Linear Programming: Beyond 4.2 (The Simplex Method)

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It is generally known that Chapter 4 of the MAT 119 textbook [10][1] is the shakiest of all chapters, especially sections 4.3 and 4.4, and leaves a lot to be desired when teaching MAT 119. This talk will put topics like LP Duality and solving Linear Programs not in standard form on firmer ground. No knowledge other than setting up Linear Programs and using the Simplex Method will be necessary.[2]

Introduction

When I was putting together my talk (titled "Linear Programming: Beyond The Simplex Method"), I realized that there were a lot of things I could talk about. However, I was only given 50 minutes to talk, and I wouldn't be able to provide all the details, or want to, for that matter. So I decided to write up a companion paper to the talk, including the examples I used on my slides, and providing full details that I could only allude to. Since there were people who couldn't make it to the talk but wanted to hear what I had to say, I decided to expand the "companion" to an expository paper.

Since the author(s)[3] of the textbook have put out a new edition, and all of my "complaints" are still valid, I figured a paper would still be appropriate. I have decided to attempt to re-create the talk in full, and I came up with the document you are now reading.

Before we continue, there are several abbreviations I will use throughout the paper: LP for Linear Program(s), BV for Basic Variable(s), and RHS for Right Hand Side (usually referring to a single entry, which is in the last column of a tableau). A lot of the material came out of Chvátal's *Linear Programming* [2].

We will assume that a LP has n variables and m inequalities, except where explicit examples are given.

This paper is organized as follows: First, the Simplex Method and reasons for its steps will be explained, along with what operations are vital and which can be altered slightly (issues related to sections 4.1 and 4.2), *assuming the LP is in standard form*;[4] properties of the Simplex Method will be discussed; LP Duality will be discussed, with emphasis towards proving a solution is correct (an elaboration on the content of 4.3); proving certain LP's are unbounded will be mentioned; and finally, solving LP's not in standard form will be mentioned (which will clarify some issues brought up in section 4.4).

Geometric Interpretation of the Simplex Method

We start with an explanation of the Simplex Method, look at its steps in detail, and then give some properties.

We define the *feasible region*, even if there are more than two variables, to be the set of all points which satisfy all the inequalities in the LP, and a *corner point* to be a feasible point where at least n inequalities are equal, where n is the number of variables. *Edges* are the sets of feasible points for which exactly $n-1$ of the inequalities are true; edges connect corner points to each other and corner points to "infinity."

The geometric interpretation of the Simplex Method is as follows: It starts off at a corner point, the origin, and looks at the edges incident with that point. Then it chooses one, and moves along an edge which makes the *objective value*[5] get larger (or stay the same). If the edge goes on forever, then the LP is unbounded; if the edge does not go on forever, it ends up at another corner point. Then the Simplex Method

[1] I have reviewed Chapter 4 of the Ninth Edition, and everything I have to say about the Eighth Edition is true for this new one as well. I could find nowhere where they made any changes related to content.

[2] Last updated: August 2006.

[3] One of the authors died since the eighth edition was put out.

[4] Briefly, LPs which can be put in standard form are those where the origin $(0, 0, \ldots, 0)$ is a feasible point.

[5] This is what you are trying to minimize or maximize and is called P in the text.

does the same thing over and over again, until it stops at a certain point. That point is the place where the objective value is the largest.

Normally, this would only lead you to a *local* maximum, instead of the *global* maximum, but this cannot happen when solving a LP. This is because the objective values and inequalities are linear in form.[6]

Analysis of the Simplex Method

The description of the Simplex Method is a combination of essential features and some degrees of freedom. Briefly, the Simplex Method consists of the following steps:

- *Adding slack variables to the LP.* This is done for two reasons: (1) Equalities are easier to deal with than inequalities; for instance, multiplying both sides of an inequality by a negative number reverses the direction of the inequality, whereas equalities do not have this problem; (2) All the inequalities together become the statement that "every variable is nonnegative." This makes no variable more important than another.[7]

- *Putting the equations into a tableau.* This is to make manipulation easier. Chvátal prefers using "dictionaries," which keep the equations as equations, where the BVs are written as a function of the nonbasic variables.

- *Choosing a column to put a pivot in.* It is actually all right to choose any column of a tableau which has a negative number in the bottom row. However, *cycling* (visiting a corner point more than once) is possible if the column is not chosen with care. I call a rule which decides which column to choose (among two or more possibilities) a *tie-breaker rule*.

  The tie-breaker rule that is used in [10] is to choose the column which has the most negative number in its column, and if two columns both have the most negative number, choose the one further to the left. This rule is called the *Largest Coefficient Rule*. Another tie-breaker rule, called the *Least Subscript Rule*, also known as *Bland's Rule*, chooses the column furthest to the left, regardless of how big the entries in the bottom row are. The Least Coefficient Rule can cause cycling; Bland's Rule prevents cycling [1].

