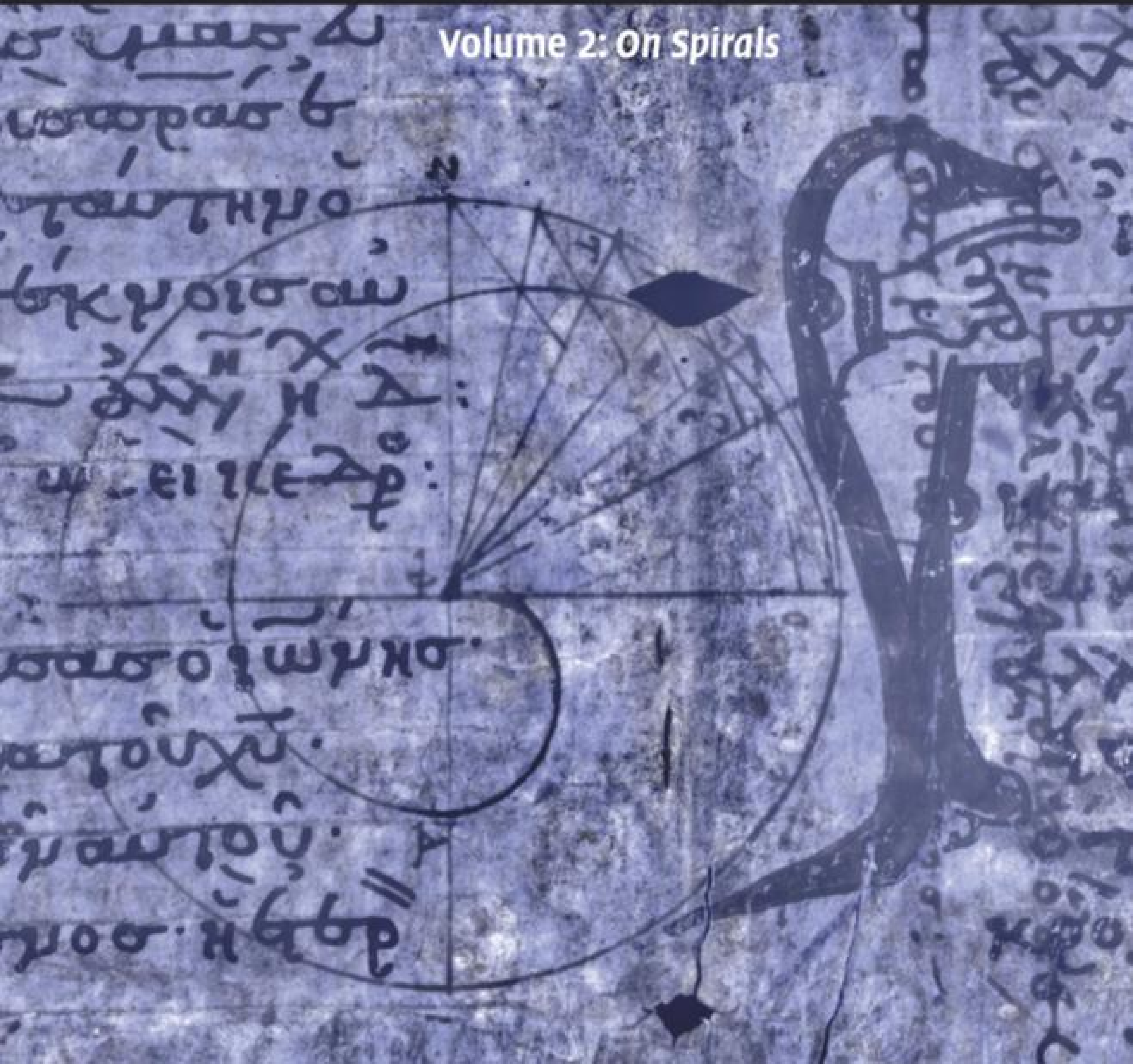
Reviel Netz

The Works of Archimedes

Translation and Commentary

Volume 2: *On Spirals*



The Works of Archimedes
Translation and Commentary

Volume II: *On Spirals*

This is the second volume of the first fully fledged English translation of the works of Archimedes – antiquity's greatest scientist and one of the most important scientific figures in history. It covers *On Spirals* and is based on a reconsideration of the Greek text and diagrams, now made possible through new discoveries from the Archimedes palimpsest. *On Spirals* is one of Archimedes' most dazzling geometrical *tours de force*, suggesting a manner of "squaring the circle" and, along the way, introducing the attractive geometrical object of the spiral. The form of argument, no less than the results themselves, is striking, and Reviel Netz contributes extensive and insightful comments that focus on Archimedes' scientific style, making this volume indispensable for scholars of classics and the history of science, and of great interest for the scientists and mathematicians of today.

REVIEL NETZ is Patrick Suppes Professor of Greek Mathematics and Astronomy at Stanford University, and is the leading scholar of Archimedes today. He has published numerous articles and books, many of which have led to new directions in the study of the history of science, including *The Shaping of Deduction in Greek Mathematics* (Cambridge University Press, 1999), *The Transformation of Mathematics in the Early Mediterranean World* (Cambridge University Press, 2004), and *Ludic Proof* (Cambridge University Press, 2009). He is also producing a complete new translation of and commentary on the works of Archimedes.

THE WORKS OF ARCHIMEDES

Translated into English, with commentary, and critical edition of the diagrams

REVIEL NETZ
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Volume II
On Spirals

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To Maya, Darya and Tamara

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SC II	The second book *On the Sphere and the Cylinder*
Eut. *SC* II	Eutocius' commentary on the above
SL	*On Spiral Lines*
