and the line AD be common to these two surfaces [see Figure 13]. Now, on the surface AB, divide the two opposite sides in half, that is, DB and the side opposite to it, continuing the points of division through line EF. And let us also divide the side AD and the one which is opposite to it on the surface AC, in half, the points of the division continuing with a straight line of which the half is GH, and let [H] be the midpoint of line AD. Similarly, divide the line EF in half at point K, and draw HK.

So then you will divide each one of the three lines EK, KF and GH according to the proportion having the mean and two extremes at the three points L, M, Q, and their greater parts will be LK, KM and GQ, all of which are manifestly equal, it being the case that all of the lines divided are equal; that is, each one of them is half the side of the cube. Then from the two points L and M, erect -- as the 12th of the XIth instructs -- lines perpendicular to the surface AB, each one of which you will make equal to line KL, and let them be LN and MP. Similarly from point Q, erect QR perpendicular to the surface AC, and you will set it equal to GQ. Now, draw the lines AL, AN, AM, AP, DM, DP, DL, DN, AR, AQ, DR, DQ. It is manifest, then, by the 5th [4th] of the XIIIth, that the two lines KE plus EL potentially, or if you wish, in power, are triple to line KL [squared][42], and so also to line LN, since KL and LN are equal. And also KE is equal to EA; therefore the two lines AE and EL are triple in power to line LN. Whence, by the penultimate of the Ist, AL is triple in power to LN, and so, by the same, AN is quadruple in power to LN. And being that every line is quadruple in power to its half, it follows, by common sense, that AN is double in length to LN. And because LM is double LK, and also KL and LN are equal, AN will be equal to LM, because their halves are equal. And because, by the 33rd of the Ist, LM is equal to NP, AN will be equal to NP. And in the same way, the three lines PD, DR and RA will be proven to be equal to one another and to the two aforementioned.

We have, therefore, the equilateral pentagon by these 5 lines, ANPDR. But perhaps you will say that it is not a pentagon, because perhaps it is not all on one same surface, the which is necessary for it to be a pentagon. And that it be all on one same surface, you will learn as follows. Line KS arises, or if you would, goes out from point K perpendicular to the surface AB, and equal to LK. And for this reason, it will be equal to each of the two lines LN and MP.

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[41] See chapter XXVII.

[42] Every times that Pacioli says **x** plus **y** are such and such in power to **z**, he means that all the terms are squared, i.e., $x^2 + y^2 = z^2$

And since it is equidistant from each of them, by the 6th of the XIth, and also, with both on the same surface, by the definition of equidistant lines, it is necessary that point S be on the line NP, and that it divide it in half. Let us therefore draw the two lines RH and HS, whence the two triangles KSH and QRH are constituted on one angle, that is, KHQ. And the proportion of KH to QR will be as of KS to QH: since, just as GH is to QR, thus KH is to QR, by the 7th of the Vth; and as RQ is to QH, thus KS is to QH, for the same reason. But GH is to QR as QR is to QH, where QR is equal to GQ. Therefore, by the 30th [31st] of the VIth, line RHS is one line. Whence by the 2nd of the XIth, all of the pentagon about which we are disputing, is on one same surface.

And I say also that this will be equiangular, which you will see as follows. EK being divided according to the proportion having the mean and two extremes, and KM being equal to its greater part, all of EM will also be, by the 4th [5th] of the XIIIth, divided according to the proportion having the mean and two extremes, and its greater part also the line EK. And therefore, by the 5th [4th] of the XIIIth, the two lines EM and MK -- and therefore the two lines EM and MP, because MP is equal to MK -- are triple in power to the line EK, and therefore also to the line AE, because AE is equal to EK. Whence the three lines AE, EM and MP are quadruple in power to line AE. Let it also be clear, by the penultimate of the Ist replicated two times, that line AP is equal in power to the three lines AE, EM and MP. Whence, AP is quadruple in power to line AE. And the edge of the cube, since it is double the line AE, is also quadruple in power to that, by the 4th of the IInd. Therefore, by common sense, AP is equal to the edge of the cube; and, since AD is one of the sides of the cube, AP will be equal to AD, and so, by the 8th of the Ist, the angle ARD is equal to angle ANP.

In the same way, you will find angle DNP to be equal to angle DRA, because you will find line DN to be quadruple in power to half of the edge of the cube. It being then the case, by these things stated, that the pentagon is equilateral and has three equal angles, it will be equiangular, by the 7th of the XIIIth. If, therefore, in this way and for the same reason, on each one of the other edges of the cube we will fabricate an equilateral and equiangular pentagon, a solid contained by 12 equilateral and also equiangular pentagon surfaces will be finished, because the cube has 12 edges.

It remains now to demonstrate that this such solid be circumscribable, or one would say exactly circumscribed, by the given sphere, which you will see as follows. That is, let us now draw out, from the line SK, two surfaces which divide the cube: one of which will divide along the line HK, and the other along the line EF. And it will be, by the 40th [38th] of the XIth, that the common division of these two surfaces divides the diameter of the cube, and thus vice versa, or if you wish, conversely, that the cube is divided by the said diameter by equals. Let line KO therefore be their common division as far as the diameter of the cube, such that point O will be the center of the cube, and let us draw the lines OA, ON, OP, OD, OR. And it is clear that each one of the two lines OA and OD is a semidiameter of the cube, and are therefore equal.

As for the line OK, it is clear by the 40th [38th] of the XIth, that it will be equal to EK, that is, to half of the edge of the cube; and, because KS is equal to KM, OS will be divided at point K according to the proportion having the mean and two extremes, and its greater part is line OK, which is equal to EK. Whence, by the 5th [4th] of the XIIIth, the two lines OS and SK will be -- and so also OS and SP, since SP, to which this demonstration does not extend, is equal to KS -- triple in power to line OK, and so to half of the edge of the cube. Whence, by the penultimate of the Ist, line OP is triple in power to half of the edge of the cube. And by the corollary of the 14th [15th] of the XIIIth, it is shown that the semidiameter of the sphere is triple in power to half of the edge of the cube, which is circumscribed, or, if you would, enclosed, by the same sphere. Whence OP is the same length as the semidiameter of the sphere which exactly circumscribes the proposed cube: for the same reason, all of the lines extended from point O to each one of the angles of all the pentagons formed on the edges of the cube, that is, to all of the angles which are proper to the pentagons, and not to those which are common to them and to the surface of the cube, that is, exactly proper, just as are the three angles N, P, R in the pentagon formed.

As for those lines which go from point O to all of the vertices of the pentagons, which are common to the pentagons and the surfaces of the cube, just as are the two angles A and D in the present pentagon, it is clear that they are equal to the semidiameter of the sphere which exactly circumscribes the cube, so that they are diameters of the cube according to the 40th [38th] of the XIth. But the semidiameter of the cube is like the semidiameter of the sphere which exactly encloses it, just as it is apparent for the reason of the 14th [15th] of the XIIIth. Therefore, all the lines emanating from point O to all the angles of the dodecahedron -- that is, of the solid contained by 12 equilateral and equiangular pentagon surfaces, which are so called in Greek -- are equal to one another, and to the semidiameter of the sphere. Whence, if the semicircle outlined over the whole diameter of the sphere, or for that matter, of the cube, if it rotates around, it will pass through all of its vertices: whence, by definition, it is circumscribable, or if you would, enclosed by the given sphere. I say also that the edge of this figure be an irrational line, that is, what is called a residue, if the diameter of the sphere which exactly circumscribes it, be rational in length or in power, which is seen as follows. It being the case that the diameter of the sphere, by the 14th [15th] of the XIIIth, will be triple in power to the edge of the cube, the edge of the cube will be rational in power if the diameter of the sphere will be rational either in length or in power. And, by the 11th [8th] of the XIIIth, it is clear that line RP divides line AD, which is the edge of the cube, according to the proportion having the mean and two extremes, and that its larger part is equal to the side of the pentagon. And, because its larger part makes a residue, by the 6th of the XIIIth, it is clear that the edge of the figure called the dodecahedron is a residue. Which is what we wished to demonstrate[43].

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[43] Apotome, defined in the 6th of the XIIIth, is called residue by Pacioli.

CHAPTER XXXI

On the rule and manner of knowing how to find the edges of the said solids.

Having been given only the diameter of the sphere, to know how to find in terms of the said diameter, the length of the edges of the five aforementioned solids, all circumscribed precisely by that sphere.

For example, let AB be the diameter of any given sphere, for which we need to find the edges of the five aforesaid solids, all of which are understood to be collocated in one same sphere, where if one of the vertices be touching, all of them will touch; that is, the said sphere exactly encloses all of them. The which we will do thus [see Figure 14]: that is, let us divide this diameter at point C, in such a way that AC will be double CB, and divide it [AB] by equals at point D, and we will make over it the semicircle AFB, to the circumference of which we draw two lines perpendicular to line AB, which are CE and DF. And we join E with A and with B, and F with B. It is clear therefore, by the demonstration of the 13th of the XIIIth, that AE makes the edge of the figure of 4 triangular and equilateral faces; and by the demonstration of the 14th [15th] of the same, that EB makes the edge of the cube, and by the demonstration of the 15th [14th], that FB makes the edge of the figure of 8 triangular and equilateral faces. Let the line AG then issue forth from point A, perpendicular to AB, and also equal to the same AB; and join G with D; and let H be the point at which GD divides the circumference of the semicircle. And raise HK perpendicular to AB. And because GA is double AD, HK will be, by the 4th of the VIth, double KD, since the two triangles GAD and HKD are equiangular, by the 32nd of the Ist, inasmuch as angle A of the larger is equal to angle K of the lesser, because each one is a right angle, and the angle D is common to both.

Now then, by the 4th of the IInd, HK is in power quadruple to KD; accordingly, by the penultimate of the Ist, HD is in power quintuple to KD. And since DB is equal to HD -- because D is the center of the semicircle -- DB will also be in power quintuple to KD[44]. And since all of AB is double all of DB, just as AC which is taken from the first AB, is double CB which is taken from the second BD, by the 19th of the Vth, there will be BC remaining from the first AB, double the CD residue of the second BD; and so all of BD is triple to DC. Therefore, the square of BD is nonuple, that is, nine times as much as the square of CD, and because this [the square of BD] was only quintuple to the square of KD, the square of DC will be, by the second part of the 10th of the Vth, less than the square of KD, and for this reason, DC less than KD. Let DM then be equal to KD, and let MN go up to the circumference, and let it be perpendicular to AB, and join N with B.

[44] This sentence (although not in the manuscript) serves as a clarifying insertion, and appeared in a different printing.

Since DK and DM are equal, the two lines HK and MN will by definition be equidistant from the center, and therefore equal to each other, by the second part of the 13th [14th] of the IIIrd and by the second part of the 3rd of the same. Whence MN be equal to MK, because HK was equal to it. And because AB is double BD, and KM is double DK, and the square of BD is quintuple the square of DK, the square of AB, by the 15th of the Vth, will be similarly quintuple to the square of KM, since the square of the double is to the square of the double, as the square of the simple is to the square of the simple. And by the demonstration of the 16th of the XIIIth, it is clear that the diameter of the sphere is thus quintuple in power to the side of the hexagon of the circle which makes the figure of 20 faces.

Now, KM is equal to the side of the hexagon of the circle of the figure of 20 faces, since the diameter of the sphere which is AB, is in power quintuple both to the side of the hexagon of the circle of that figure, and to KM. And also, by the demonstration of the same, it is clear that the diameter of the sphere is composed of the side of the hexagon plus two sides of the decagon of the circle of the figure of 20 faces[45]. Now, KM being the same length as the side of the hexagon, and also, AK equal to MB, since they are the residues or if you wish remainders, of equals; if equals were taken away, then MB will be the length of the side of the decagon. And because MN is the same length as the side of the hexagon, since it is equal to KM, then, by the penultimate of the Ist and by the 10th of the XIIIth, NB will be as the side of the pentagon of the circle of the figure of 20 faces. And because, by the demonstration of the 16th of the same, it turns out that the side of the pentagon of the circle is the edge of the same figure of 20 faces, it is clear that line NB is the edge of this figure.

Now, divide EB -- which is the edge of the cube precisely circumscribed by the given sphere -- according to the proportion having the mean and two extremes, at point P, and let the larger part be PB. It is clear now, by the demonstration of the preceding, that PB is the edge of the figure of 12 faces. The edges of the five premised, or I would say previously given solids are therefore found, solely by means of the diameter of the sphere given to us, the which edges are these; that is, AE for the pyramid of 4 faces, EB the edge of the cube, FB the edge of the 8-faced figure, NB the edge of the 20-faced, and the line PB the edge of the 12-faced figure. And which among these edges are larger than the others, is worked out thus. It is clear that AE is larger than FB, because the arc AE is larger than the arc FB, and also FB is larger than EB, and EB larger than NB. And also I say NB to be larger than PB, since, AC being double CB, by the 4th of the IInd, the square of AC will be quadruple the square of CB; and, by the second part of the corollary of the 8th of the VIth, and by the corollary of the 17th of the same, it is clear that the square of AB is triple to the square of BE. But by the 21st of the VIth, the square of AB to the square of BE, is as the square of BE to the square of CB, since the proportion of AB to BE is as BE to BC, by the second part of the corollary of the 8th of the VIth. Whence, by the 11th

[45] See Chapter XXIX.

of the Vth, the square of BE is triple to the square of CB, and because the square of AC is quadruple to the same square, as has been shown, the square of AC will be, by the first part of the 10th of the Vth, greater than the square of BE. And so line AC is larger than line BE, and yet AM is larger still. And it is already manifest by the 9th of the XIIIth, that, if line AM will be divided according to the proportion having the mean and two extremes, line KM will be its larger part, which is equal to MN; and also, when BE is divided according to the same proportion, that is *habens medium et duo extrema*, its larger part is the line PB. Since all of AM is larger than all of BE, MN, which is equal to the larger part of AM, will be larger than PB, which is the larger part of EB. And this is shown by the 2nd of the XIVth book, the which is fortified without the help of any which follow. Therefore, by the 19th of the Ist, NB is much larger than PB.

Whence it appears that the length of the edges of the five aforementioned solids are ordered in descending order almost in the same order as the number of faces of the solids. There is only one out of order, and that is the case of the cube and the octahedron, that is, in the 8 faces, because the length of the edge of the octahedron exceeds the length of the edge of the cube; while it happens that the construction and formation of the cube precedes that of the octahedron, as appears in the XIIIth, and this is not without mystery. Hence, in the formation of the cube, the octahedron is presupposed, because by the same division of the diameter of the proposed sphere, the edge of the pyramid of 4 triangular faces and the edge of the cube is found. Therefore, AE, the edge of the pyramid, is larger than the edges of all of the other solids. And after that, FB, the edge of the 8 faces, is larger than the edges of all of the other solids which follow after it. And EB, the edge of the cube, follows in third place in size; and in fourth place is NB, the edge of the 20 faces, that is, the icosahedron. And the least of all is PB, the edge of the dodecahedron, that is, of the 12 pentagonal faces.

CHAPTER XXXII

On the proportion of the said regulars to one another and to their dependents.

Having understood the sufficiency of the said five regular solids, and shown the impossibility of there being more than five of them, while their dependents proceed to infinity, we must next find a way of presenting their proportions to one another, the magnitude of their capacity or volume, and of their surface area; and then the inclusion of one in the other and conversely, and first their physical appearance.

The proportions of one to the other will always be irrational by reason of our above adduced proportion, which interposes itself in their composition and formation, as was said; except that in the cases of the tetrahedron, the cube and the octahedron, -- precisely because of their particular proportions relative to the diameter of the sphere in which they are inscribed, --

sometimes their proportions might be rational. But the proportions of the icosahedron and of the dodecahedron, to which the others need to be compared, can never be rational, for the reason given. And so, it does not appear to me, excellent Duke, necessary to say more about them here, because the volume of infinite irrationalities would have to grow, in which the intellect would more likely be confounded, rather than take pleasure, to which end our study is always intended. And what we have said in our work up to here on the treatment of this subject, seems to me ought to suffice, since we have written another specific work on the subject of the regular solids[46], the which work being widely distributed to everyone, can be easily accessed. And using the dimensions put forward in that work, one can, according to the flight of their genius, draw from it much use and delight. And I say the same of all their dependents, of which I have given a good number of examples in that same work.

It is the case that, by the 10th of the XIVth, the proportion of [the volumes of] the dodecahedron to the icosahedron, when both are made in the same sphere, is found to be exactly as that of all of the surfaces of the one added together, to all of the surfaces of the other added together[47]. And the 16th of the same, shows the octahedron to be divisible into two pyramids of equal height, which is equal to the semidiameter of the sphere in which it is constructed, and their bases are square. The which square is sub-double [one half] in surface area to the square of the diameter of the sphere[48]. And knowing the measure of this, is quite useful for us; and by means of that, it is possible to arrive at many other measurements.

CHAPTER XXXIII

On the proportion of all their surface areas, each to the others.

Likewise their surfaces, excellent Duke, we can say in the same way, to be proportional to each other, as was said of their corporeal masses [volumes], that is, irrational by the mischief of the pentagon which interposes itself in the dodecahedron. But as for the other surfaces, such as those of the tetrahedron, cube and octahedron, they can sometimes be rational, due to being triangles and squares, and measured in proportion with the diameter of the sphere in which they are formed, as was seen above. The conclusion of the 8th [6th] of the XIVth is true: [the proportion of the area of] all of the surfaces of the 12 pentagonal faces, to all of the surfaces

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[46] ***Summa***, Distintio Ocatava, De Corporibus Regularibus, pp.69-75 of manuscript.