- *Choosing a row to put a pivot in.* The only requirement for choosing a row is that we choose a row so that when we turn a specific entry into a pivot,[8] that all the entries in the RHS column are nonnegative, so that the point represented by the tableau is still feasible. This is also why the smallest ratio is chosen. Once again, if there are two or more rows which work, then the tiebreaking rule needs to be used to specify which one becomes the pivot row. If two rows have the same minimum ratio, the RHS column will have a zero in it after pivoting, a situation known as *degeneracy*. When degeneracy is present, it means: (1) That corner point satisfies *more* than n of the inequalities in the original LP with equality, and (2) In the next iteration, P may not go up at all.

- *The row operations used to turn an entry into a pivot.* The values of the variables can be read off quickly from a tableau if there is a pivot in every row; otherwise, a system of linear equations needs to be solved to determine the values. When turning an entry into a pivot, another pivot will be destroyed,[9] but the specific row operations mentioned will prevent any other pivots from being destroyed.

Properties of the Simplex Method

- *It is known to "cycle".* An example of this is provided below, using the Largest Coefficient Rule. After six iterations, the tableau is the same as the original one; hence the Simplex Method will never terminate for this LP.

[6] This is closely related to the fact that the feasible region is *convex*: If x and y are in the feasible region, then so is $tx + (1-t)y$, for any t between 0 and 1.

[7] Because of this reason, I would prefer calling the extra variables x_{n+1}, x_{n+2}, $\ldots$, x_{n+m} instead of s_1, s_2, $\ldots$, s_m. However, I will stick with the book's notation in this paper.

[8] We do that by dividing that row, the *pivot row*, by an appropriate constant, then adding/subtracting multiplies of the pivot to other rows to zero out the entries above and below the pivot.

[9] specifically, the one in the same row

$$\begin{aligned} &\text{maximize } 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ &\tfrac{1}{2}x_1 - \tfrac{11}{2}x_2 - \tfrac{5}{2}x_3 + 9x_4 \le 0 \\ &\tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 \le 0 \\ &x_1 \le 1 \\ &x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$$\begin{array}{c|c|ccccccc|c} \text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \text{RHS} \\ \hline s_1 & 0 & \boxed{1/2} & -11/2 & -5/2 & 9 & 1 & 0 & 0 & 0 \\ s_2 & 0 & 1/2 & -3/2 & -1/2 & 1 & 0 & 1 & 0 & 0 \\ s_3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline P & 1 & -10 & 57 & 9 & 24 & 0 & 0 & 0 & 0 \end{array}$$

$$\longrightarrow$$

$$\begin{array}{c|c|ccccccc|c} \text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \text{RHS} \\ \hline x_1 & 0 & 1 & -11 & -5 & 18 & 2 & 0 & 0 & 0 \\ s_2 & 0 & 0 & \boxed{4} & 2 & -8 & -1 & 1 & 0 & 0 \\ s_3 & 0 & 0 & 11 & 5 & -18 & -2 & 0 & 1 & 1 \\ \hline P & 1 & 0 & -53 & -41 & 204 & 20 & 0 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{c|c|ccccccc|c} \text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \text{RHS} \\ \hline x_1 & 0 & 1 & 0 & \boxed{1/2} & -4 & -3/4 & 11/4 & 0 & 0 \\ x_2 & 0 & 0 & 1 & 1/2 & -2 & -1/4 & 1/4 & 0 & 0 \\ s_3 & 0 & 0 & 0 & -1/2 & 4 & 3/4 & -11/4 & 1 & 1 \\ \hline P & 1 & 0 & 0 & -29/2 & 98 & 27/4 & 53/4 & 0 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{c|c|ccccccc|c} \text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \text{RHS} \\ \hline x_3 & 0 & 2 & 0 & 1 & -8 & -3/2 & 11/2 & 0 & 0 \\ x_2 & 0 & -1 & 1 & 0 & \boxed{2} & 1/2 & -5/2 & 0 & 0 \\ s_3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline P & 1 & 29 & 0 & 0 & -18 & -15 & 93 & 0 & 0 \end{array}$$

$$\longleftarrow$$

$$\begin{array}{c|c|ccccccc|c} \text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \text{RHS} \\ \hline x_1 & 0 & -2 & 4 & 1 & 0 & \boxed{1/2} & -9/2 & 0 & 0 \\ x_4 & 0 & -1/2 & 1/2 & 0 & 1 & 1/4 & -5/4 & 0 & 0 \\ s_3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline P & 1 & 20 & 9 & 0 & 0 & -21/2 & 141/2 & 0 & 0 \end{array}$$