[47] Additionally, the 6th and the 8th of the XIVth tells us that both the surface areas and the contents (volumes) of the dodecahedron and the icosahedron are proportional respectively to the side (edge) of the cube and the side of the triangle making the icosahedron, all inscribed in the same sphere, therefore, they are proportional to each other.

[48] See Plato's ***Meno*** dialogue for a pedagogical on the diagonal of the square as the geometric mean between 1 and 2, or √2.

of the 20 triangular faces, that is, of the dodecahedron to those of the icosahedron, is as that of the edge of the cube to the side of the triangle of the solid of 20 faces, when all of the said solids be exactly contained, or circumscribed, by one same sphere. I do not think one can pass over in silence the reason for this wonderful propriety between them of their faces, that is, that the faces of the dodecahedron and those of the icosahedron, are each exactly circumscribed by one same circle, as the 5th [2nd] of the XIVth shows. And this is particularly worthy of note when both are to be constructed within the same sphere.

And the proportion of the surface area of the entire tetrahedron to the surface area of the octahedron, is the proportion noted by the 14th of the said XIVth; since a face of the tetrahedron is one and one third as much as one of the faces of the octahedron, that is, in the **sexquiterza** proportion, which is when the larger contains the lesser, one and one third times, just as 8 to 6, and 12 to 9. And the proportion of the total surface area of the octahedron to the total surface area of the tetrahedron, when they are both inscribed in the same sphere, makes a **sexquialtera**, that is, one and a half as much, such as, if that of the octahedron were 6, then the [tetrahedron] would be 4, which is when the larger contains the lesser, one and one half times, both being of one same sphere. And the total surface of the tetrahedron added to that of the octahedron, compose a surface area called the **mediale**, as the 13th of the cited XIVth argues. And the total surface area of the hexahedron, that is, the cube, equals double the square of the diameter of the sphere which circumscribes it, and the perpendicular which is drawn from the center of the sphere to each one of the faces of the said cube, will always be equal to half of the edge of the said cube, by the last of the XIVth. That is, if the said diameter were 4, the total surface area would be 32; and if the said perpendicular [from the center of the sphere] were 1, the edge of the cube would be 2[49]. Having fully treated these proportions and surfaces in our earlier work[50], let those be a supplement to what we have given here, with regard to the dependents, which can, with diligence, be worked through in all ways using algebra.

CHAPTER XXXIV

On the inclusions [nesting] of the five regulars, one in the other and the other in the one, and how many there are in all, and why.

It follows now to clarify how these five essential solids, that is, regulars, be contained by the other, and which yes and which no, and why. Whence, speaking first of the tetrahedron, it is

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[49] Thus, the diameter of a sphere enclosing a cube of edge 2, would be √12, and the total surface area of the cube would be 24. Likewise, a cube inscribed in a sphere of diameter 4, would have a value for each edge of 4/√3, etc.

[50] Henceforth, whenever Pacioli refers to "our work," he is referring to his ***Summa de Arithmetica, Geometrica, Proportioni et Proportionalita***.

shown that it can in no way receive in itself any other than the octahedron, that is, the solid of 8 triangular faces and of 6 solid angles, since within it, there are neither sides nor faces nor angles in which it would be possible to support the sides of the cube, nor its angles nor its surfaces, in such a way that they would touch them equally, as required by their true inscription, as its material form demonstrates to the eye, and is manifested by true science in the 1st of the XVth. Nor could either of the other two, that is, the icosahedron and dodecahedron, be inscribed in the tetrahedron.

But if we wish to inscribe or form the said octahedron within the solid of four equal faces, the tetrahedron, we will do it in this way. That is, first we will construct the said tetrahedron as we have instructed above; and that being thus completed, we will then divide each one of its edges in half, and we will connect all their mid-points with straight lines, the one with the other and vice versa. And that being accomplished, without doubt we will have precisely situated the said solid in such a way, that its 6 solid angles will be supported equally upon the 6 edges of the said tetrahedron. The which will be made clear by construction, and is proved in the 2nd of the XVth.

CHAPTER XXXV

How the said tetrahedron is to be formed and placed in the cube.

The said tetrahedron will be collocated in the cube in the following way: that is, first we will make the cube according to the methods given above, then on each one of its 6 square surfaces, we will draw the diagonal or diameter, and the objective will be concluded as the 1st of the XVth demonstrates, since the said tetrahedron, as we stated, has 6 edges corresponding to the number of the 6 surfaces of the cube, and those come to be its 6 diagonals traced on its surface. And the 4 vertices of the pyramid come to rest in 4 of the 8 [vertices] of the said cube. The which also the teacher of all things, sacred experience, renders clear in their material form.

CHAPTER XXXVI

On the inclusion of the octahedron in the cube.

And wishing to form the 8 faces, that is, the octahedron, in the hexahedron, first it is necessary to have constructed in the cube the triangular equilateral pyramid, whose edges, as was said, are the 6 diameters of its faces. And so, if we will divide each one of the said diameters in half, and join those mid-points one with the other with straight lines, without doubt the octahedron is formed exactly within the proposed cube. And each one of its solid angles [vertices] comes to rest exactly on the faces of the said cube by the 3rd of the XVth.

CHAPTER XXXVII

On the fabrication of the hexahedron in the octahedron.

The hexahedron or cube, will be made in the octahedron in this way; that is, first we will make the said octahedron according to the method given above in this work. The which thus formed, find the center of each one of its triangular faces, by the 5th of the IVth; the which 8 centers we will then join to one another by means of 12 straight lines. And we will have achieved our purpose, and each one of the solid angles of the cube will come to rest on the face of the said octahedron, as the 4th of the XVth declares.

CHAPTER XXXVIII

On the inscription of the tetrahedron in the octahedron.

If we will wish to form the equilateral triangular pyramid, that is, the tetrahedron, in the octahedron, we will first make the cube within the latter, in accordance with what was said in the preceding chapter; and then we will construct the tetrahedron within that cube, in the said manner. And thus we will have similarly collocated the tetrahedron within the given octahedron, as the 5th of the XVth states.

CHAPTER XXXIX

On the formation of the dodecahedron in the icosahedron.

The icosahedron, as was said, has 12 solid angles, each one contained by the 5 surface angles of its 5 triangles. And so, wishing to make in that the dodecahedron, it would be useful first, to make, according to our instructions in this work, the said icosahedron. And when this has been duly carried out, find the center of each one of its triangular faces by the 5th of the IVth; and then we will join those centers with 30 straight lines, in such a way that they will form of necessity 12 pentagons, each one opposite to one solid angle of the said icosahedron. And each one of the edges of the said pentagons is opposed, in crosses, to each one of the edges of the said icosahedron; and just as on the icosahedron there are 12 solid angles, so on the dodecahedron there are 12 pentagons; and just as in the former there are 20 triangular faces, so in the said dodecahedron there are 20 solid angles produced in the said faces, by means of the said lines. And just as in the former there are 30 edges, thus in the dodecahedron there are 30 edges opposite to those, in crosses as was said -- all of which is manifested by their [physical] form, as also the 6th of the XVth concludes.

CHAPTER XL

On the placement of the icosahedron in the dodecahedron.

If you wish to form the icosahedron in the dodecahedron, first construct the latter according to the method given above, and find the center of its 12 pentagonal faces, as the 14th of the IVth instructs; and connect those to one another with 30 lines in such a way that, in so doing, they will produce 20 triangles and 12 solid angles, each one contained by 5 surface angles of the said triangles. The vertices of which will be at the 12 centers of the 12 pentagons; and similarly its 30 lines will oppose in crosses the 30 of the dodecahedron, just as was said above, and also results from the 7th of the said XVth.

CHAPTER XLI

On situating the cube in the dodecahedron.

We will also make the cube easily in the said dodecahedron, considering that it is itself formed on the 12 edges of the cube, as contained in the 17th of the XIIIth. Since, if to each one of its 12 pentagons, according to the requirements of the above, 12 cords are drawn, without doubt 6 equilateral quadrangular surfaces will be formed, and to each one of those, two solid angles of the said dodecahedron will be opposed, and in the 8 solid angles of the dodecahedron, there will be formed 8 solid angles of the inscribed cube, in such a way that what remains on each face of the cube, is a **corpo seratile**[51]. All of which is made clear by the 8th of the XVth.

CHAPTER XLII

On how the octahedron is to be formed in the dodecahedron.

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51 Archimedes says a prism on a triangular base is called 'seratile' in *De momentis aequalibus,* Prop IV.

If the cube is first arranged in the dodecahedron, as was said in the preceding, then the octahedron will easily be formed in the said dodecahedron. Since we will divide in half, the 6 edges of the dodecahedron opposite to the 6 surfaces of the cube, that is, those edges which make a sort of **colmo[52] al seratile**, which are exactly 6 in number. And those 6 mid-points we will extend by 12 straight lines, all of them in such a way that they will come to generate 6 solid angles, each one contained by 4 surface angles of 4 of the triangles of the octahedron. And each one touches one of the said 6 sides of the dodecahedron, and consequently the problem is shown to be resolved, just as is contained in the 9th of the Xvth.

CHAPTER XLIII

On the inclusion of the tetrahedron in the said dodecahedron.

The tetrahedron will also be placed in the same dodecahedron, if the cube is formed in it first, as was described, and then the tetrahedron is placed in the cube, as was also shown. The which things being done, our objective will be precisely completed, as follows. That is, since the solid angles of the cube are placed in the solid angles of the dodecahedron, and the solid angles of the tetrahedron come to rest in those of the cube, it follows that the said tetrahedron is duly included in the said dodecahedron; and this is made clear by our experience, in building the solids in their material form, which we have presented to your Grace, -- and it is also scientifically demonstrated by the 10th of the said XVth.

CHAPTER XLIV

On the fabrication of the cube in the icosahedron.

The cube is formed in the icosahedron if, in that, the dodecahedron is first made as we said above. And then, in that dodecahedron, make the cube in the prescribed manner; the which things done, our purpose will turn out to be achieved by what we said above: since the solid angles of the dodecahedron all fall upon the centers of the faces of the icosahedron, and the solid angles of the cube fall upon the said solid angles of the dodecahedron. And consequently, the intent is carried out, which is also explained to us by the 11th of the XVth.

[52] colmo = peaked, or ridge-pole.

CHAPTER XLV

On the manner of forming the tetrahedron in the icosahedron.

There is no doubt: if in the said icosahedron the cube be formed, as we instruct above, and then, in that cube, the tetrahedron be constructed, then of necessity, that will also come to be inscribed in the said icosahedron. Since the solid angles of the pyramid of 4 triangular faces touch those solid angles of the cube, and those of the cube touch those of the icosahedron, it follows *de primo ad ultimum* that those of the tetrahedron touch likewise those of the icosahedron. And consequently our proposal is concluded by the 12th of the XVth. And this is all respecting their proposed inclusions.

CHAPTER XLVI

Why it is not possible for there to be more of the said inscriptions.

Whence, excellent Duke, from the things we have discussed, it is manifest that, there being 5 regular solids, if it were possible, as one might suppose, to inscribe each one in the other, it would follow that each one of them should receive 4, and consequently, there would be 20 inscriptions, that is, 4 times 5. But, because each one does not receive each one, as was said, there are but 12 inscriptions. That is: only one in the tetrahedron, the octahedron; and two in the cube, that is, the tetrahedron and the octahedron; and two also in the octahedron, that is, the cube and the tetrahedron. And there are three of them in the icosahedron, that is, the dodecahedron, the cube and the tetrahedron; and there are four in the dodecahedron, that is, the icosahedron, the cube, the octahedron and the fourth, the tetrahedron. And all these add up to twelve. This is because, in the pyramid of 4 faces, there are neither edges nor vertices nor surfaces in which it were possible to support the vertices of the 3 other regulars, while they do support those of the octahedron. The cube is also only able to receive in itself the pyramid and the octahedron. And the octahedron, only the cube and the pyramid. And in none of these is it possible to collocate either of the other two, that is, the icosahedron nor the dodecahedron. And it happens that the icosahedron gives reception to three; it refuses only the octahedron, and this is because of that glorious sign which makes all demons tremble, that is, the Sacred Cross. For, there is no place in the icosahedron to support 3 lines which cross each other at right angles, and would extend from one vertex diametrically across to the other, as required for the proper inscription of the octahedron. But the dodecahedron is, of all of them, endowed with the singular prerogative, as receptacle to all, of not prohibiting or forbidding lodging to any. And for that reason also, the ancient Plato along with others mentioned, attributed it to the Universe.

CHAPTER XLVII

How the sphere is to be formed in each one of the said regular solids.

Above, as was seen, excellent Duke, we have demonstrated that each of the said five regular solids can be inscribed in a given sphere, by which it is circumscribed. It is now fitting to demonstrate how a given sphere could also be inscribed in each one of them, which we will present here, with unmistakable clarity: that is, conversely, how the sphere can be inscribed in each of them. Proceed as follows.

From the center of the sphere which circumscribes each one of these particular solids, draw or trace to every one of its faces, the perpendiculars, which, of necessity, will fall in the centers of the circles which precisely circumscribe those same faces; now, since all the circles which precisely circumscribe the said faces are equal, these perpendiculars will be equal. Whence, if we take the length of one of them, and describe the circle around the center of the sphere which circumscribes them, and turn its semicircle around until it returns to the place from which it commenced to move, since it must pass through the end points of all of the perpendiculars, we will prove by the corollary of the 15th [14th] of the IIIrd, that the sphere described by the motion of this semicircle, is contingent upon or exactly touches all of the faces of the given solid at the point of their intersection with the perpendiculars, since the sphere cannot touch more faces of the solid, than the semicircle touches when it moves. Whence it is clear that we have inscribed the sphere inside the given solid, just as we proposed to do.

CHAPTER XLVIII

On the form and the arrangement of the plane tetrahedron, solid or hollow; on the truncated, solid or hollow; and on the elevated, solid or hollow.

The plane tetrahedron, solid or hollow, is formed by 6 equal lines, which contain 12 surface angles and 4 solid angles; and between them they make 4 equilateral and equiangular triangular faces. [see Plates I, II]

On the truncated or diminished. [see Plates III, IIII]

The truncated, or we would say diminished tetrahedron, plane solid or hollow, is contained by 18 lines which bring into being 36 surface angles and 12 solid angles. And 8 faces circumscribe it, of the which 4 are hexagons, that is, of 6 equal sides, and the other 4 are triangles, similarly equilateral and also equiangular. But of the said 18 lines, 12 are common to the triangular and to the hexagonal faces, which nonetheless all belong to those hexagons, because of necessity, those 4 hexagons joined together with some of their sides, bring into being those 4 triangles, just as the evidence of its own material form makes clear to our eyes. And it arises from the preceding [tetrahedron], when its edges are cut by uniform thirds.

On the elevated solid. [see Plates V, VI]

The elevated, or we would say, stellated[53] tetrahedron, solid or hollow, similarly has 18 lines of which 6 are common. And it has 36 surface angles and 8 solid angles, of which 4 are cones of the surface pyramids and 4 are common to the 5 pyramids; that is, to that interior pyramid which the eye cannot see, but only the intellect apprehends, and to the other 4 exterior ones. The said solid is composed of these 5 pyramids, if the faces are equilateral and equiangular triangles, as its material form demonstrates to us. And its surfaces which cover it, which are not properly called bases, altogether are 12 in number, all triangular. And it is not possible to make a elevated solid of the truncated tetrahedron, due to the defect of the hexagons, which do not make solid angles.

CHAPTER XLIX

On the plane hexahedron, solid or hollow; truncated, solid or hollow; plane elevated and elevated truncated.

The hexahedron, or we would say cube, plane, solid or hollow, has 12 lines or sides [edges] or ribs, and 24 surface angles and 8 solid angles and 6 faces or surfaces which contain it, all square, equilateral and also equiangular, similar to the form of the devilish instrument otherwise called dice or **taxillo**. [see Plates VII, VIII]

On the truncated or diminished. [see Plates VIIII, X]

The truncated or diminished plane hexahedron, similarly solid or hollow, has 24 lines which bring into being around them, 48 surface angles, of which 24 are right angles and the others acute. They have 12 solid angles, and are contained by 14 surfaces or faces, that is, by 6 squares and 8 triangles. And all of the said lines are common to the squares and to the trigons, because those 6 squares joined together at the vertices, of necessity produce 8 triangles, just as the hexagons would, in the truncated tetrahedron. And it emerges from the cube cut uniformly at the middle of each one of its edges, as its actual material form demonstrates to the eye.

On the elevated. [see Plates XI, XII]

[53] **pontuto** -- in mod. It., appuntito: lit. "pointed"; i.e., what we would call today "stellated."

In order to construct the elevated hexahedron, solid or hollow, 36 lines must converge, the which applied together, bring into being 72 surface angles and 6 solid pyramidal angles, each one contained by 4 surfaces. And it is covered by 24 triangular surfaces, which properly should not be called bases. And of those lines, 12 of them are common to all those surface triangles which contain and circumscribe it. And the said solid is composed of 6 extrinsic quadrilateral pyramids, all of which present themselves to the eye according to the position of the solid; and also of the intrinsic cube upon which the said pyramids are placed, and only the intellect imagines it, because all is concealed to the eye by the superposition of the said pyramids on it. And of that cube, its 6 square surfaces are bases of the said 6 pyramids which are all of the same height, and are concealed from the eye, and in this hidden way, circumscribe the said cube.

On the elevated truncated solid. [see Plates XIII, XIIII]

The elevated truncated hexahedron, solid or hollow, has 72 lines or sides or ribs. And these make 144 surface angles, and 14 solid angles, all pyramidal. Of the which, 6 are of quadrangular pyramids, and 8 are pyramids with a trilateral base. And of the said lines, 24 of them are common to the trigonal and tetragonal pyramids. And it has 48 faces or surfaces which circumscribe it, all triangular: and this thus-constructed solid is composed of the intrinsic, truncated, solid hexahedron, perceptible only by the intellect, and of 14 pyramids as was said. And when thrown onto a plane surface, it always comes to rest upon 3 pyramidal cones or points, as the form demonstrates.

CHAPTER L

On the plane octahedron, solid or hollow; and the truncated, solid or hollow; and on the elevated, solid or hollow.

The plane octahedron, solid or hollow, receives in itself 12 lines and 24 surface angles, and it has 6 solid angles; and it is contained by 8 equilateral and likewise equiangular trilateral faces, as it presents itself to us in its own material form. [see Plates XV, XVI]

On the truncated plane solid. [see Plates XVII, XVIII]

The truncated or cut octahedron, plane solid or hollow, has 36 lines which make 72 surface angles, that is, 48 are formed by the hexagons and 24 by the squares. And it contains 24 solid angles and has 14 faces, of the which 8 are hexagons, that is, of 6 sides, and 6 of them are tetragons, that is, squares. But of the said lines, 24 of them are common, that is, to the squares and the hexagons. And those such squares are formed by the hexagons when all 8 are uniformly contiguous, and the truth of all this, with their material forms, will be seen by the eye and made known to the intellect. And again of this one it is not possible to form its stellation with equal faces, having the same defect of the hexagons, which, as we said of the truncated tetrahedron, cannot make a solid angle. And [this truncated one] is formed from the octahedron, by uniformly cutting each one of its edges in thirds.

On the elevated, solid or hollow. [see Plates XVIIII, XX]

The elevated octahedron, solid or hollow, has 36 lines of equal length, and has 72 surface angles and 8 solid pyramidal angles. And it is contained by 24 surfaces, all equilateral and equiangular trigons, the which exactly circumscribe it. But, of those lines, 12 of them are common to all of the triangles of the pyramids. And this such solid is composed of 8 pyramids of triangular sides, equilateral and equiangular of the same height, which are all apparent outside, and also of the intrinsic octahedron, perceptible only by the imagination of the intellect, of the which octahedron, the faces are bases of the said 8 pyramids, as its material form manifests to us.

CHAPTER LI

On the plane icosahedron, solid or hollow; on the truncated, solid or hollow; and on the elevated, solid or hollow.

The plane icosahedron, solid or hollow, contains 30 lines or edges, all of them equal, and these generate in it, 60 surface angles and 12 solid angles; and they also form in it 20 faces, all triangular, equilateral and equiangular. And each one of the said solid angles are made or contained, by 5 surface angles of the said triangular faces, which the model of this solid demonstrates. [see Plates XXI, XXII]

On the truncated, plane solid. [see Plates XXIII, XXIIII]

The truncated icosahedron, plane or if you like solid, has 90 edges or lines, and it has 180 surface angles, of which 120 are angles of the triangles which gather together in the composition of this solid, and 60 of which are angles of pentagons which also meet, all equilateral. And these lines form around the said solid, 32 faces, of which 20 are hexagons, that is, of 6 equal sides, and 12 of them are pentagons, that is, of 5 equal sides. And each one of them in its place, is equilateral and also equiangular, that is, all the hexagons are of equal angles, and likewise all the pentagons. But all the edges, those of the pentagons as well as the hexagons, are equal to one another. Only in their angles are the pentagons and the hexagons different. And the solid so composed emerges from the preceding regular, when each one of its sides is cut uniformly by its third part. And through such cuts, 20 hexagons and 12 pentagons come into being, as was said, and [30] solid angles. But of the said lines, 60 of them are common to the hexagons and pentagons, because the 20 hexagons uniformly joined together, of necessity, produce 12 pentagons. And neither of this solid, is it possible to make the elevated, because of the defect of the said hexagon, as we said above concerning the truncated tetrahedron and the truncated octahedron.

On the elevated solid. [see Plates XXV, XXVI]

The elevated icosahedron, solid or hollow, has 90 lines within itself, and has 180 surface angles and 20 solid pyramidal; and it has 60 faces or surfaces, which circumscribe it, all triangular, equilateral and also equiangular. But of the 90 lines, 30 of them are common to each one of the surfaces of its 20 pyramids. And the said solid is composed of 20 triangular pyramids, equilateral and equiangular, of equal height, and of an interior, integral icosahedron, perceptible only through the imagination by the intellect; and its faces are similarly bases of the said pyramids, all of which again its own material form makes obvious.

CHAPTER LII

On the plane dodecahedron, solid or hollow; and on the truncated, solid or hollow; and on the elevated, solid or hollow, and on the elevated truncated, solid or hollow; and its origin or dependence.

The plane dodecahedron, solid or hollow, has 30 equal lines or edges, which bring into being 60 surface angles. And it has 20 solid angles and 12 faces or surfaces, which contain it, and these are all pentagons of equal edges and angles, as is apparent from its model. [see Plates XXVII, XXVIII]

On the diminished or truncated. [see Plates XXVIIII, XXX]

The truncated or diminished dodecahedron, plane, solid or hollow, has 60 lines, all of equal length, and it has 120 surface and 30 solid angles. But of the 120 surface angles, 60 are of the triangles and 60 are of the pentagons; and those of the triangles, are of necessity brought into being by the said pentagons, when they are angularly joined to each other: as was said in the creation of the angles of the truncated tetrahedron and octahedron, which from hexagons, quadrangles and triangles formed themselves, and likewise in those of the truncated icosahedron, pentagons are formed by hexagons, as the solid figure demonstrates.

And each one of the said solid angles is made and contained by 4 surface angles, of the which two are triangles and two are pentagons which meet at one same point. And all of its lines or edges, are common to the triangles and to the pentagons, since, if they are applied properly together, the one is brought into being by the other, that is, the triangles by the pentagons, and the pentagons by the triangles. And just as the 12 equilateral pentagons angularly joined, form 20 triangles in the said solid, so also we can say that 20 equilateral triangles, angularly joined among them, bring about 12 pentagons similarly equilateral. And from this, all of the said lines among them are shown to be common, as said. And the surfaces which circumscribe this are 32, of the which 12 are pentagons, equilateral and equiangular, and 20 are triangles, also equilateral and equiangular, all among them, as we have said, are reciprocally brought into being. And this is clear from looking at the physical form; and this solid is derived by cutting the preceding in the middle of each edge.

On the elevated. [see Plates XXXI, XXXII]

The elevated dodecahedron, solid or hollow, has 90 lines and 180 surface angles, and 12 solid elevated pentagonal pyramids; it has also 20 bases of hexagonal structures. And it has 60 surfaces, all triangles, equilateral and equiangular. But, of the said 90 lines, 12 are common to the 12 bases of the pentagonal pyramids, whose bases likewise shall be pentagons. And these are the intrinsic bases of the regular dodecahedron, which come together in its construction, which the intellect comprehends only by the imagination. And it is only these 30 common lines which together bring into being the 20 low-lying solid angles, which, as was said, are hexagonal, that is, there are altogether 6 lines in their formation. And the said solid is formed from the aforementioned intrinsic regular dodecahedron and from 12 pentagonal pyramids, equilateral and equiangular and of equal height; and their bases are the same bases of the intrinsic dodecahedron, *ut supra*.

On the truncated elevated. [see Plates XXXIII, XXXIIII]

The elevated truncated dodecahedron, solid or hollow, has edges or lines 180 in number, of the which 60 are elevated to bring about the pentagonal pyramids, and 60 are elevated to construct the triangular pyramids; the other 60 are edges of the bases of each one of the said pyramids, that is, of the pentagons and triangles. And this solid thus made, is composed of the intrinsic planecut dodecahedron, offered to the intellect only by the imagination, and of 32 pyramids, of which 12 are pentagonal, all equal to each other in height, and the other 20 are triangular, also equal to one another in height. And the bases of these pyramids are the surfaces of the said truncated dodecahedron, referring each to its own, that is, the trigons to the triangular pyramids, and the pentagons to the pentagonal pyramids. And dropped onto a plane, this always stops on 6 points or pyramidal cones; of the which cones, one is a pentagonal pyramid and the other 5 are triangular pyramids.

When this solid is suspended in mid air, it appears absurd to the eye that these points apices should be on one same plane, but this attribute, excellent Duke, is of such great abstraction and profound science, that whoever grasps it, I know, will not make of me a liar. And one finds its dimensions by means of very subtle methods, especially of algebra and *almucabala*, which, although known by few people, we have demonstrated well in our work[54] with methods that permit them to be easily apprehended. Of similar subtlety is the cut icosahedron, where intermingled hexagons and pentagons make all measurements difficult.

CHAPTER LIII

On the solid of 26 faces and its origin, plane, solid or hollow; and on the elevated, solid or hollow.

There is to be found excellent Duke, another solid said to have 26 faces[55], quite dissimilar from those already mentioned, deriving from a most joyful beginning and origin. 18 of its faces are squares, equilateral and right-angled, and 8 are triangular, similarly equilateral and equiangular. And this particular solid has 48 edges or lines, and it has 96 surface angles, of which 72 are right angles, and are those of its 18 square faces; and 24 are acute and are those of its 8 equilateral triangles. And these 96 concur in the composition of 24 solid angles, each one of which consists of a surface angle of the triangle and of three right angles of 3 squares. [see Plates XXXV, XXXVI]

And of its 48 lines, 24 are common to the trigons and to the squares, since from those 18 squares fittingly joined together, of necessity there turn out to be 8 triangles formed, just as in the other truncated ones, described above. And this solid originates from the uniform hexahedron, each of its edges cut in the same way that the physical form of it visually presents itself to us[56]. And the knowledge of this is very useful in many applications, for those who know how to deal with it, especially in architecture. And so much for this solid, plane and hollow.

Of the elevated, solid or hollow. [see Plates XXXVII, XXXVIII]

In its formation, the 26 faces, solid or hollow, elevated, bring together 144 lines, which, according to the appropriately ordered arrangement, will generate in it 288 surface angles and 26 elevated solid pyramids, of the which 18 are contained by 4 acute surface angles, and 8 are contained by 3 acute surface angles. And the said solid is composed of 26 lateral pyramids, of the which 18 are quadrangular and 8 triangular, all of which the eye can see around it from the outside; and of the 26 intrinsic solid plane faces mentioned above, comprehended only by the

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54 See note chapter VIII above.

55 The figure of 26 faces is otherwise called, *vigintisex basium* or *icosahexahedron*, or, in modern terms, small rhombic cuboctahedron.

56 This figure of 26 faces can be derived by dividing each of the 12 edges of a cube, by 2 points, into three parts, in the ratio of 1: $\sqrt{2}$: 1, and truncating the cube accordingly. The edges of all of the equilateral triangles and squares of the figure will then have the proportional value of $\sqrt{2}$.

imagination. And its 26 faces are likewise bases of the aforementioned 26 pyramids, that is, the 18 quadrangular bases [of the 18 quadrangular pyramids] and the 8 triangular bases of the 8 triangular pyramids. And in whatever way one will throw this solid onto a plane, it will always come to rest upon three points or pyramidal cones. And this experience from its material form will be satisfying to the eye.

CHAPTER LIV

On the solid of 72 faces, plane, solid and hollow.

It is fitting, excellent Duke, to collocate here the solid of 72 faces[57], which our philosopher from Megara fully describes in the 14th [17th] of his XIIth. This solid, since its faces are plane, lateral, angular and irregular, cannot be said to be dependent upon, nor derived from any of the regular solids, but is formed and created, as our philosopher demonstrates in the said location, only by means of the duodecagon figure, that is, of twelve equal sides. And of its aforementioned faces, 48 are quandrangular, inequilateral and inequiangular, and they have only the two opposite edges extended toward the one and the other pole, or we would say cone, equal between them; and the other of its 24 faces are similarly inequilateral triangles. And of these, 12 of them are around one of the cones, and 12 around the other, and each one of them has two edges equal, that is, those that extend to the point of the lower and upper pole. [see Plates XXXVIIII, XL]

Of this solid also, it will always be possible to make its elevated form, as was done in the others, but, because of the deformity of its faces, its study will be difficult, although it would present to the eye a not-mediocre subtlety. And there would be brought into being within it, 72 pyramids according to the number of its 72 faces, of which pyramids the bases will be the same as that face imagined inside. The elevated form of it, I have not taken the trouble to draw materially here, so as to leave to the reader also his part, whose ingenuity I do not doubt. And these 72 bases are employed by many of the architects in their dispositions of buildings, in order to make a very fitting form, especially where it is necessary to make a tribune or other vault, or if you wish, a ceiling. And it happens that they would not always make exactly that many faces in the structure they were building; nonetheless, they take this figure as their starting point, quartering it, or trisecting it, in whatever way necessary, depending upon the location and position where they intend to put such a structure. And you can find it very commonly, in various parts, placed and constructed, such as in the inestimable ancient temple, the Pantheon, which today is called the Rotunda by the Christians in the **capo del mondo**. The which is arranged with so much skillful industry and such consideration of proportions, that with the light of one single eye opened at its apex, everything glistens, splendid and luminous.

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[57] The solid of 72 faces is otherwise called *Setuaginta Duarum Basium.*

Not to mention many other famous and glorious cities, like Florence, Venice, Padua, Naples and Bologna, in which many structures, both sacred and secular, small or large, are made as a mirror of this. Moreover, here in your city of Milan, in the worthy shrine of San Scetro, the ornate chapel is cut from a part of this solid, with a certain amount of discreet convexity applied to the walls; and a rosette is added in each base for adornment. And the vault at the first altar, in your devout and most sacred temple of the Graces, and those of the side apses, are nothing other than a part of this same model, to the bases of which, here also these rosettes have been added to increase delight.

And, although many construct and draw the forms as they please, knowing neither Vitruvius nor any other architect, yet they are practicing the art of architecture, without knowing it: just as Aristotle said of the crude peasants, that *Solegizant et nesciunt se solegizar*[58], so these [untutored craftsmen] *Utuntur arte et nesciunt se uti*[59]. Also the common tailor and the shoemaker use geometry and know not what it is. And so masons, cabinet makers, smiths and all kinds of artisans, use measure and proportion and do not know it, because, as has been said before, everything consists of number, weight and measure.

But what shall we say of the modern edifices, ordered and arranged according to various and diverse models, which, as long as they are scale models, appear somewhat attractive to the eye, because they are small; but then when they are built, do not support their own weight? Far from lasting a thousand years, they fall into ruin before three. And by their negligence, these charlatans are more occupied with re-doing than doing; incurring unnecessary expenses. And they call themselves architects, never having seen even the cover of that most excellent volume of our very distinguished architect and great mathematician Vitruvius, who composed ***On Architecture*** with the fullest documentation on each kind of structure. And whosoever deviates from that treatise, digs in the water, and makes their foundations on sand, and soon enough ruins the art of architecture. These pretenders who call themselves architects, do not know the difference between a point and a line; how could they then know the different kinds of angles, without which knowledge it is impossible to build anything well. And this is manifested by what the aforesaid Vitruvius said, about the great joy and supreme delight of Pythagoras, when, with certain science, he found the true proportion of the two straight lines that contain the right angle of the square. And to celebrate this discovery of Pythagoras, they made a great sacrifice and feast to the gods of an hundred oxen. And this angle is of such excellence that it can never change, and the best geometers call it by another name, *Angulum iustitiae,* the angle of justice, since without its knowledge, it is not possible to know right from wrong in any of our operations, nor without it, is it ever possible in any way to provide an exact

[58] They were speaking correctly without knowing it.

[59] They were practicing the art without knowing it.

measure. While the modern cobblers in their constructions, do not seem to make anything, except outside the right and proper ancient tradition. No inconvenience interferes with their foolishness, while they criticize those -- few as there are still to be found -- who practice the true and ancient methods. And it is these true architects who delight in our mathematical disciplines, imitating the true guide of all builders in the work of Vitruvius. When these principles are not followed, we find the condition of our modern buildings, both religious and profane, crooked and twisted. And for this, Your Excellency's motto and its effect would be most appropriate, the one about the axe which straightens everything crooked.

And continuing the construction already begun, your city of Milan, no less attractive than Florence, will soon lose the false impressions caused by abominable and inept works, by removing their perpetrators. Because in truth, his Grace has a better comprehension of the art while sleeping, than they do while looking with a thousand eyes. This is demonstrated as well by your close relative, the illustrious Duke of Urbino, in the admirable workshop of his noble palace, mentioned earlier. And with condolences for those who might have taken badly what has been said up to this point, let this suffice for the said solid.

CHAPTER LV

On the way of knowing how to form more, beyond the said ones, and how their forms proceed to the infinite.