$$\uparrow$$

$$\begin{array}{c|c|ccccccc|c} \text{BV} & P & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & \text{RHS} \\ \hline s_1 & 0 & -4 & 8 & 2 & 0 & 1 & -9 & 0 & 0 \\ x_4 & 0 & 1/2 & -3/2 & -1/2 & 1 & 0 & \boxed{1} & 0 & 0 \\ s_3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline P & 1 & -22 & 93 & 21 & 0 & 0 & -24 & 0 & 0 \end{array}$$

$$\uparrow$$

- *It can wander.* This means that it can visit a large number, even all, of the corner points. Klee and Minty [8] provided a family of examples of this, again with the Largest Coefficient Rule:

$$\begin{gathered} \text{maximize } \sum_{j=1}^{n} 10^{n-j} x_j \\ \left(2 \sum_{j=1}^{i-1} 10^{i-j} x_j \right) + x_i \le 100^{i-1}, \quad i = 1, \dots, n \\ x_1, x_2, \dots, x_n \ge 0 \end{gathered}$$

For $n = 4$, this gives the LP:

$$\begin{gathered} \text{maximize } 1{,}000x_1 + 100x_2 + 10x_3 + x_4 \\ x_1 \le 1 \\ 20x_1 + x_2 \le 100 \\ 200x_1 + 20x_2 + x_3 \le 10{,}000 \\ 2{,}000x_1 + 200x_2 + 20x_3 + x_4 \le 1{,}000{,}000 \\ x_1, x_2, x_3, x_4 \ge 0 \end{gathered}$$

The feasible region for this LP is a 4-dimensional cube which had had some of its (3-dimensional) faces pushed in at different angles. The Simplex Method will visit all $2^4 = 16$ vertices, using the Largest Coefficient Rule.

Why do we use the Simplex Method, if it has these flaws? First, it will usually find a solution quickly.[10] Second, it is the easiest LP solution method to describe. Third, it was the first one discovered, by G. B. Dantzig in 1947 [3]. Fourth, it terminates unless it cycles.

Some alternatives are the *interior point methods*, where the current point moves through the interior of the feasible region, instead of around its boundary. One of these is called the Ellipsoid Method [7].[11] There are also the Primal-Dual algorithms,[12] which approach the maximum value of P from above *and* below.

LP Duality

Motivation

The book provides no motivation for LP duality, other than the fact that a LP and its dual both have the same *value* (i.e., the minimimum when trying to minimize, or the maximum when trying to maximize). In fact, there is much more here on a theoretical level. So we will movtivate LP duality here, with a very practical problem.

Consider the LP:

$$\begin{aligned} &\text{maximize } 2x_1 + 3x_2 \\ x_1 + x_2 &\leq 3 \\ x_1 &\leq 2 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned} \tag{$\mathcal{P}$}$$

We solve this LP by using the Simplex Method (putting it into a tableau, and performing pivoting operations). After two iterations, we get the final tableau

$$\begin{array}{c} \text{BV} \\ x_1 \\ s_2 \\ x_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccccc} P & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\ \left[\begin{array}{c|cccccc|c} 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ \hline 1 & 0 & 0 & 2 & 0 & 1 & 8 \end{array}\right] \end{array}$$

which represents an optimal solution: $x_1 = 1$, $x_2 = 2$, and $P = 8$. The proof that is given for optimality is that the bottom row, turned into an equation and solved for P, yields $P = 8 - 2s_1 - s_3$,[13] which means the maximum is 8, since s_1 and s_3 are bigger than or equal to 0, and the only way to guarantee optimality is to have $s_1 = 0$ and $s_3 = 0$ (which means $x_1 + x_2 = 3$ and $x_2 = 2$, i.e., $(x_1, x_2) = (1, 2)$).

But we've forgotten one thing: What if I made a mistake during the pivoting?

Simple Error Checking

There are a few ways to check whether you have made any mistakes during pivoting. Most of them are quick checks:

- *Test your final answer for feasibility.* This is along the lines of the "Check your answer whenever possible" philosophy. In fact, at least n of the inequalities (which include the nonnegativity conditions $x_i \geq 0$) must actually be equalities; this is because the Simplex Method moves from corner point to corner point.
- *Make sure $P = 2x_1 + 3x_2$.* In other words, plug x_1 and x_2 into the original objective value.

[10] within $O(n \ln m)$ iterations
[11] Chvátal describes this algorithm in an appendix of his textbook [2].
[12] developed by Ford and Fulkerson [5], from earlier contributions by J. Egerváry [4] and H. W. Kuhn [9]
[13] This is a consequence of P being $2x_1 + 3x_2$, and the equalities in the LP; it is an equivalent expression.

- *The first column always stays the same.* This can be proven by induction, and the fact that a new pivot *never* is in the bottom row.[14]
- *The numbers in the RHS column are always nonnegative.* This is of course assuming you're not using the "alternate pivoting strategy" for LPs which can't be put into standard form. Negative numbers in the RHS column indicate a bad choice of pivot row and/or arithmetic errors.
- *You should always have exactly one pivot in each row.* Unless you're in the middle of turning an entry into a pivot, that is. In which case every row other than the pivot row should have a pivot in it.