It does not seem necessary to me, excellent Duke, to elaborate further on the said solids, since the process of their construction tends towards the infinite, by the continuous and successive truncation of the solid angles, and as a result, their various forms come to multiply. And this can easily be done on one's own, since the way has been opened to them by what we have already said, as in the old proverb, *Quod facile est inventis addere,* it is not difficult to add to things already invented: and hence by cutting something here, adding something there, to the aforesaid solids, it will be easy to make other ones. And we have only continued this up till now, to show how from the five regular solids, the virtue is always instilled in their dependents, similar to the way in which the five simple elements go into forming everything which is created. For that reason -- as indicated earlier, to the five simple elements, that is to earth, air, water, fire and heaven, Plato was compelled to attribute the five chosen regular forms, as is apparent throughout his ***Timaeus***, where he dealt with the nature of the universe.

And to the element of earth he attributed the cubic form, that is, that of the hexahedron, since no other figure needs more violence to put it into motion. And among all the elements, what is to be found more fixed, constant, and firm, than earth? And to the element of fire, he gave the form of the tetrahedron, since, in flying upward, it produces the pyramidal form, which is similar to our fire, as apparent to the eye, since we see that which is wide at the base, and decreases uniformly as you go up, such that the flame ends in a point at the top, just as does the cone of every pyramid. To the air, he attributed the form of the octahedron, since, just as with the slightest movement, the air follows fire, so the octahedron's pyramidal form is next, for ease of motion, after the form of the pyramid. And the figure of 20 faces, that is the icosahedron, he assigned to water, since, it being enclosed by more faces than any of the others, it appeared to him that this was fitting to the sphere, that which more quickly spills down in motion than that which ascends. And to heaven, he attributed the form of the 12 pentagonal faces, just as to that which is the receptacle of all things: this dodecahedron is likewise receptacle and abode of all the other four regular solids, as comes into view in their inscriptions the one in the other. And again, as said by Alcinoveus, on the ***Timaeus*** of Plato: because, just as in the heaven there are 12 signs of its zodiac, and each one of those is divided into 30 equal parts, which in its entire annual revolution makes 360, thus this dodecahedron possesses 12 pentagonal faces, of the which each one is resolved into 5 triangles by fixing a point in the middle, and each of the said triangles into 6 scalene triangles, which, in all the faces, are 30 triangles each, which altogether makes 360, as in the said zodiac.

And these such forms are much commended by the very celebrated philosopher, Calcidius, expounding on the said ***Timaeus***; and thus also by Macrobius, Apuleius and very many others, because in fact, they are worthy of all commendation, for reasons which are demonstrated in their constructions, showing the sufficiency of the said five forms, just as of the five elements, which could not in any way be more in number; for just as the number of the said elements could by nature not be more than they are, likewise it is not possible for there to be more than five regular solids, where all the faces, edges and angles are equal, and contained in one sphere, where if one vertex is touching the sphere, they all do. Because, if in nature it might be possible to assign a sixth simple form, the Great Artificer would come to have been diminished in his works, and without prudence in judgment, not having known, in principle, all of nature's relevant requirements. Moved by this certainty, and not by any other, Plato, as was said, comprehending these forms, attributed them to each of the five elements, arguing from the standpoint of an excellent geometer and a most profound mathematician. Seeing that there could not by any means be any other than the five different regular forms which would fit within a sphere, and have, as was said, equal faces and angles, as shown in the penultimate [18th] of the XIIIth, and posed by us before[60], Plato proved convincingly that the said five simple elements derive from the five forms and every other form depend upon them. It turns out that only these five are called regular -- not, however, to exclude the sphere, being above all others most regular, all the others deriving from it, as from the cause of the most sublime causes. And in the sphere, there is not any dissimilarity, but uniformity throughout, and in every place it has its beginning and end, its left and right, whose form and its cause, here we will discuss next,

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[60] Chapter XXV.

putting an end to our discussion of the dependents. And after that, we will discuss all the oblong solids, that is, those that are longer than they are wide.

CHAPTER LVI

On the formation of the spherical solid.

What the sphere[61] might be has been defined by many; principally by the worthy mathematician, Dionysus. Yet, our Euclid describes it most concisely in his XIth, and that description[62] is cited by everyone after him, where he states thus: "the trace of the semicircle makes the sphere." The sphere will be that which contains the trace of the arc of the circumference of the semicircle. Each time and in whatsoever way one takes the semicircle, keeping the line of the diameter firm, it turns around the said arc until it returns to the place whence it started to move: that is, a semicircle made over any line, keeping that line still, that semicircle is conducted around with its entire revolution. Such a solid, which is thus described, is called a sphere, whose center will be the center of the said semicircle thus led around. [see Plate XLI]

Demonstration of the said definition

Let the semicircle C be constructed on the line AB, the center constructed at point E, and its entire arc be part of the circumference ADB [see Figure 15]. I say that, fixing the said line AB, which makes the diameter of the said semicircle, and rotating the arc above it, starting from the point D, going toward the lower part and returning toward the upper part to the said point D, whence it first moved, or [rotating] the arc in the opposite direction, going up first and returning towards the lower, to the given point D, imagining the given semicircle, for example, to be something physical which is capable of cutting, the round object made by the rotation of the said semicircle will be a spherical body and a sphere, which otherwise would not form a solid, because the arc alone led around, would not even make a trace, the line being without thickness or depth. And let this suffice for the description of the sphere and its cause.

CHAPTER LVII

How all five regular solids are to be collocated in the sphere.

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[61] *Sphera solida.*

[62] Book XI, Definition 14.

And in this sphere, excellent Duke, all the five regular solids are imagined in this way. First the tetrahedron: if we mark or imagine on the sphere's surface, that is, its shell or covering, 4 points, equidistant in every way from one another, and if those points be joined together by 6 lines, which, of necessity, will pass inside the sphere, there will be formed in it precisely the aforementioned solid. And whoever would make the cut, by imagination, with a plane surface in each direction, according to the said protracted straight lines, there would remain uncovered precisely the said tetrahedron. So that by this the others might be better understood —- [imagine] the said sphere to be a cannonball, and on it the said 4 points were marked off equidistant from one another: if a stone cutter or cobblestone maker with his tools were to cut it away or remove everything, leaving those 4 points, then he would have made of that said cannonball, precisely the tetrahedron.

Similarly, if, on the said spherical surface, we mark off [8] points, equidistant from each other, and joined them by 12 straight lines, there will be collocated within the said sphere, by imagination, the second regular solid, called hexahedron or cube, that is, the figure of the diabolical instrument called dice. The which points similarly marked off on a cannonball, and a stonecutter proceeding as above, he will have reduced the said ball to the form of a cube. And if 6 points be marked on the said surface, also all equidistant from one another, and we continue, or we would say join those points together with 12 straight lines, there will be made precisely in the said sphere the third regular solid called octahedron. A model of this, made on a stone by a stone cutter, will have made a solid with 8 triangular faces. And likewise, if 12 points are marked, and those conjoined by 30 straight lines, there will likewise be made in the said sphere, the fourth solid called icosahedron. And the same stone-cutter will have reduced the cannon ball to the solid with 20 triangular faces. And if 20 points be marked in the same manner, also joining them with 30 straight lines, there will be formed in the said sphere, the fifth and most noble regular solid, called the dodecahedron, that is the solid with 12 pentagonal faces. And thus the stone cutter would have made of the said ball that very same form. Whence, with similar imagination, all will be placed within the sphere in such a way that their vertices will be situated on the surface of the sphere, if the sphere touches one of its vertices, it will immediately touch them all; and it is not possible in any way, that one vertex would touch the sphere, and the others would not, when the said solid is collocated within the sphere.

And by this infallible science, Your Excellence will be able occasionally, as we used to do, to have some fun with the more incompetent stone cutters, in this way proving their ignorance: ordering them to make from stones like these, some form -- but not one of the five regulars -- with equal sides, faces and angles; for example, ordering them to make a capital, base, or cornice of some column, which is to have 4 or 6 equal faces, as we said, and that one of the 4 faces not be triangular, or those of 6 not be square. And likewise, of 8 and 20 faces, that none be triangular, or of 12 that none be pentagonal, the which things are all impossible. But like all foolhardy braggarts, they will promise you the moon*, roma et toma, maria et montes,* since there are many to be found who don't know or care to learn, in spite of the moral dictum which says: *Ne* *pudeat quae nescieris te velle doceri*[63]. Similarly those inept carpenters, when asked what they would do if they could not find a plane, respond that they would make one of them with another plane. Another boor said that his square was too large to measure such a small right angle, presupposing that right angles vary from one to the other. Or that man, who, having two sticks of equal length, in the form of a *tau*, that is, T, put before his eyes, judges now one, and now the other to be longer. And there are many other similar **capassoni** [mental tricks] [64].

One of these amateurs was present, during the construction of the palace of the dear departed Count Girolimo in Rome. As it happened, there were present many worthy craftsmen from different disciplines, among others, a painter called Melozzo da Frullì. In order to give enjoyment to the imagination, Melozzo and I exhorted the Count, that he should have a certain capital [of a column] made, in one of these forms, not making clear to the Count the difficulty of the task, but only that it would be a worthy thing. And assenting to this, the Count called his master builder to him, and asked if he knew how to make it. And he responded that this was a small undertaking, and that he had already made such a thing several times, the which made the Count doubt whether this were something so worthy as we were recommending. Yet we stuck with what we said, adding openly, that the builder could not do it, because the shape required was impossible. And recalling the said lapicide — which is what they still called them at that time — I asked him again if he had ever made it. Then, with a supercilious smile, he said immediately that to prove the point, he was ready to tackle the job at any time. The Count said to him: "If you don't make it, what would you like to lose?" And that clever one responded not badly: "My Lord, as much again that it will please Your Excellence, to have paid me." And they remained satisfied. He gave him 20 days for the task, and he, who had asked for only 4, turned out to lay waste many a marble block, and made a zero for a capital. Finally, the Count had him stop, so as not to waste more marble blocks. And the stone cutter ended up in ridicule, but never gave up wishing to know the origin of the proposal. Once he found it to be myself, the friar, henceforth he bore me not a little rancor. Finding me, he said: "Sir, sir, I will not pardon you for the injury you did to me, unless you teach me how to do it." I offered to teach him as much as he wanted, and since I was staying in Rome for a few days, did not play him the villain, but taught him this and other things pertinent to his craft; and he courteously sent an excellent cape to me with his compliments. Thus I say to Your Excellence, that at such times, there are opportunities to make some others aware of their mistakes, that they should not brag so much and be so self-satisfied, being contemptuous of almost everyone else.

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[63] Do not be ashamed to wish to be instructed in that which you do not know.

[64] Pacioli was the author of various works dedicated to mathematical and non-mathematical games, puzzles and amusements, such as ***De Ludis*** or ***Schifanoia***, and ***De Viribus Quantitatis***, otherwise known as ***La Forza della Matematica***.

This is what Hiero once did with the poet Simonides, as Cicero tells in his ***De Natura Deorum***. That is, Simonides recklessly committed himself within the space of one day, to say precisely what was God, and he said this was not as difficult to know as others make it out to be. To whom, Hiero, at the end of the said time, asked if he had found it; Simonides said, not yet, and could he be granted somewhat more time. After which time, the same thing happened, and in brief, after several more such occasions, he confessed to understand it less than before, and remained confused with his foolhardiness. This much for placing the solids in the sphere.

CHAPTER LVIII

On the oblong solids, that is, longer or higher than wide.

Next, excellent Duke, so that our treatise should be complete, it is necessary to say something about the oblong solids, that is, of those which are longer or higher than wide, such as columns and their pyramids. Of which several kinds are to be found, and so, first, we will speak of the columns and their origins, then, of their pyramids.

The columns are of two constructions, that is round and lateral. Just as with plane figures, some are curvilinear, and are those that are contained by curved or bent lines, and others are called rectilinear, and are those that are contained by straight lines. The round column[65] [see Plate XLII] is a solid contained between two equal circular bases, which are equidistant between them. The which is defined by our philosopher, in the [21st definition of the] XIth, as follows, that is: the round corporeal figure of which the bases are two plane circles at the extremities and equal in **crassitudine**, that is height, made by the tracing of the rectangular parallelogram, with the side which contains the right angle being fixed, and the said surface rotated until it returns to its original place. And this figure is called the round column. Whence, of the round column, the sphere and the circle, there is one same center.

For example, let ABCD be the parallelogram, that is, quadrangular surface of equidistant sides [see Figure 16]. And fix the side AB, the which, thus fixed, the entire parallelogram is carried around until returning to its place whence it began to move. The figure, thereby corporeal, described by the motion of this parallelogram, is called the round column, whose bases are two circles. And the center of one circle is the point B, and the other is that which the line DA makes in its motion or turning, and its center is the point A. And the axis of this column is said to be the line AB, which stays fixed in the movement of the parallelogram. And if we will imagine the parallelogram ABCD, when it reaches with its rotation the position ABEF [see Figure 17], which then joins the place whence it began to move by the continuation of the plane

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[65] *Columna rotunda solida.*

surface, that is, that the entirety is a parallelogram DCEF, and that we have made in it the diameter DE, the which diameter DE, will also be the diameter of the column.

When one says that the column, the circumscribed sphere and the circle, are of one common center, it must be understood that these be of one same diameter. For example, we have said that DE is the diameter of this column, and therefore the sphere and the circle, whose diameter is the line DE, must necessarily have one same center with the center of the proposed column. Accordingly let the line DE divide the line AB at point G, and G will be the center of the column, since it divides the axis of the column by equals, and also [divides] the diameter of the column by equals; which is proven by the 26th of the Ist, because the angles which are at G are equal, according to the 15th of the Ist, and the angles which are at A and at B are right angles by the hypothesis. And the line AD is also equal to line BE; whence DG is equal to EG, and thus AG is equal to GB. And, being that C and F are right angles, if, above point G, by the distance DG, and also on the line DE, is made a circle, it will pass, by the converse of the first part of the 30th [31st] of the IIIrd, through points C and F. Whence point G is the center of the circle of which the diameter is the diameter of the column, and so also of the sphere, and in this way, it is manifested that, to each rectangular parallelogram, the circle, and to each column, the sphere, can be circumscribed. And thus this theorem is made clear which our philosopher wished to propose to us in the said definition of the round column, of the which this much is sufficient. And next we will speak on the laterals, as was promised.

CHAPTER LIX

On the lateral columns, beginning with the trilateral. [see Plates, XLIII, XLIIII]

The other species or sort of columns are called laterals. Of these, the first is triangular[66]: of which the bases, that is, upper and lower, are two triangles, equidistant from each other, according to the height of the column, as the one drawn here [see Figure 18], of the which the upper base is the triangle ABC, and the lower is the triangle DEF. And this same figure our author says is called the **seratile** solid, and is similar to the peak of a roof of a house that has 4 faces or walls, by which its roof slopes on only two cants, as the eye shows. And it is possible for the bases to be equilateral [or not equilateral]. And of similar columns, the 3 faces are always parallelograms, that is, of 4 edges and rectangular, so that the said **seratile** solid is contained by 5 surfaces, of which three are quadrangles and two are triangles.

CHAPTER LX

[66] *Columna laterata triangula.*

On quadrilateral columns. [See Plates XLV and XLVI]

The second kind are quadrilateral[67], and are those that have two quadrangular bases in the said manner; and four other surfaces that surround it are also quadrilaterals, the opposite equidistant from each other. [see Figure 19]. And these also are sometimes equilateral and sometimes not, according to the disposition of their bases, since, of the rectilinear quadrilateral plane figures, four kinds are designated. One is called square, and is that whose sides are all equal and angles are all right, like figure A [see Figure 20]; the other is called the long tetragon, and is that which has equal opposite sides and angles similarly right, but is longer than wide, like figure B [see Figure 21]. The third kind is called *elmuaym*[68], which is an equilateral figure but not rectangular, and by another name is called rhombus, like figure C here [see Figure 22]; the fourth kind is said to be similar to the *elmuaym,* also called rhomboid, of which only the opposite sides are equal and equidistant from each other, and it does not have right angles, as shown in figure D [see Figure 23]. All 4-sided figures, other than the preceding, are called *elmuariffe,* that is, irregular, as are the figures labeled E [see Figure 24]. Now the said quadrilateral columns can be varied according to these different types of bases, but since we always wish to have the same distance between their bases, they must be distinguished by their height. And we can call these regular, if their faces are so, and the others irregular or *elmuariffe.*

CHAPTER LXI

On the pentagonal columns. [see Plates XLVII, XLVIII]

In the third place are the pentagonal columns[69], that is, those of 5 faces [and two bases], as seen in figure AB here [see Figure 25], where each of the [vertical] sides is a tetragon or quadrilateral. And the bases of these such columns are always two pentagons, that is two rectilinear figures of five sides or angles, since in all the rectilinear figures the number of the angles equals the number of its sides; and it cannot be otherwise. And these also could be equilateral or inequilateral as their bases permit, just as a little earlier was said of the quadrilateral laterals. It is the case that some pentagons are equilateral and equiangular, and others inequilateral and consequently inequiangular, but any pentagon which has 3 of it angles equal, if it be [also] equilateral, of necessity will be also equiangular, as the 7^{th} of the $XIII^{th}$ shows. This is said because it would be possible for the pentagon to have equal sides with two angles equal to each

[67] *Columna laterata quadrangula.*

[68] Again, Arabic terminology is maintained by Pacioli, as it was by Campanus, for example.

[69] *Columna laterata pentagona.*

other; but still the angles would not all have to be equal. And these two pentagons, that is, upper and lower, must also be understood as equidistant from each other, whether the column be equilateral or inequilateral, as one might wish.

And for this reason, excellent Duke, the species of lateral columns can grow infinitely, according to the variety of the rectilinear figures of more or fewer sides, since, of each lateral column, its upper and lower bases, are necessarily similar rectilinear figures which agree in number of sides, that is, that the one would not be a triangle and the other a tetragon; and again if one is equilateral and equiangular, so is the other, for the uniformity of the column, however diversely their varieties might be made, sometimes equilateral and other times inequilateral; for which reason, it does not seem to me necessary to extend myself further on the subject, but only to induce the memory that their denomination always derives from their bases: that is, whatever their bases are called, so they are called. For example, if the bases are triangles as they were above in the **seratile** solid, they would be called triangular; and if they were tetragons or quadrilaterals, they would be called quadrangular; and if pentagons, pentagonal; and if of 6 sides, they would be called hexagonal [see Plates XLVIIII, L] , and so on. But the bases being of whatever quality they wish, the vertical faces of each one will always be rectangular tetragons. In order that their material forms render perceptible that which was said up to now, Your Excellence will be able to see each of them at the number assigned to them in the table, and also, below, in our work, on a plane, in perspective, with the same numeration.

CHAPTER LXII

On the method of measuring all kinds of columns, and first the round ones.

It would now seem fitting to me to put forward the method by which to know how to measure all kinds of columns. It happens that we have treated them fully in our work, yet I will summarize it here, just to indicate it to Your Excellency; beginning with the general rule for all the round ones. [see Plate XLII] First one measures one of its bases, reducing it by squares, according to the approximating method found by the noble geometer Archimedes, located in his work entitled, ***On the Quadrature of Circles,*** and cited in our work with its demonstration, that is, thus: having found the diameter of the base, and multiplied that by itself, one takes 11/14 of the product, that is, eleven fourteenths; and that multiplied by the height of the column, this last product is the corporeal mass [volume] of the entire column. For example, to better understand it, let the round column be ABCD, whose height AC or BD is 10, and the diameters of each one of the bases — the one AB and the other CD — is 7 [see Figure 26]. I say that to square this[70] and every other one similarly, you take one of the said diameters,

[70] As in squaring the circle, i.e. finding the aprox. area.

whichever it may be, AB or CD, which makes no difference, they being equal, that is, 7; and this 7 must be multiplied by itself. That will make 49, and of this I say one takes 11/14ths, which is 38 1/2. And this I say is multiplied by the height or length, of the entire column, that is, by BD or AC, which we put at 10. That will make 385, and this much we will say to be the capacity or volume of that entire column. And this case, excellent Duke, means that if these numbers represent **braccia** of whatever sort one wishes, in that measure there will be 385 small square cubes, that is, like dice of one **braccio** each way, that is, one **braccio** in length, one **braccio** in width and one **braccio** in height, as the figure here shows [see Figure 27]. And thus, if the said numbers signify **piedi***,* the same goes as was said with the **braccia**, and if **passa**, then **passa**, and if **palmi** then **palmi**, and so forth. And resolving the said column into cubes, it would make 385 of them; and let this suffice for our present purpose. Nonetheless, to square the aforesaid circular bases and find their dimensions, many other ways are given, which all amount to the same, which we have taken up one by one in our said work.

The reason the said 11/14ths is taken, that is, of the 14 parts of the multiplication of the diameter by itself in each circle, is because, it was found, with many approximations by Archimedes, that the circle in comparison with the square of its diameter, is as 11 to 14. That is, if the square of the diameter were 14, the circle would be 11, even though it is still not known with precision by any scientist. But it is off by little, as here, to the eye, it appears in the figure, that the circle is less than the said square by as much as are the angles [corners] of the said square [see Figure 28], which area is lost by the circle; the which corners of the entire square are the 3/14ths, that is, 3 of the 14 parts. And the 11 comes to be contained by the circular space, as it appears in the square ABCD, whose sides are equal to the diameter of the circle, that is, to the line EF, which divides it in half, passing through the point G, called the center of the said circle, as our philosopher tells us at the beginning of his Ist book. And so much for round columns.

CHAPTER LXIII

On the means of knowing how to measure all lateral columns.

Having shown the method of finding the dimension of the round columns, that of the laterals follows. For the which, let this be the general rule, with precision; that is, that you always square one of its bases, whichever one wishes, and then multiply the result by the height or length of the said column. And this last product makes exactly its volume or capacity, the vertical faces being as many as you might wish, and it never fails. As, for example, let the lateral tetragon column [see Plates XLV, XLVI] be AB, which is 10 units high, and each one of its bases be 6 each way [see Figure 29]. I say that first one of the said bases is squared, that, it being equilateral, one of its sides is multiplied by itself, that is, 6 by 6 makes 36, and this makes exactly the area of the base. Now I say that this is multiplied by the height or length of the entire said column, that is, by 10. This will make 360. And the column will be exactly so many [cubic] **bracia** or **piedi**, as in the way said above of the round columns. And likewise if its bases were inequilateral or otherwise irregular, still according to the rules given by us in the said work, it is always squared, and this product is multiplied by its height, and this will solve the problem infallibly in each case. And to do all the others, this same rule must be applied, whether they be trigons, pentagons, hexagons or heptagons, and so on; that is, that the areas of its bases must first be measured, according to the requirements of measuring those bases: if they are triangles, by the rule of triangles, and if pentagons, by the rule of pentagons and similarly if hexagons. Of the which forms and figures, the rules are given throughout our said work, which is so easily available, because of the many copies printed, universally by now distributed, that I do not have to repeat it here. And thus we will be able to finish the discussion of the said columns, and next we will speak of their pyramids.

CHAPTER LXIV

On the pyramids and all their different types.

Next in order, excellent Duke, we should speak of the pyramids and their diversity, dealing first with those called round pyramids[71], and then one by one with all the others. And to fully identify it, we will state along with our philosopher in [the 18th definition of] his XIth, that the round pyramid be a solid figure, made by the trace of a right-angled triangle, one of its sides fixed which contain the right angle, and rotated until it returns to the point whence it began to move. And if the fixed side be equal to the rotated base, it will be a right-angled figure; and if it will be longer, it will be acute-angled; and if it will be shorter, it will be obtuse-angled. And the axis of the said figure is the fixed side, and its base will be a circle. And this is called the pyramid of the round column. [see Plate LI]

For example, to understand this better, let the triangle be ABC, of which the angle B is right, and let the side which is fixed be AB [see Figure 30]. The which side fixed, rotate the said triangle around until it returns to the place whence it began to move. That such figure, thereby corporeal, the which is described or formed by the movement of this triangle, is the said round pyramid. Of the which there are three *differentia* or species, since some are right-angled, some acute-angled, the third type obtuse-angled. And the first is formed when the side AB were equal to the side BC; and let the line BC, as the triangle turns, reach the location of line BD in such a way that point C falls upon point D and becomes one same line. And in this, it is understood

[71] *Pyramis rotunda.*

as the line then rejoins the position from which it began to move, according to its **rectitudine**.[72] And this line will be as the line CBD. And because, by the 32nd of the Ist and the 5th of the same, the angle CAB is half of a right angle, the angle CAD will be a right angle, and so this such pyramid will be called a right-angled pyramid. But if the side AB be longer than the side BC, it will be acute-angled, because, by the 32nd of the Ist and by the 19th of the same, angle CA[B][73] will be less than half of the right angle, and the entire angle CAD is less than right, and therefore acute. Whence the said pyramid is acute-angled. And if the side AB be less than side BC, angle CAB will be greater than half of a right angle, by the 32nd of the Ist and by the 19th of the same, and the entire CAD, which is double that of CAB, greater than right and obtuse. Therefore, the pyramid is fittingly called obtuse-angled, and line AB of this pyramid is called the axis, and its base, the circle described by the line BC thus led around on the center B. And this said pyramid is called the pyramid of the round column, that is, of that which would make the parallelogram that grows out of the two lines AB and BC, side AB staying fixed, as was said above of the round column. And this satisfies the intent on the round pyramid and its *differentia.* And we will now speak on the others.

CHAPTER LXV

On the lateral pyramids and their diversity.

The lateral pyramids[74], excellent Duke, are of infinite kinds, just as the variety of their columns whence they originated, as shortly we will establish. But first, we put forward the explanation of our philosopher in [the 12th Definition of] his XIth, where he states the lateral pyramid to be a corporeal figure, contained by surfaces which are raised from one surface up to a point opposite. The reason is shown by noting, that, in every lateral pyramid, all the surfaces which enclose it, except its base, are raised to one point, the which is the said cone of the pyramid; and all these such lateral surfaces are triangles, and, most of the time, the base is not triangular, as appears here: [see Plates LII, LIII] the triangular pyramid A, whose cone is B; and the quadrilateral pyramid D and its cone E; and the pentagonal pyramid F and its cone G [see Figure 31]. And likewise, in the appendix, you will find for all the pyramids, correctly drawn in their own material form, the Plate numbers LII, LIII, LIIII, LV, LVI, LVII solid and hollow, in the plane by perspective, [labeled to correspond] to the same numbers. And the derivation of these, is from the lateral columns of which we spoke above, and they are born in this

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[72] "right-angledness" so to speak i.e., the vertex angle has the quality of a right angle.

[73] CAD in the original.

[74] Pyramis laterata.

manner: that is, fixing a point actually on one of the bases of the lateral column, or imagining it, and joining that by straight lines, with each one of the rectilinear angles of the other opposite base of the said column. Thereby, the pyramid of the said column will be precisely formed, by as many triangular surfaces as there would be lines or sides contained in the base of the said column. And the column and its pyramid would be denominated by the same number; that is, if such lateral column will be trilateral or triangular, the pyramid will also be called trigonal or triangular; and if the said column be quadrilateral, its pyramid will be called quadrilateral; and if pentagonal, pentagonal, and so forth.

What is shown, as was stated earlier of the said lateral columns, is that their species could be infinitely multiplied according to the diversity and variation of their rectilinear bases; this is what we say must happen to their lateral pyramids, since to each column or cylinder, there corresponds a pyramid, be it round or lateral. And that point thus fixed on its base, does not have to be situated exactly in the middle of the said base — it matters not even if it were outside of it — since, with the said protracted lines, the pyramid is also generated; it happens that the one drawn exactly to the mid-point, would be called a right angled pyramid to the plane, and the others would be called sloping or inclined [see Plates LVIII, LVIIII]. There are some others called short or truncated, and they are those which do not arrive at a point to a cone, but lack the peak, and are called truncated or cut. And there are as many of these as there are their respective whole ones, and likewise in name, whether round or lateral, as appears here in outline [see Figure 32], the round truncated A, the short triangular B, the cut quadrangular C. And this seems to me to be sufficient for their knowledge. And next at hand, we will speak of their elegant measurement.

CHAPTER LXVI

On the manner and way of knowing how to measure each pyramid.

The just and precise volume and measure, excellent Duke, of each one of the whole pyramids, whether round or lateral, will be obtained from the volume of their columns in this way. First we will find the area or space of the base of the pyramid, which we intend to measure, by means of the rules given above for finding the volume of all columns, round and lateral; and, that found, we will multiply it by the axis, that is, the height of the said pyramid: and that which it makes, will be the capacity of its entire column. And of this last multiplication, we will always take 1/3, that is, its third part, and that much exactly, is the volume of the said pyramid, and it never fails.

For example, let ABC be a round pyramid, of the which the base is the circle BC — whose diameter is 7 — and its axis AD, which is 10 in height [see Figure 33]. I say that, first, the base is squared as was done above with the round column, since, as was said of the columns and the pyramids, they would have the same bases and the same heights. [Thus] we will have 38 ½ for the surface of the base, which area multiplied by the axis AD, that is, by 10, will make 385 for the capacity of its entire column. Now, of this, I say that 1/3 is taken. Of that comes 128 1/3, and this is the volume of the said pyramid. The reason for the adduced precision[75] is to note, that the value of round columns must correspond in number to the proportion found before between the diameter and the circumference, and by that proportion stated above, between 11 and 14. Which, as we said above, was not found precisely, but approximately by Archimedes. But, it is no less true what we have said, that the round pyramid in quantity will be exactly 1/3 of its round column, even though we do not yet know, because of our ignorance, which number might exactly express the quadrature of the circle. But 1/3 of this, it is. And the said column is its exact triple, that is, three times as great as its pyramid, as is proven by the 9th [10th] of the XIIth. But, the others, all laterals, can be figured out exactly in value by their rectilinear bases. And so, as was done with the round, similarly the same must be observed of the laterals, since, likewise concerning these, in the 8th [9th] of the XIIth, it is proven that they are triple, that is, three times their pyramids. And let this suffice on their dimensions.

CHAPTER LXVII

How it is clearly shown that each lateral pyramid is subtriple [one-third] of its column.

In the 6th [7th] of the XIIth, excellent Duke, our philosopher determines the **seratile** body, or triangular prism -- the which is the first of the species of polygonal columns, as was said above -- to be divisible into 3 equal pyramids, of the which each one of the bases is a triangle, and, consequently, the said solid will be triple to each one of those. And with this evidence each pyramid is shown to be one third of its cylinder or column. And from here emerges the rule given above, that 1/3 of the volume of the entire column is taken, the which appears clearly in the rectilinear columns, since all of those are resolvable into as many triangular solids, or prisms, as triangles can be distinguished in their bases, and they are always said to be composed of as many prisms as there are triangles, as is proven in the 8th [9th] of the XIIth. Whence the quadrilateral column, whose base, by being quadrilateral, resolves itself into two triangles if you draw a diagonal line from one angle to the opposite one [see Figure 34]; and above these such triangles one imagines, or actually makes, two prisms. And because each one is triple to its pyramid, it follows for both of these to be triple to their two pyramids. But, both of the prisms make the entire quadrilateral column: therefore, the two pyramids of the two prisms make 1/3 of the entire said quadrilateral column, and these two pyramids total exactly one [pyramid]

[75] I.e., paradoxically, the volume of the round pyramid is precisely 1/3 that of the corresponding column, even though the value is transcendental due to the value of π.

relative to the entire quadrilateral column, just as those two triangular prisms make the entire quadrilateral column, these two parts being equal and integral of the said column. It is the case that the rule given cannot fail, for all of the reasons adduced. And similarly, the same effect is manifested in each one of the other lateral columns, as also of the third species of lateral columns called pentagon, of the which the base is resolvable into 3 triangles [see Figure 35]. And, for that reason it is stated, the entire column is divided into 3 triangular prisms, of the which each one is triple to its pyramid, and, for this reason, all 3 are triple to all 3 of their pyramids, and these together represent one entire column, just as their three triangular prisms make up the entire pentagonal column. And thus the discussion is the same on all the others.

And the said resolution of bases into triangles, is demonstrated in [Proclus' corollary to] the 32nd of the Ist, where each polygonal figure is demonstrated, that is, figures of more angles and sides, to be always resolvable into as many triangles as there are angles or sides, minus two. For example, the quadrilateral has 4 angles, and, consequently, 4 sides: that is resolvable into two triangles at the least, that is, the smallest resolution which appears if in that quadrilateral, one draws a straight line from one of its opposite angles to the other, as is seen here in the figure of the tetragon ABCD, the which is divided into the two triangles ABD and BCD by the line BD, the which in art is called the diagonal line and also diameter. And likewise, the pentagon resolves itself into at the least 3 triangles, that is, by the general rule, into two triangles less than there are of its angles or sides. Which will be the result, if you draw two straight lines from any vertex to the two others, opposite, as the pentagon described here in the figure ABCDE is made. In which, two lines drawn from angle A to the opposite angles C and D, resolve the pentagon into the 3 triangles ABC, ACD and ADE. And each one of the said lines, in the art is called the chord of the pentagonic angle. And likewise the hexagons resolve themselves into 4 triangles, and so forth. Thus we are much indebted, excellent Duke, to the ancients, who, with their constant vigils have enlightened our minds, especially our Euclid of Megara, who brought together in an ordered way the knowledge of the past, and to them added his own, in these very excellent disciplines and mathematical sciences, with many of his careful demonstrations, as appears throughout his sublime work, demonstrating a genius, not human, but divine, especially in his Xth book, in which, truly, he showed as much of the divine as is permitted to man. And I do not know how he might have more nobly said anything about those most abstract, irrational lines, whose science is profound, above any other, in the judgment of those who know most about it. And on the different aspects of the integral pyramids, let our purpose be completed here.

CHAPTER LXVIII

How the truncated pyramids are to be measured.

For the short or truncated pyramids, their measure is found by means of their integrals from which they are derived, as the imperfect from the perfect, in the following manner. First, we will convert the said short pyramid to the whole, up to its cone, with the method given in our published work, and we will measure that whole, by the methods stated above. And its entire volume will be clear to us, which we will note down. From there, using the same method, we will measure the volume of that small, pyramid which was added to the truncated in order to make it whole, and subtract that volume from the volume of the whole which we had noted. The portion remaining must necessarily be the exact volume of the said truncated pyramid. And of all the methods of determining the volume, this one is the shortest and most sure; and the same method is applied, whether they be round or lateral.