More Sophisticated Error Checking

Since the inequalities restrict what values the variables can have, shouldn't they be able to provide a bound on how big P can get? The answer to this question is a resounding yes.

For instance, our LP named $(\mathcal{P})$[15] requires that x_1 and x_2 be at most 2. If we multiply the first inequality ($x_1 \leq 2$) by 2 on both sides, and multiply the second inequality ($x_2 \leq 2$) by 3 on both sides, and then add them together, we get the following inequality:

$$\begin{array}{rcr} x_1 \leq 2 & \rightarrow & 2x_1 \leq \ 4 \\ x_2 \leq 2 & \rightarrow & 3x_2 \leq \ 6 \\ \hline & & P = 2x_1 + 3x_2 \leq 10 \end{array}$$

So $P \leq 10$. We can do even better. Since $x_1 + x_2 \leq 3$, we also can deduce that

$$P = 2x_1 + 3x_2 \leq 3x_1 + 3x_2 \leq 3 \cdot 3 = 9.$$

So we do not necessarily need to end up with $2x_1 + 3x_2$ on the left-hand side, in order to get an upper bound on P; a larger coefficient is good enough.

Enough messing around. Let's get organized, and try to find the best possible upper bound. What we will do is associate with each inequality a variable y_i, which will end up being a *dual variable*, and multiply both sides of the i^{th} inequality by y_i. To make sure that the inequality stays the same, we will require that $y_i \geq 0$, for all i. Using our LP $(\mathcal{P})$ as an example,

$$\begin{array}{ccccc} (& x_1 + x_2 & \leq 3 &) & \cdot\, y_1 \\ (& x_1 & \leq 2 &) & \cdot\, y_2 \\ (& x_2 & \leq 2 &) & \cdot\, y_3 \end{array}$$

Adding the inequalities together results in:

$$x_1y_1 + x_2y_1 + x_1y_2 + x_2y_3 \leq 3y_1 + 2y_2 + 2y_3,$$

or

$$x_1(y_1 + y_2) + x_2(y_1 + y_3) \leq 3y_1 + 2y_2 + 2y_3.$$

Now, to say something about $2x_1 + 3x_2$, we need to have

$$\begin{aligned} y_1 + y_2 &\geq 2 \\ y_1 + y_3 &\geq 3 \end{aligned}$$

Since we want the best lower bound, we want to make $3y_1 + 2y_2 + 2y_3$ as small as possible. Thus, we need to solve the following problem:

$$\begin{gathered} \text{minimize } 3y_1 + 2y_2 + 2y_3 \\ y_1 + y_2 \geq 2 \\ y_1 + y_3 \geq 3 \\ y_1, y_2, y_3 \geq 0 \end{gathered} \qquad (\mathcal{D})$$

[14] The entry in the pivot column in the last row is negative, by definition.
[15] $\mathcal{P}$ is for "primal", synonym for "original."

This is the *dual LP* (of $(\mathcal{P})$).

It is possible to show that the dual LP of $(\mathcal{D})$ is $(\mathcal{P})$,[16] so the name "dual" is well-earned. Note that the dual LP of a maximization problem is a minimization problem, and vice versa.

Clearly, the value (the minimum) of the dual LP can be no less than the value (maximum) of the primal, since the optimum solution to the primal problem is feasible. But can there be a gap? The answer, happily, is no:

> **Von Neumann's Duality Theorem[17].** *If a LP has a maximum value of v, then its dual LP has a minimum value of v.*

A version of this is also true if the primal LP does *not* have a solution. If the primal LP is unbounded, then the dual LP must be infeasible; a feasible point for the dual LP would provide an upper bound. If the primal LP has no feasible points, then the dual LP can be unbounded (having a minimum of $-\infty$ or a maximum of $+\infty$) or infeasible; however, in this case, the dual LP cannot have a finite minimum, because Von Neumann's Duality Theorem also applies to minimization problems.

The Dual Variables

Where are y_1, y_2, and y_3? Do we need to solve another LP to find them? Fortunately not. They're in the bottom row of the final tableau, under the columns corresponding to the slack variables:

$$
\begin{array}{c}
\text{BV} \\ x_1 \\ s_2 \\ x_2 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} P & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{c|cccccc|c}
0 & 1 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 2 \\
\hline
1 & 0 & 0 & \boxed{2} & \boxed{0} & \boxed{1} & 8
\end{array}\right]
\end{array}
$$