CHAPTER LXIX

On the measure of all the other regular and dependent solids.

Next, we should say something of the dimensions of the regular solids, and their dependents; but as for the said regulars, I do not need to further expatiate on them here, because I have already composed a specific treatise on them, dedicated to your Excellency's illustrious relative, Guido Ubaldo, Duke of Urbino. And the reader has easy access to that book, since it has come into common use, as was said above, and many copies can be found in this, your illustrious city. Their measure is more speculative, just as they themselves are more excellent and perfect than the other solids, certainly material for a good mind; not for a fool. All this was treated adequately there, in my prior work. But the method for getting the dimensions of the dependent solids, those that are derived from the regular solids, is similar to that method given for the truncated pyramids; that is, it is necessary first to convert them to their perfect wholes, and then to diligently measure and note down those wholes, by the rules given in our cited work. And that quantity set aside, next measure separately, according to the rules for measuring pyramids, one of the parts that was added to convert it to a whole, regular solid, and all of these must be subtracted from the entire volume of that regular solid. What remains will be exactly the volume of the said dependent.

When the said dependent is one of the truncated solids, such as the truncated tetrahedron, which is lacking the vertices relative to its original, those vertices become little pyramids, all equal and uniform, and therefore we can measure one of them quickly, and by that one, all the others will be known, according to the number of sides or faces or other, that we are dealing with -- which we must always allow to guide us in practice -- and having them, you can subtract them, as above, from the whole. But if the said dependent were one of the elevated solids, then, to get its volume, you must add to the volume of the regular solid, the volume of all of its little pyramids, whose number by necessity will turn out to be the same as the number of faces of its original. To summarize briefly, when dealing with these derived solids, we must use the regulars as the guiding light, adding to them or subtracting from them, as necessity demands. To depend on any other method, would bring about inextricable chaos. And hence with regard to these other solids, this book is the appropriate document; for I have no doubts about the rare talents and speculative intellects gathered here, who are ready for this and any other faculty, which sagacity we have always assumed to be addressing in our discussion, and especially in excellence and antonomasia supreme among all others, your Excellency, the Duke, whom we have addressed in our discourse in no way as a novice, on this or other subjects. Your Excellency being gifted and cultivated in all things alike, so that if I wished to extend myself on the subject, not the paper, but my entire life would not suffice. "*Sed quod patet expresse non est probare necesse*."[76] When, with a mere look, your Excellency[77] makes healthy and happy each troubled countenance[78]; this is truly that sun which warms and illuminates both poles. And what more can one say today about God's reign among mortals through your Excellency? If not that it is peaceful and refreshing, not only for Italy, but for all of Christendom.

[The Virgin Mary][79] shows herself splendid, generous, magnificent and magnanimous to everyone; in her is mercy, in her is pity, in her, magnificence, in her is gathered together whatever is good in every creature[80]. Demosthenes yields with Cicero and Quintilian to gracious

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[76] "On the contrary, what is so clearly evident, it is not necessary to prove."

[77] Because nouns have gender in Italian, Pacioli can do here what is impossible in English, i.e., he begins here the use of the feminine pronoun "*quella*," to a point that one can no longer pin it down to the Duke's excellence (feminine), the Virgin Mary, or God's grace (feminine) which hover over this whole section of the text. The result is that by antonomasia, "your excellence, or his excellency" has become more than just the Duke, the physical person, much as in English today, "his honor" transcends the judge – a standard of excellence up to which every mortal who clothes that office must live.

[78] Dante's Inferno, XI, 91.

> *O sol che sani ogni vista turbata,*
> *tu mi contenti sì quando tu solvi ...*

Here Dante compares Virgil his guide, to the sun.

[79] Pacioli says "quella" which literally would translate as "she", or (f) "that one" — which here refers to the Virgin Mary. For clarity's sake, rather than using "she", we have identified the noun to which the pronoun refers. [RSS]

[80] This echoes word for word, the prayer of St. Bernard to the Virgin Mary which opens the last canto of Dante's Paradiso, except that Bernard's "thou," is "she" here.

> *In te miscericordia, in te pietate,*
> *in te magnificenza, in te s'aduna*
> *quantunque in creatura è di bontate.*

Compare to Pacioli's:

> *in quella è miscericordia, in quella è pietade,*
> *in quella magnificentia, in quella s'aduna*
> *quantunche in creature de bontade.*

discourse, a fountain from which flows such a vast river of speech[81], nectar to those who are good, and a sharp knife to the wicked. Your Excellency, mindful of every religious order, and of their temples not only restorer, but assiduous builder; devoted to all the daily and nightly observance of the divine offices, not with less respect than do the faithful along with the holy prelates, which is manifest by your Excellency's most worthy devotional chapel[82] dedicated to divine worship, ornate with excellent choirs, and other particular works of devotion. Your benignity lends its merciful ear without delay to every supplicant, especially to the pious, and not only helps those who ask, but responds many times before the asking[83]. For the which reasons, not without merit, He who has seen nothing new[84], has allowed your Excellency, among all others in our times, to participate in the entire universe of His graces. Therefore it is fitting that, just as Octavian made his temple of universal peace in Rome, your Excellency has constructed in his glorious city of Milan, the most sacred Temple of Graces dedicated to the memory of so many graces received, and each day your Excellency continues to adorn it in all ways, acting as benefactor whenever needed. And I pray that the reader not attribute my brief discourse here to flattery, which is foreign to both my nature and my profession; for if I were to do other than to tell the simple truth, you would, yourself, be more guilty of envy and rancor towards his Excellency, than I flattery, by not realizing and admiring his many excellent and heavenly gifts.

"*Sed quod oculis vidimus testamur*[85]". And your Excellence is not witnessed by myself alone, but with my entire most sacred seraphic religious Order, whose superior and singular head and shepherd, is our most reverend father Maestro Francesco Sansone da Brescia, the most worthy General, at our General Conference of this current year, conducted in your illustrious city of

---

[81] Dante's Inferno, I, 79-80. Here Pacioli again compares the Duke of Milan to Virgil, the fountain whom Dante calls his "*maestro*".

[82] Santa Maria delle Grazie in Milan, built in part by Bramante. It is in the refectory of this church that Leonardo painted the Last Supper, commissioned by the Duke. Thanks to massive sandbagging, this wall was miraculously the only wall left standing after the WW II Allied bombing of Milan.

[83] See again the opening prayer of Canto XXXIII, 16-18 of Paradiso.

In praising the Virgin Mary, Bernard says:

> *La tua benignità non pur soccorre*
> *a chi domanda, ma molte fiate*
> *liberamente al dimandar precorre.*

[84] Dante's Purgatorio, X, 94-96:

> *Colui che mai non vide cosa nova*
> *produsse esto visibile parlare,*
> *novello a noi perchè qui non si trova*

[85] But we witness what we have seen with our eyes.

Milan: at which assembly there gathered a great number of very famous and celebrated men, doctors and bachelors of sacred theology and other sciences, from throughout the world, and from every nation which is under heaven, during which assembly, every day assiduously, there would be lectures and public debates, always attended by a large number of people, devoted to those who serve your Excellency, the Duke, together with the most reverend Lord Monsignor, your cousin, Hippolyto Cardinal Estense, the worthy Archbishop of Milan; and many others who are part of your very splendid magistracy. I will not speak here of the affluence and rich abundance in everything, received from the hands of your Excellency, to provide generously for the needs of such a multitude, and not only for those present at that time, but also several months worth of provisions for those not yet arrived. For whose health and happy state, the whole troupe of the minor orders, with folded hands, spread their praise towards the most High, and I in particular, unworthy and miserable sinner, who commend myself devoutly and continuously to your Excellency.

CHAPTER LXX

The order in which the said solids are to be found, as they are located in this work, made in perspective, and also the physical models on public display with their corresponding labelled plates.

Where there is no order, there will always be confusion, so, to make our compendium more fully intelligible, and to be able to find all the appropriate figures, viewed in perspective, in the following appendix, and also the physical models by their displayed plate, your Excellence will observe this method: that is, when you read above in a chapter about a particular solid and its creation and formation, you will see there, in the margin of the book, a number assigned by the old abacus that is, commencing from the beginning at the 48th chapter, stating I, II, III, IIII, V and so on to their end. And you will come to find that same exact number in this appendix, where the said solids are all portrayed in order; the which number will likewise be placed in the margin, referring I to I, II to II, III to III, and thus with all of them. And that such figure will be of the said solid, depicted in the plane, with all the perfection of perspective, by our skilled Leonardo da Vinci. And these same numbers you will also look for, among the hanging physical models made of the said solids, with their names in Greek and Latin, placed on a label affixed above each one on its cord between two pieces of black amber, also referring each one, as was stated, to the number placed in the margin, where that particular solid is treated. And your Excellence will have them at his disposal, in both one way and the other; the which would merit being constructed not of base material -- as I was obliged to do for reasons of poverty -- but of precious metal and adorned with fine gems. But your Excellence will take into consideration the affection and the spirit of your perpetual servant.

CHAPTER LXXI

On how one should understand the vocabulary used by mathematicians; that is, hypothesis, hypotenuse, corausto, pyramidal cone, pentagonic cord, perpendicular, catheto, diameter, parallelogram, diagonal, center, saetta.

These are some terms, excellent Duke, employed by scholars of the mathematical disciplines, so that they can understand each other without equivocation; which terms would annoy those who are not expert in them; and we often employed those terms above in this our compendium, as you have found in reading it; we observed them in order not to deviate from the ancients. It seems to me not without utility then, to give to the reader a brief definition of them. And first on hypothesis.

On hypothesis

By hypothesis one must understand the presupposition, admitted and conceded among the parties, author and adversary, by means of which one intends to arrive at a conclusion, or, [if] negated, a conclusion does not follow. And that is why it [the hypothesis] is not usually admitted, if it is not possible.

What is the hypotenuse in geometry

By the hypotenuse, especially in all rectilinear figures, one understands the line which is opposite to the largest angle. But, more precisely, it is customary to understand the side opposite to the right angle in right or orthogonal triangles, as they are called in art, which of necessity, are always the half of the square figure or of the long tetragon, that is, the rectangular figure of 4 sides, longer than wide.

What is the corausto [elevated horizontal] between straight lines.

The corausto is to be understood as a straight line, which joins the end points of two [vertical lines] elevated in height. And there can be more or fewer of them, depending upon the number of elevated lines.

On the cone or pyramidal vertex.

The cone of the pyramid means the highest point of the summit, where the lines, which go out from its base, come together.

On the pentagonic chord.

By pentagonic or pentagonal chord, or shall we say the entire pentagonic angle, one understands a straight line drawn within the figure of the pentagon, from any of its vertices, to any other opposite to that, as was done several times.

Perpendicular.

The perpendicular means a straight line elevated or situated, squarely above another, that is, it makes one or more right angles around itself, and again when it is situated in the said way, above a plane surface. And people usually look for it in triangles in order to measure them, as we said in our cited work in the relevant place.

Catheto

Catheto means the same as the perpendicular, and in dealing with triangles, in the vernacular it is commonly called the **saetta** [ray] of a triangle. This word comes from the Greek.

On the diameter.

Properly, the diameter is understood as a straight line in the circle, which passes through its center, and, with its end points, touches the circumference by each end, and divides the circle into two equal parts. But it is customary also in squares to say the diameter, and so, in order not to equivocate, one says diameter of circle and diameter of square, to differentiate between one and the other.

On the parallelogram.

Parallelogram is understood as a surface of equidistant sides. The which, properly, are quadrilateral, that is, those 4-species which you had above in the 59th chapter, called square and long tetragon, rhombus and rhomboid, and, by other name, *elmuaym* [diamond shaped] and similar to the *elmuaym*. And even though every regular figure with an even number of sides has opposite equidistant sides, such as the hexagon, octagon, decagon, dodecagon and others similar, nonetheless, when speaking of parallelograms, it is those with 4 sides which are meant.

On the diagonal line.

A diagonal is principally understood as a straight line drawn from one angle to the other opposite in the long tetragon, which divides it into two equal parts, to differentiate it from the square. And it is also called such when used thus in the rhombus and the rhomboid.

On the center of the circle.

The center is properly called, in the circle, that mid-point at which, resting the foot of the compass immovably, [and] turning the other foot, the circle is described by the line called circumference or periphery. And all the lines emanating from that point to the said circumference, are equal to each other. But the word center is also used when describing other, rectilinear figures — to mean the mid-point of their surface: as in triangles, squares, pentagons, hexagons and other equilateral and equiangular figures, because the straight lines drawn from that point to each of their vertices, will likewise all be equal to one other.

Saetta

A **saetta** is that straight line which, from the mid-point of the arc of each portion of the circle, moves and falls at a right angle in the middle of its chord. And it is called the **saetta** with respect to the part of the circumference which is called arc, similar to the physical arch which also uses the said three names, that is, chord, arc and **saetta**.

On many other terms.

And inasmuch as there be many other terms used, which we have amply treated in our *magnum opus*, I will not take the trouble to cite them here; but only these did I deem necessary for the intelligibility of the present compendium. The which, if it be not complete in spite of its great number of pages, nonetheless, lacks not in substance and lofty speculations. And truly, excellent Duke, not misleading your Excellence, I say that the speculation of the mathematicians can extend virtually no higher, whether the length of a work happens to be sometimes more and sometimes less. And in these, our philosopher from Megara came to a conclusion, and ended his whole work on Arithmetic, Geometry, Proportions and Proportionality, in XV parcelled books, subdivided so as to make it clear to the intellect. And hence not a little grace and dignity will accrue to your beautiful and worthy library, as we said earlier in our epistle to you, it being the one and only composed of such order and subject matter, known to no one until now — save your Excellency — in the entire universe. It is here in your illustrious, great city of Milan, with not ordinary efforts and long vigils, under the protective shade of your Excellency, and of your, as if son, my, unworthy that I am, personal and singular patron, his eminent lordship Galeazzo Sanseverino de Aragon, second to none in the military arts, and a great lover of our disciplines, especially in the day's work of his assiduous studies, where he tastes of the most useful and sweet fruit.

And let the conclusion of our discussion be the humble supplication and due reverence by the perpetual servant of your Excellency, to whom, infinitely and in every manner, [this work] is commended. *Quae iterum atque iterum ad vota felicissime valeat*, once again with all the best wishes.

The solids to the reader:
The philosophers were forced to come to us
To find the sweet fruit
Which feeds the mind, and delights the intellect

Disticon:

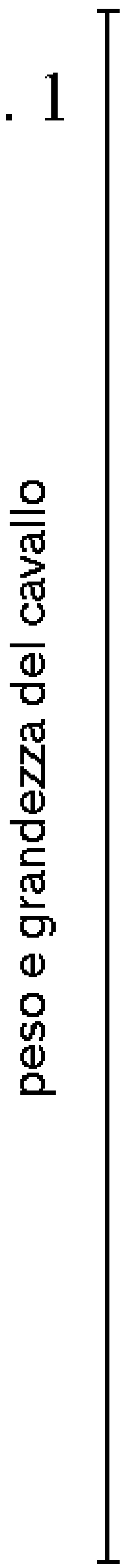
Querere de nobis fructus dulcissimus egit
Philosophos causam mens ubi laeta manet[86]

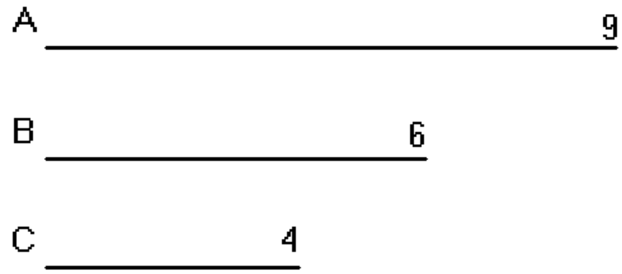
—F I N I S—

On December 14 in Milan, in our worthy monastery, the entire province being governed by the most Reverend P., its worthy minister, professor of sacred Theology, Maestro Francesco from Mozzano.
McccLxxxxviii, the Sovereign Pontiff Alexander VI seated in the 7th year of his pontificate.

[86] This Latin is a rough translation of the preceding lines.