If we multiply both sides of the first inequality of $\mathcal{P}$ (the one which we added s_1 to, in order to get an equality) by y_1, multiply both sides of the second inequality by y_2, etc., and add these new inequalities together, we get:

$$
\begin{array}{rcr}
x_1 + x_2 \le 3 & \to & 2x_1 + 2x_2 \le 6 \\
x_1 \le 2 & \to & 0 \le 0 \\
x_2 \le 2 & \to & x_2 \le 2 \\
\cline{3-3}
 & & P = 2x_1 + 3x_2 \le 8,
\end{array}
$$

which means that $P \le 8$. Also, it is easy to check that $(x_1, x_2) = (1, 2)$ is feasible, and $P = 8$ for this point. Thus $(1, 2)$ *must* be a maximum, because P cannot be made any bigger than 8, QED, even if we did make mistakes elsewhere.[18]

Let us look again at the final tableau.[19] If $y_i \neq 0$, then there cannot be a pivot in the s_i column (since the only pivot in the bottom row is in the P column), and so $s_i = 0$. Conversely, if s_i is nonzero, then s_i must be a Basic Variable, and so there must be a pivot in the s_i column, which is not in the bottom row, and hence $y_i = 0$.

Thus, in either case, we have $s_i y_i = 0$, for all i. If we consider solving the dual LP, we can introduce more slack variables t_1, t_2, ..., t_n, to turn those inequalities into equalities. By analogy, we must also have $t_i x_i = 0$, for all appropriate values i. These two sets of conditions are known as the *complementary slackness conditions*, and they help tremendously in solving LPs:

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[16] We leave this as an exercise for the overzealous reader.

[17] Originally proven by John von Neumann and George Bernard Danzig, formalized in [6] by Gale, Kuhn, and Tucker.

[18] If we did, they don't matter; we've solved the problem in spite of them.

[19] not mentioned during the talk, due to time constraints

Necessary and Sufficient Conditions for Optimality. *If $(x_1, x_2, \ldots, x_n)$ is a feasible point of a LP, and $(y_1, y_2, \ldots, y_m)$ is a feasible point of that LP's dual, then the following are equivalent:*

(1) *These points are optima of their respective LPs;*
(2) *The objective value of the primal LP at x is the same as the objective value of the dual LP at y; and*
(3) *$y_i s_i = 0$ for all $i = 1, 2, \ldots, m$, and $x_i t_i = 0$ for all $i = 1, 2, \ldots, n$.*

Dual variables come in handy in other situations as well. For instance, suppose you solved $(\mathcal{P})$, like your boss told you to, but when you're presenting your results at a meeting, your boss says, "That inequality $x_2 \leq 2$ should actually have been $x_2 \leq 2.1$. I hope that doesn't change your answer too much."

What can you do?[20] Fortunately, dual variables can help you out here. If you have solved $\mathcal{P}$, then you can approximate the solution to the modified LP:

$$\begin{aligned} &\text{maximize } 2x_1 + 3x_2 \\ x_1 + x_2 &\leq 3 + \varepsilon_1 \\ x_1 &\leq 2 + \varepsilon_2 \\ x_2 &\leq 2 + \varepsilon_3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

It turns out that $P = 8 + \varepsilon_1 y_1 + \varepsilon_2 y_2 + \varepsilon_3 y_3$, if ε_1, ε_2, and ε_3 are "small enough."[21] If you consider 0.1 to be "small enough", you can say that the optimum is approximately $8 + 0.1(1) = 8.1$.

Proofs of Unboundedness[22]

It is also possible to prove that a LP (which is being maximizied) is unbounded, i.e., that the maximum is $+\infty$.

To see how, consider the LP

$$\begin{aligned} &\text{maximize } 2x_1 + x_2 + x_3 \\ -2x_1 + x_2 - 2x_3 &\leq 4 \\ 2x_1 - 2x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

After setting up the initial tableau and pivoting twice, the following table results:

$$\begin{array}{c} \text{BV} \\ s_1 \\ x_1 \\ P \end{array} \begin{array}{c} \begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & \text{RHS} \end{array} \\ \left[\begin{array}{c|cccccc|c} 0 & 0 & -1 & -1 & 1 & 1 & 8 \\ 0 & 1 & -1 & 1/2 & 0 & 1/2 & 2 \\ \hline 1 & 0 & -3 & 0 & 0 & 1 & 4 \end{array}\right] \end{array}$$

The Simplex Method says that this LP is unbounded, because all entries in the x_2 column are nonpositive. But once again, how do we know we didn't make a mistake?

Here's how we can check our answer: Let all nonbasic variables except for x_2 be 0, let $x_2 = t$, and convert the rows of the tableau above to equations:

$$\begin{aligned} -t + s_1 &= 8 \\ x_1 - t &= 2 \\ P - 3t &= 4. \end{aligned}$$

[20] Other than supressing the urge to kill your boss, of course.

[21] How small is "small enough"? It's along the line of: small enough to make $\dfrac{f(x+h) - f(x)}{h}$ close to $f'(x)$.

[22] Not given during the talk, due to time constraints.

This means that $(x_1, x_2, x_3) = (2+t, t, 0)$, because $x_3 = 0$, as x_3 is not a Basic Variable. These points are feasible for all $t \geq 0$, because

$$\begin{aligned} -2x_1 + x_2 - 2x_3 &= -2(2+t) + 1 \cdot t - 2 \cdot 0 = -4 - t \leq 4, \text{ and} \\ 2x_1 - 2x_2 + x_3 &= 2(2+t) - 2 \cdot t + 0 = 4 \leq 4. \end{aligned}$$

Furthermore, $P = 2x_1 + x_2 + x_3 = 2(2+t) + t + 0 = 4 + 3t$, so $P \to \infty$ as $t \to \infty$; hence the LP is unbounded, QED.

Mixed Constraints

Problems Not In Standard Form

Up until now, we have assumed that the LP is in standard form. For LPs which are not in standard form, the Simplex Method cannot be used right away, because the initial point (the origin) is infeasible. So a feasible point needs to be found, whereupon the Simplex Method can take over and work towards a solution.

The "alternative pivoting strategy" mentioned in section 4.4 moves from one "pseudo-corner" to another. (I'm definining a *pseudo-corner* to be a point where n of the LP's inequalities are equal, without any condition on feasibility. A feasible pseudo-corner is thus a corner point.) Its progress is best described as "stumbling around in the dark." According to Sultan [11], this "stumbling" will eventually reach a corner point, unless there are no feasible points at all.

An alternative is the *Two-Phase Simplex Method*. To solve the LP[23]

$$\begin{aligned} &\text{maximize } 3x_1 + x_2 \\ &x_1 - x_2 \leq -1 \\ &x_1 + x_2 \geq \ \ 3 \\ &2x_1 + x_2 \leq \ \ 4 \\ &x_1, x_2 \geq 0 \end{aligned}$$

we first solve

$$\begin{aligned} &\text{maximize } -x_0 \\ &x_1 - x_2 - x_0 \leq -1 \\ &x_1 + x_2 - x_0 \leq -3 \\ &2x_1 + x_2 - x_0 \leq \ \ 4 \\ &x_1, x_2, x_0 \geq 0 \end{aligned}$$

to get a corner point for the Simplex Method to work with. Note that we are trying to minimize x_0, and if this minimum is equal to zero, we can find a point where the Simplex Method can continue with the original problem.

The new LP has an initial tableau of the form

$$\begin{array}{c} \text{BV} \\ s_1 \\ s_2 \\ s_3 \\ \\ P' \end{array} \begin{array}{c} \begin{array}{cccccccc} P' & x_0 & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\ \left[\begin{array}{c|cccccc|r} 0 & -1 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 & -3 \\ 0 & -1 & 2 & 1 & 0 & 0 & 1 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

This tableau represents an infeasible "pseudo-corner," but we can create a feasible solution by putting a pivot in the x_0 column, in the second row (the row with the smallest entry in the RHS column):

$$\begin{array}{c} \text{BV} \\ s_1 \\ x_0 \\ s_3 \\ \\ P' \end{array} \begin{array}{c} \begin{array}{cccccccc} P' & x_0 & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\ \left[\begin{array}{c|cccccc|r} 0 & 0 & 2 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 0 & -1 & 1 & 7 \\ \hline 1 & 0 & -1 & -1 & 0 & 1 & 0 & -3 \end{array}\right] \end{array}$$

[23] This is not the example I used during the talk. I didn't go into details, so I don't feel that I need to keep the same problem.

Now we can perform the Simplex Method as usual. We get a final tableau of:

$$
\begin{array}{c}
\text{BV} \\ x_1 \\ x_2 \\ s_3 \\ P'
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} P' & x_0 & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{c|cccccc|c}
0 & 0 & 1 & 0 & 1/2 & -1/2 & 0 & 1 \\
0 & 1 & 0 & 1 & -1/2 & -1/2 & 0 & 2 \\
0 & -2 & 0 & 0 & -1/2 & 3/2 & 1 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

We can see that $x_0 = 0$ for this tableau, since x_0 is not a BV, and so the original problem is feasible. (If the second, auxiliary LP stopped with a nonzero entry in the lower right-hand corner, that would say that the original LP is infeasible—that it has no feasible points whatsoever.) We now take this tableau, remove the P' and x_0 columns, and the final row, and insert it into the tableau for the original LP:

$$
\begin{array}{c}
\text{BV} \\ x_1 \\ x_2 \\ s_3 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} P & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{c|ccccc|c}
0 & 1 & 0 & 1/2 & -1/2 & 0 & 1 \\
0 & 0 & 1 & -1/2 & -1/2 & 0 & 2 \\
0 & 0 & 0 & -1/2 & 3/2 & 1 & 0 \\ \hline
1 & -3 & -1 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