Fig. 1





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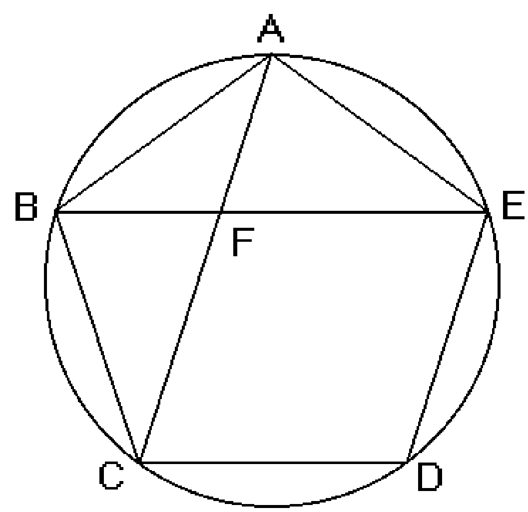

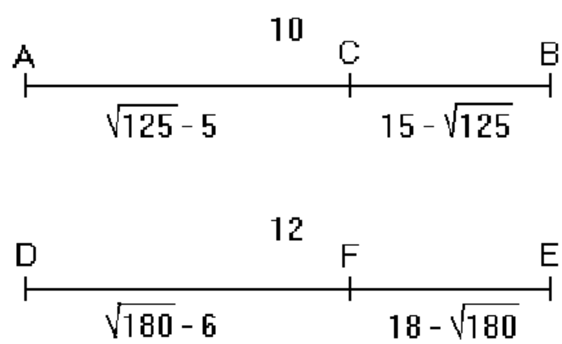

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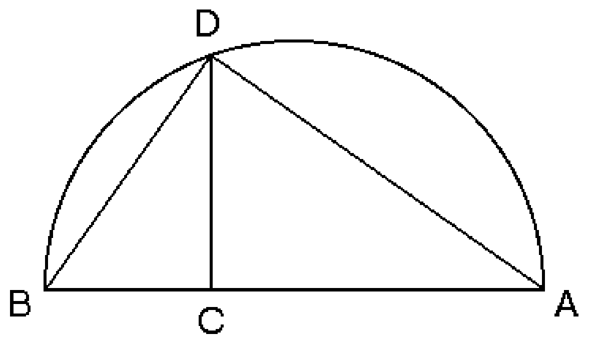

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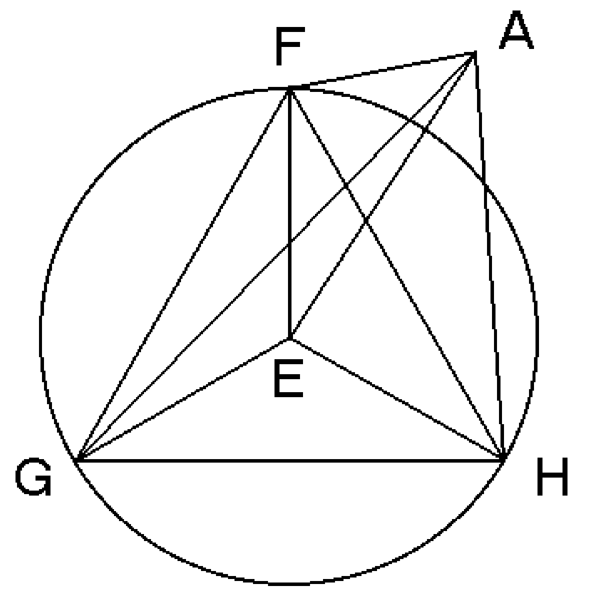

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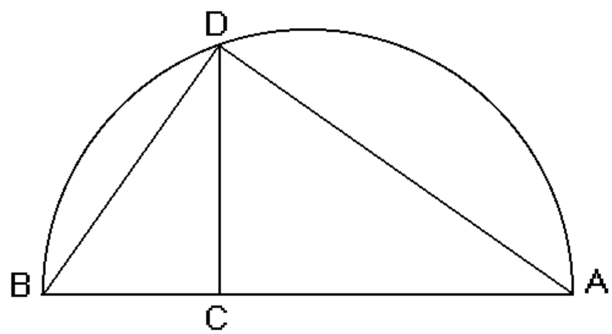

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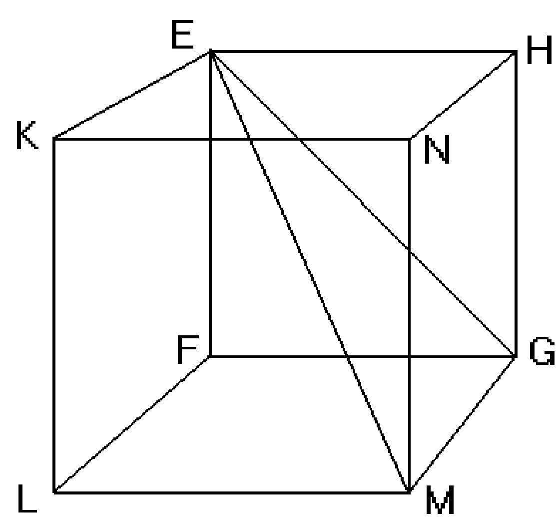

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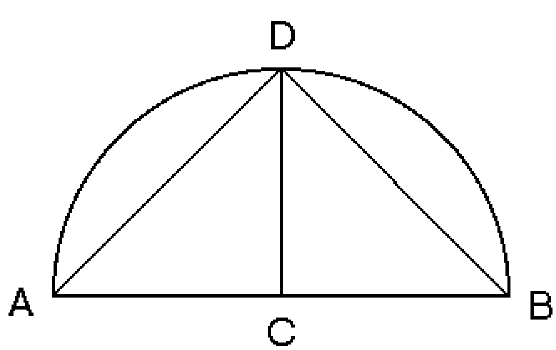

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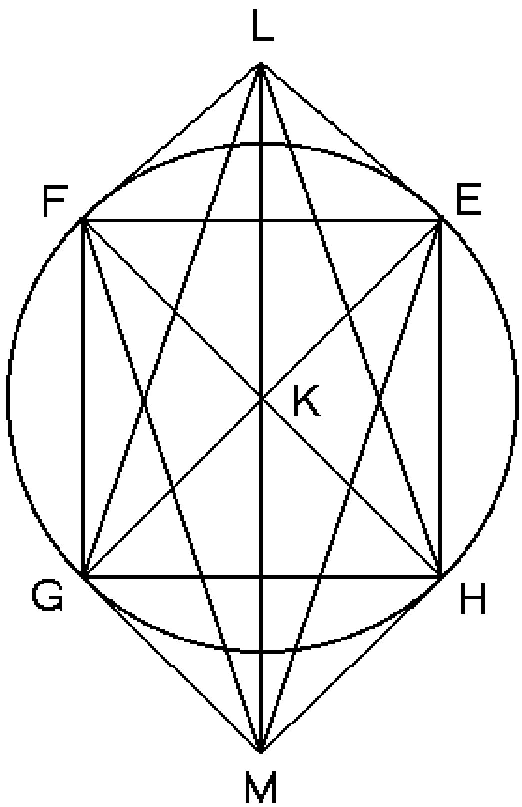

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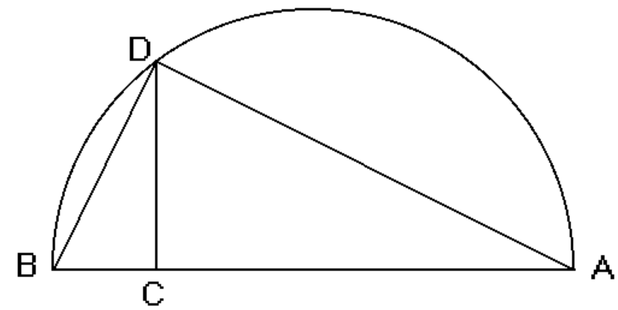

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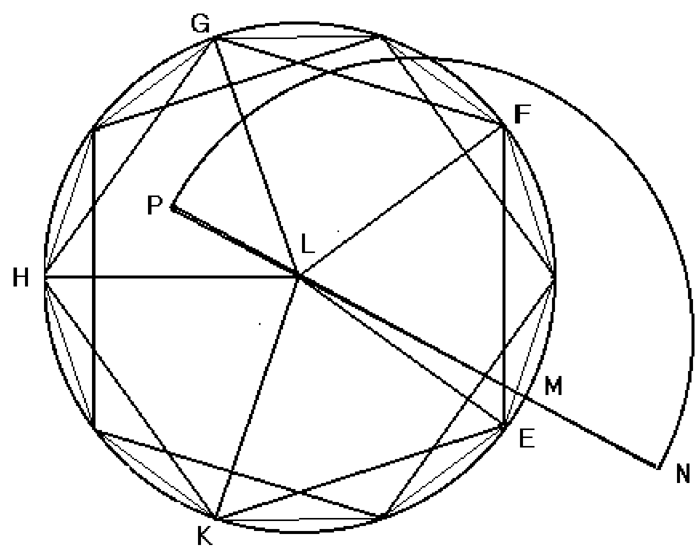

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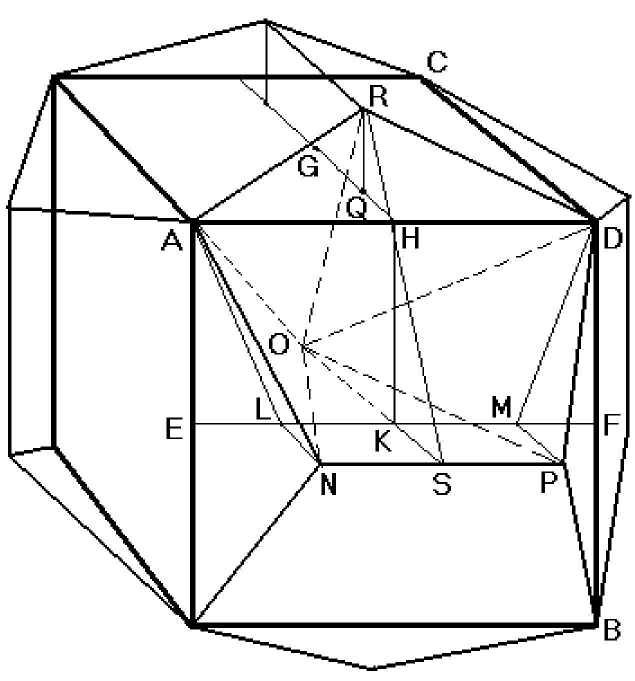

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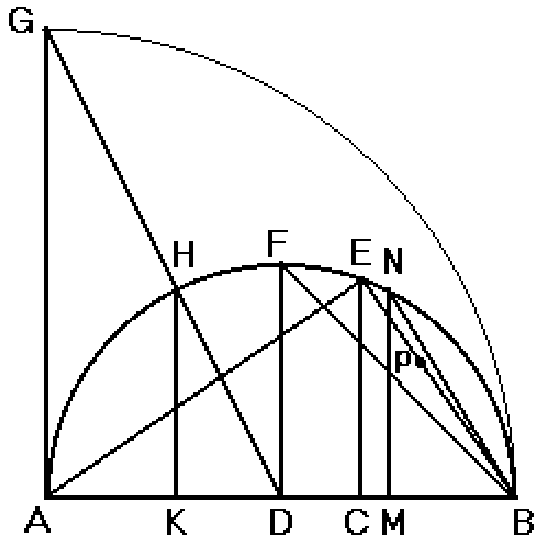

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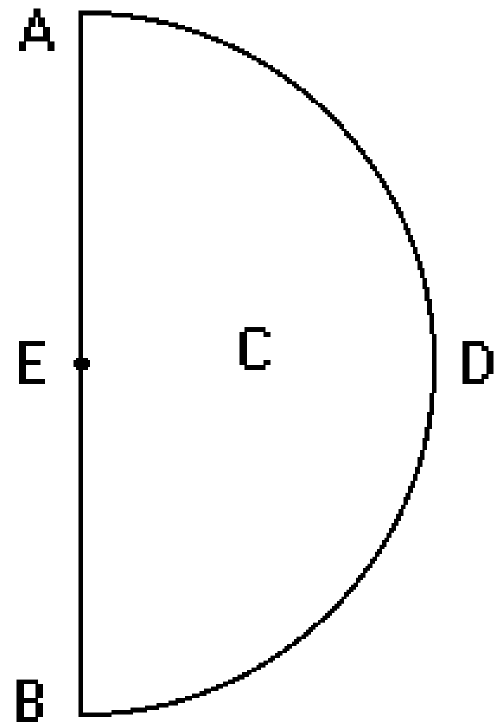

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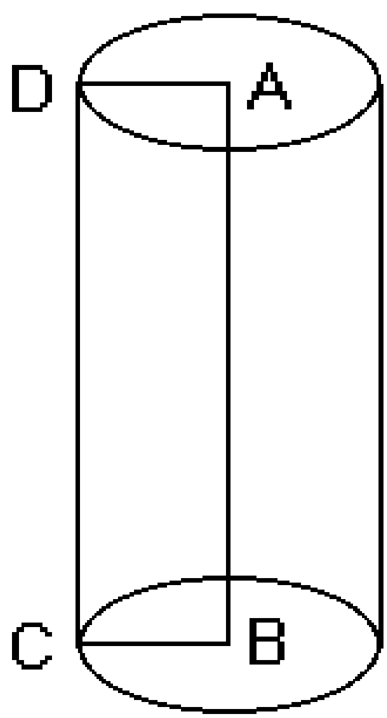

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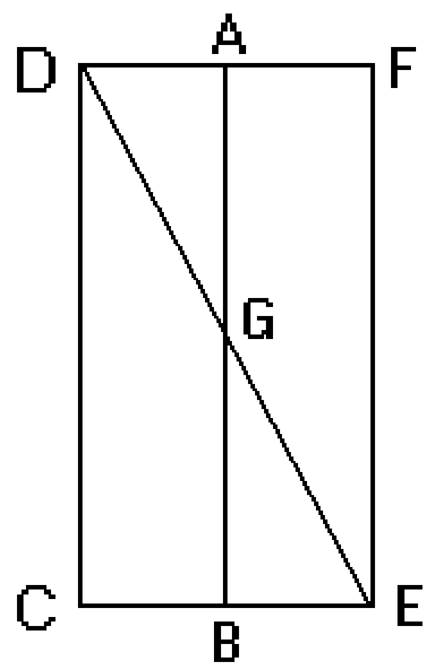

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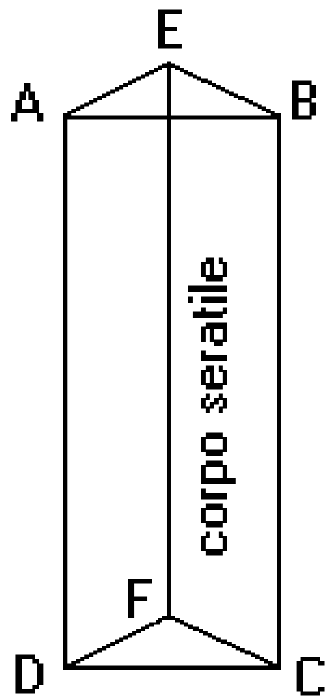

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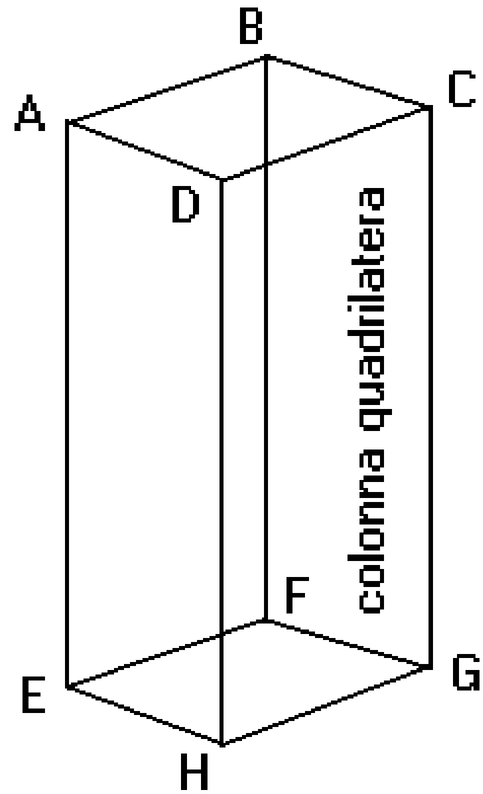

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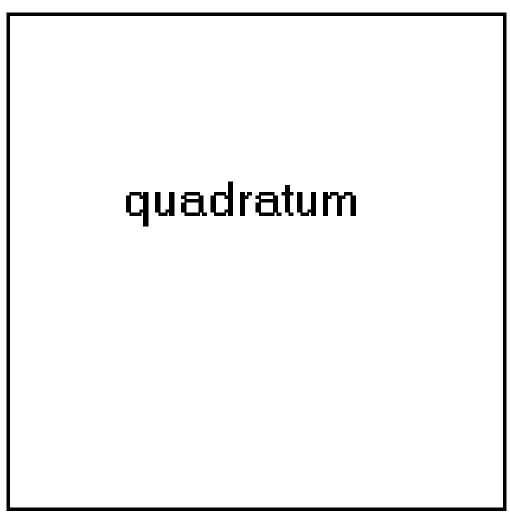

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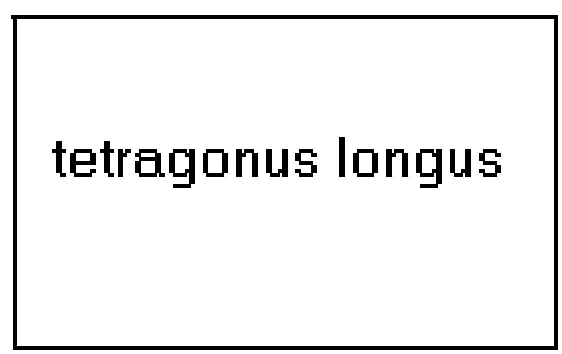

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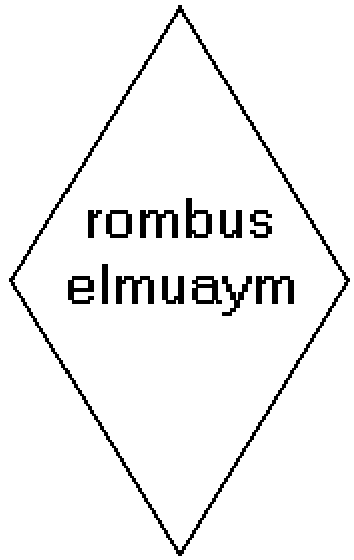