The x_1 and x_2 columns no longer have a pivot in them, but that can be fixed by adding multiples of rows 1 and 2 to row 4:

$$
\begin{array}{c}
\text{BV} \\ x_1 \\ x_2 \\ s_3 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} P & x_1 & x_2 & s_1 & s_2 & s_3 & \text{RHS} \end{array} \\
\left[\begin{array}{c|ccccc|c}
0 & 1 & 0 & 1/2 & -1/2 & 0 & 1 \\
0 & 0 & 1 & -1/2 & -1/2 & 0 & 2 \\
0 & 0 & 0 & -1/2 & 3/2 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & -2 & 0 & 0
\end{array}\right]
\end{array}
$$

Now we have found a corner point and can continue with the Simplex Method to find the solution $(x_1, x_2) = (1, 2)$, with $P = 5$.[24]

Linear Programs With Equalities

The book handles an equation like

$$x_1 + x_2 + x_3 = 10$$

by replacing it with the inequalities

$$x_1 + x_2 + x_3 \leq 10$$
$$x_1 + x_2 + x_3 \geq 10$$

and adding a slack variable to each, resulting in two equalities:

$$
\begin{aligned}
x_1 + x_2 + x_3 + s_1 &= 10 \\
-x_1 - x_2 - x_3 + s_2 &= -10
\end{aligned}
$$

This is silly, since we must obviously have $s_1 = s_2 = 0$.

An alternative is to put the original equality into the tableau, without adding a slack variable to it, and turn one of the entries in that row into a pivot. The simplex method works as usual from here on.[25]

For example, consider Example 3 from Section 4.4 of [10]:

[24] We leave the rest of this exercise to the overzealous reader.

[25] It may not be possible to put the LP in standard form, so the Two-Phase Simplex Method or the alternate pivoting strategy must be used. Note that if you have to use the Two-Phase Simplex Method, then you would have to use the alternate pivoting strategy (if you use the book's method), so you are not making the procedure needlessly long.

$$\begin{aligned}
\text{minimize } 7x_1 + 5x_2 + 6x_3 \\
x_1 + \ \ x_2 + \ \ x_3 &= 10 \\
x_1 + 2x_2 + 3x_3 &\le 19 \\
2x_1 + 3x_2 \qquad\quad &\ge 21 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

Putting it into a tableau, we get:[26]

$$\begin{array}{c} \text{BV} \\ \\ s_1 \\ s_2 \\ P \end{array}
\begin{array}{c}
\begin{array}{cccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & \text{RHS} \end{array} \\
\left[\begin{array}{c|cccccc|c}
0 & 1 & 1 & 1 & 0 & 0 & 10 \\
0 & 1 & 2 & 3 & 1 & 0 & 19 \\
0 & -2 & -3 & 0 & 0 & 1 & -21 \\
\hline
1 & 7 & 5 & 6 & 0 & 0 & 0
\end{array}\right]
\end{array}$$

This is not a valid tableau, because there is no pivot in the first row. However, we can put a pivot in that row, in the column of the variable with the same sign as the RHS[27] and smallest subscript, and put a pivot in its (x_1's) column and the first row.[28] We now get:

$$\begin{array}{c} \text{BV} \\ x_1 \\ s_1 \\ s_2 \\ P \end{array}
\begin{array}{c}
\begin{array}{ccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & \text{RHS} \end{array} \\
\left[\begin{array}{c|cccccc|c}
0 & 1 & 1 & 1 & 0 & 0 & 10 \\
0 & 0 & 1 & 2 & 1 & 0 & 9 \\
0 & 0 & -1 & 2 & 0 & 1 & -1 \\
\hline
1 & 0 & -2 & -1 & 0 & 0 & -70
\end{array}\right]
\end{array}$$

There is a negative sign in the RHS column, which we can fix with the Two-Phase Simplex Method, or the alternate pivoting strategy; the latter only requires one pivoting operation, in the third row and x_2 column:

$$\begin{array}{c} \text{BV} \\ x_1 \\ s_1 \\ x_2 \\ P \end{array}
\begin{array}{c}
\begin{array}{ccccccc} P & x_1 & x_2 & x_3 & s_1 & s_2 & \text{RHS} \end{array} \\
\left[\begin{array}{c|cccccc|c}
0 & 1 & 0 & 3 & 0 & 1 & 9 \\
0 & 0 & 0 & 4 & 1 & 1 & 8 \\
0 & 0 & 1 & -2 & 0 & -1 & 1 \\
\hline
1 & 0 & 0 & -5 & 0 & -2 & -68
\end{array}\right]
\end{array}$$

The standard Simplex Method can now be used. After two pivoting operations, the final answer is reached: $(x_1, x_2, x_3) = (1, 9, 0)$, with $P = -52$, indicating that the minimum is 52.[29] This was done with one fewer pivoting operation than in the book, on a tableau which had one less row and two fewer columns.

Multiple Solutions[30]

The Simplex Method will only supply one solution to the LP; however, there are situations where all solutions are sought. In general, if $\vec{x}_1, \ldots, \vec{x}_k$ are the corner points where the optimum value is attained, then any convex combination of these corner points will be solutions; that is, every solution is of the form

$$\sum_{i=1}^{k} \lambda_i \vec{x}_i, \quad \text{where } \lambda_i \ge 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1,$$

[26] The third inequality has been reversed, and the minimization of $7x_1 + 5x_2 + 6x_3$ turned into the maximization of $P = -7x_1 - 5x_2 - 6x_3$.

[27] If all the signs in the pivot row—except for the one in the RHS column—are different from the one in the RHS column, then the LP has no feasible points, using the same logic as for the alternate pivoting strategy.

[28] a la the alternate pivoting strategy, or Bland's Rule

[29] A LP with an equality does have a dual LP, except with one noteworthy difference: The dual variable corresponding to the equality can be positive, negative, or zero.

[30] This section was added August 2006 and is based on e-mail conversations with Frederic Ferrero.

for some real values of λ_i, and every point which can be written in this form is an optimal solution. Thus the geometric method solves this version of the problem; with some work, the Simplex Method (along with the complimentary slackness conditions) can do the same.

Specifically, let's consider the following LP:

$$\begin{aligned} &\text{maximize } x_1 + x_2 + x_3 \\ &x_1 + \ \ x_2 + 3x_3 \leq 1 \\ &x_1 + \ \ x_2 \qquad\ \ \leq 1 \\ &\qquad 2x_2 - 5x_3 \leq 1 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

whose dual is

$$\begin{aligned} &\text{minimize } y_1 + y_2 + y_3 \\ &y_1 + \ \ y_2 \qquad\ \ \geq 1 \\ &y_1 + \ \ y_2 + 2y_3 \geq 1 \\ &3y_1 \qquad - 5y_3 \geq 1 \\ &\qquad y_1, y_2, y_3 \geq 0 \end{aligned}$$

A dual solution (obtained by solving the Primal LP with the Simplex Method) is $(y_1, y_2, y_3) = (1, 0, 0)$. According to the Necessary and Sufficient Conditions for Optimality, any optimal solution (x_1, x_2, x_3) to the Primal LP has

$$x_1 + x_2 + 3x_3 = 1, \tag{E1}$$

since dual variable y_1 is not zero. (The other inequalities may be strict, since $y_2 = y_3 = 0$.)

We also need to check the inequalities in the Dual LP and see which are strict; the corresponding primal variable has to be 0. In this case, the first two inequalities are actually equal for the dual solution $(1, 0, 0)$, and the third is strict $(3 > 1)$. That means we must have

$$x_3 = 0. \tag{E2}$$

Now we solve the system of linear equations consisting of (E1) and (E2), and find the restrictions on the free variables to make (x_1, x_2, x_3) feasible. In our case, we have

$$\begin{aligned} x_1 &= 1 - t \\ x_2 &= t \qquad , \quad \text{where } t \text{ is a free variable.} \\ x_3 &= 0 \end{aligned}$$

The conditions on t to make (x_1, x_2, x_3) feasible are

$$\begin{array}{r} x_1 + x_2 + 3x_3 \leq 1 \\ x_1 + x_2 \leq 1 \\ 2x_2 - 5x_3 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{array}, \quad \text{so} \quad \begin{array}{r} (1 + t) + t + 0 \leq 1 \\ (1 - t) + t \leq 1 \\ 2t - 5 \cdot 0 \leq 1 \\ 1 - t \geq 0 \\ t \geq 0 \\ 0 \geq 0 \end{array}, \quad \text{or} \quad \begin{array}{l} 1 \leq 1 \quad \text{(always true)} \\ 1 \leq 1 \\ t \leq 1/2 \\ t \leq 1 \\ t \geq 0 \\ 0 \geq 0 \end{array} .$$

So t must be in $\left[0, \frac{1}{2}\right]$. This, along with the equations for x_1, x_2, and x_3, parameterizes all optimal solutions to the LP.

To parameterize all solutions to the *Dual* LP, you exchange the roles of the primal and dual solutions. You use the Simplex Method to find a solution to the *Primal* LP, use the complementary slackness conditions to deduce which inequalities of the *Dual* LP should be equalities and which *dual* variables should be zero.

Conclusion

The subject of Linear Programming extends beyond the Simplex Method algorithm, much as Linear Algebra extends beyond Gaussian Elimination, and the theory behind it has enough substance to make study worthwhile. This theory helps to explain why the Simplex Method proceeds as it does, suggests alternate approaches to solving LPs, and can be used to formally prove that a certain solution is an optimum. It is hoped that this paper has introduced people to this young (50-year old) field, and that further results can follow with its study.

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