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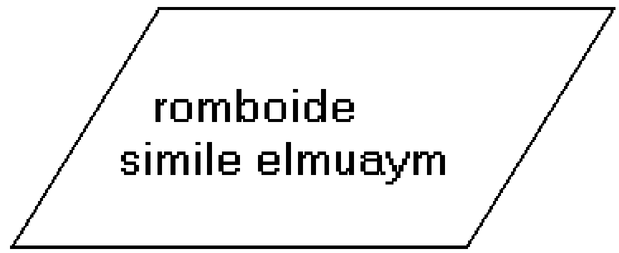

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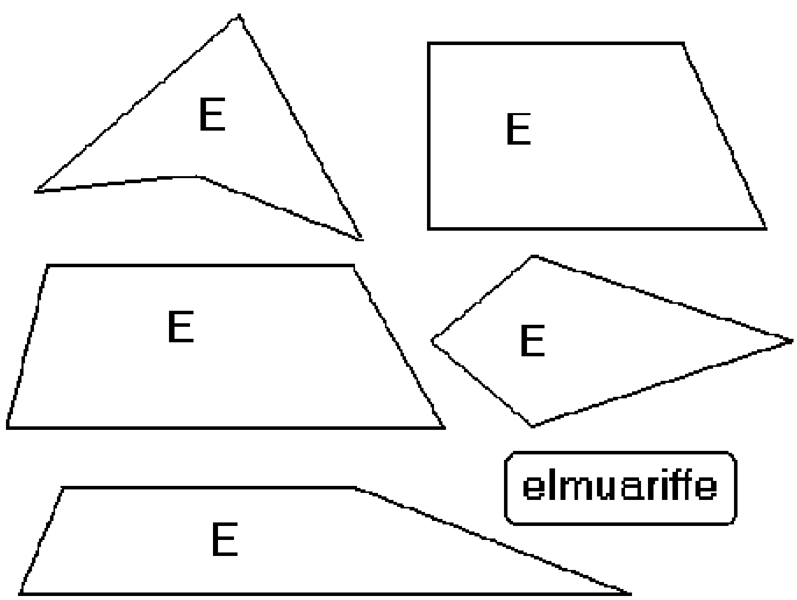

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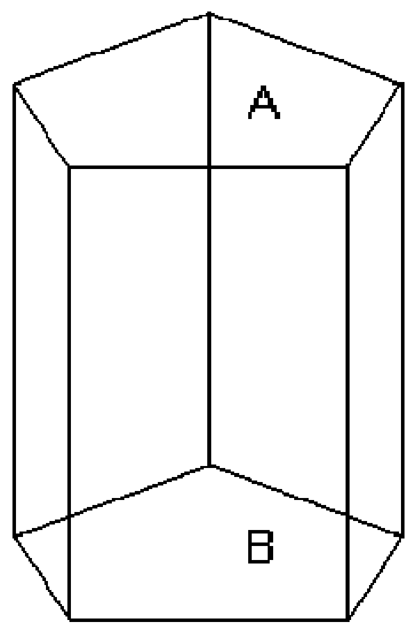

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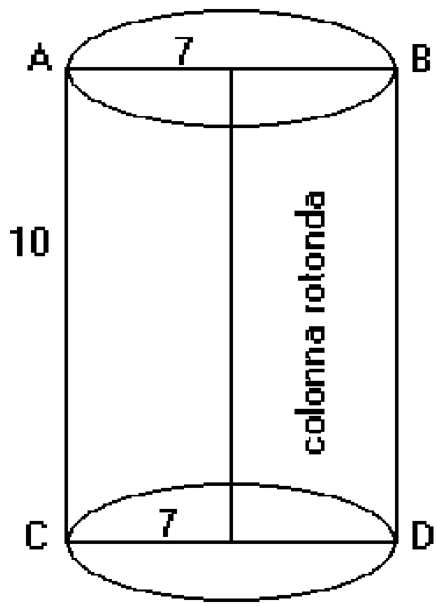

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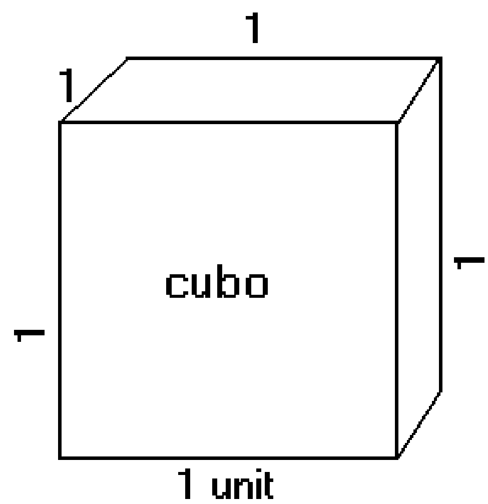

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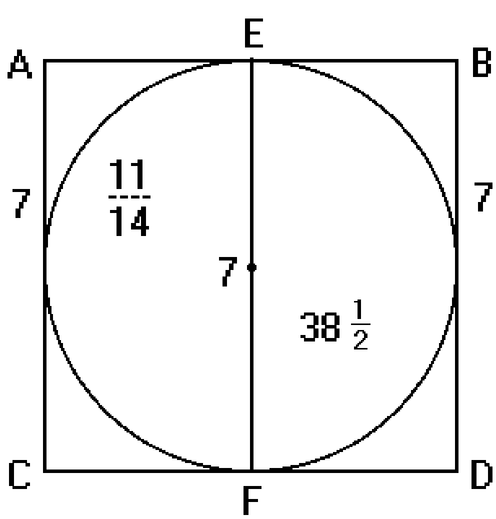

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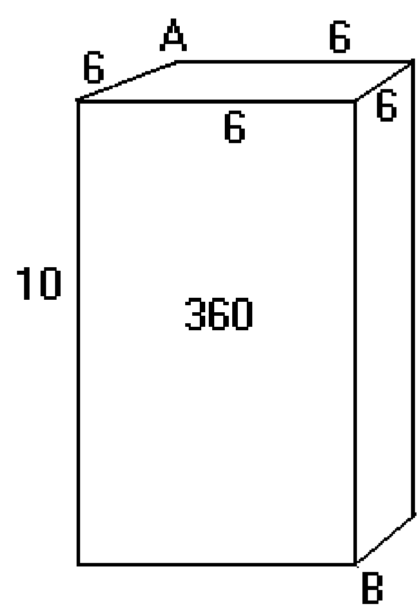

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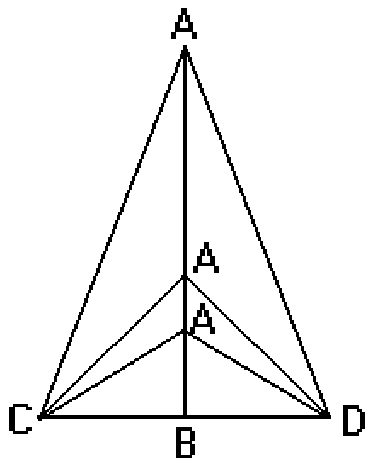

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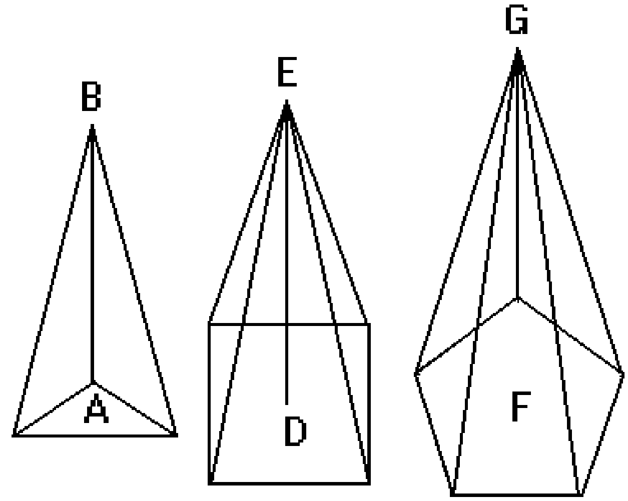

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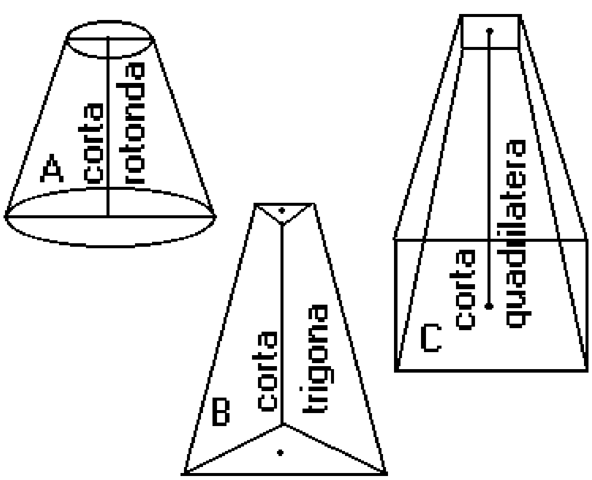
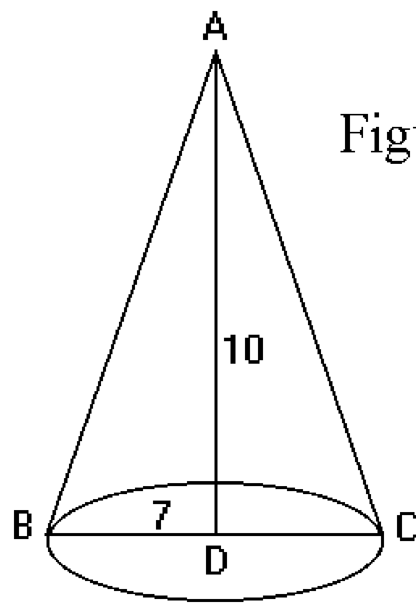
Figure 32





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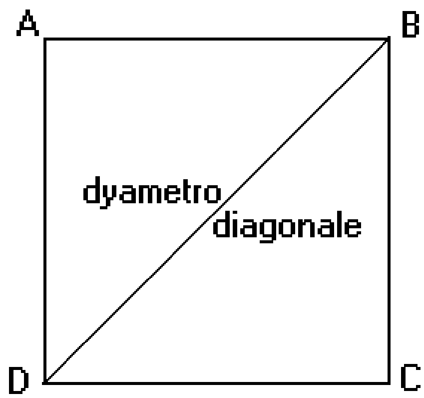

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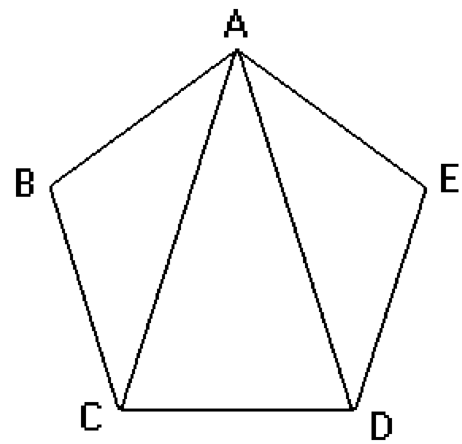

Figure 35

LIST OF PLATES
DRAWN BY LEONARDO DA VINCI
FOR THE 1498 FIRST EDITION OF *DE DIVINA PROPORTIONE*

English
Latin
Greek

Octaedron Apotetmimenon Stereon

XVIII. Hollow Truncated Octahedron

Octaedron Abscisum Vacuum

Octaedron Apotetmimenon Cenon

XIX. Solid Elevated Octahedron

Octaedron Elevatum Solidum

Octaedron Epirmenon Stereon

XX. Hollow Elevated Octahedron

Octaedron Elevatum Vacuum

Octaedron Epirmenon Cenon

XXI. Solid Plane Icosahedron

Icosaedron Planum Solidum

Icosaedron Epipedon Stereon

XXII. Hollow Plane Icosahedron

Icosaedron Planum Vacuum

Icosaedron Epipedon Cenon

XXIII. Solid Truncated Icosahedron

Icosaedron Abscisum Solidum

Icosaedron Apotetmimenon Stereon

XXIV. Hollow Truncated Icosahedron

Icosaedron Abscisum Vacuum

Icosaedron Apotetmimenon Cenon

XXV. Solid Elevated Icosahedron

Icosaedron Elevatum Solidum

Icosaedron Epirmenon Stereon

XXVI. Hollow Elevated Icosahedron

Icosaedron Elevatum Vacuum

Icosaedron Epirmenon Cenon

XXVII. Solid Plane Dodecahedron

Dodecaedron Planum Solidum

Dodecaedron Epipedon Stereon

XXVIII. Hollow Plane Dodecahedron

Dodecaedron Planum Vacuum

Dodecaedron Epipedon Cenon

XXIX. Solid Truncated Dodecahedron

Dodecaedron Abscisum Solidum

Dodecaedron Apotetmimenon Stereon

XXX. Hollow Truncated Dodecahedron

Dodecaedron Abscisum Vacuum

Dodecaedron Apotetmimenon Cenon

XXXI. Solid Elevated Dodecahedron

Dodecaedron Elevatum Solidum

Dodecaedron Epirmenon Stereon

XXXII. Hollow Elevated Dodecahedron

Dodecaedron Elevatum Vacuum

Dodecaedron Epirmenon Cenon

XXXIII. Solid Elevated Truncated Dodecahedron

Dodecaedron Abscisum Elevatum Solidum

Dodecaedron Apotetmimenon Epirmenon Stereon

XXXIV. Hollow Elevated Truncated Dodecahedron

Dodecaedron Abscisum Elevatum Vacuum

Dodecaedron Apotetmimenon Epirmenon Cenon

XXXV. Solid Plane Icosahexahedron (or Small Rhombic Cuboctahedron)

Vigintisex Basium Planum Solidum

Icosiexaedron Epipedron Stereon

XXXVI. Hollow Plane Icosahexahedron

Vigintisex Basium Planum Vacuum

Icosiexaedron Epipedon Cenon

XXXVII. Solid Elevated Icosahexahedron

Vigintisex Basium Elevatum Solidum

Icosiexaedron Epirmenon Stereon

XXXVIII. Solid Elevated Hollow Icosahexahedron

Vigintisex Basium Elevatum Vacuum

Icohexahedron Epirmenon Cenon

XXXIX. Solid Seventy Two Sided

Septuagintaduarium Basium Solidum

Hebdomecontadissaedron Stereon

XL. Hollow Seventy Two Sided

Septuagintaduarum Basium Vacuum

Hebdomecontadiffaedron Cenon

XLI. Solid Sphere

Sphera Solida

Sphaera Sterea

XLII. Solid Round Column

Columna Rotunda Solida

Cion Strongylos Stereos

XLIII. Solid Triangular Lateral Column

Columna Laterata Triangula Solida

Cion Pleurodis Trigonos Stereon

XLIV. Hollow Triangular Lateral Column

Columna Laterata Triangula Vacua

Cion Pleurodis Trigonos Cenis

XLV. Solid Quadrangular Lateral Column

Columna Laterata Quadrangula Solida

Cion Pleurodis Tetragonos Stereos

XLVI. Hollow Quadrangular Lateral Column

Columna Laterata Quadrangula Vacua

Cion Pleurodis Tetragonos Cenos

XLVII. Solid Pentagonal Lateral Column

Columna Laterata Pentagona Solida

Cion Pleurodis Pentagonos Stereos

XLVIII. Hollow Pentagonal Lateral Column

Columna Laterata Pentagona Vacua

Cion Pleurodis Pentagonos Cenos

XLIX. Solid Hexagonal Lateral Column

Columna Laterata Exagona Solida

Cion Pleurodis Hexagonos Stereos

L. Hollow Hexagonal Lateral Column

Columna Laterata Exagona Vacua

Cion Pleurodis Hexagonos Cenos

LI. Solid Round Pyramid

Pyramis Rotunda Solida

Pyramis Strongy Sterea

LII. Solid Triangular Lateral Pyramid

Pyramis Laterata Triangula Solida

Pyramis Pleurodis Trigonos Sterea

LIII. Hollow Triangular Lateral Pyramid

Pyramis Laterata Triangula Vacua

Pyramis Pleurodis Trigonos Ceni

LIV. Solid Quadrangular Lateral Pyramid

Pyramis Laterata Quadrangula Solida

Pyramis Pleurodis Tetragonos Sterea

LV. Hollow Quadrangular Lateral Pyramid

Pyramis Laterata Quadrangula Vacua

Pyramis Pleurodis Tetragonos Ceni

LVI. Solid Pentagonal Lateral Pyramid

Pyramis Laterata Pentagona Solida

Pyramis Pleurodis Pentagonos Sterea

LVII. Hollow Pentagonal Lateral Pyramid

Pyramis Laterata Pentagona Vacua

Pyramis Pleurodis Pentagonos Ceni

LVIII. Solid Inequilateral Triangular Lateral Pyramid

Pyramis Laterata Triangula Inequilatera Solida

Pyramis Pleurodis Trigonos Auisopleuros Sterea

LIX. Hollow Inequilateral Triangular Lateral Pyramid

Pyramis Laterata Triangula Inequilatera Vacua

Pyramis Pleurodis Trigonos Auisopleuros Ceni