CS	*On Conoids and Spheroids*
DC	*Measurement of the Circle (Dimensio Circuli)*
Eut. *DC*	Eutocius' commentary on the above
Aren.	*The Sand-Reckoner (Arenarius)*
PE I, II	*Planes in Equilibrium*
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QP	*Quadrature of the Parabola*
Meth.	*The Method*
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CF II	The second book *On Floating Bodies (de Corporibus Fluitantibus)*
Bov.	*The Cattle Problem (Problema Bovinum)*
Stom.	*Stomachion*

ABBREVIATIONS OF CITED MANUSCRIPTS AND PRE-HEIBERG EDITIONS

Manuscripts

A (reconstructed source manuscript of DEGH4)
B (Moerbeke's Latin translation, partly of A, partly of a different manuscript) Ottob. Lat. 1850
C (The Archimedes Palimpsest)
D Laur. XXVIII.4
E Marc. 305
G Par. Gr. 2360
H Par. Gr. 2361
4 Vat. Gr. Pii II.16

Editions

Commandino
Commandino, F. 1558. *Archimedes Opera non nulla*. Venice.
Torelli
Torelli, J. 1792. *Archimedis quae supersunt Omnia*. Oxford.
Nizze
Nizze, E. 1824. *Archimedes von Syrakus vorhandene Werke*. Stralsund.

ACKNOWLEDGMENTS

Work on this volume proceeded alongside the publication of *The Archimedes Palimpsest* (Cambridge University Press, 2011). In many ways, this translation would have been impossible without the palimpsest. Indeed, the palimpsest grounds the new text, in particular that of the diagrams. It is a pleasure to have the opportunity to thank again the wonderful team that made work on the palimpsest possible – and a pleasure. I thank my co-editor Nigel Wilson, curator of manuscripts Abigail Quandt, and the three lead imagers – Roger Easton, Bill Christens-Barry and Keith Knox. Above all I thank my co-author and curator of manuscripts, now Director of the Schoenberg Institute for Manuscript Studies at the University of Pennsylvania, William Noel.

This is the second volume of my translation of Archimedes. The first volume got published in 2004, and I have been receiving e-mails for several years now asking where the second volume was! I apologize. Delay was due to various technical reasons, mostly outside of the author's sway, and it is reasonably hoped that future volumes will be published at a faster clip. Meanwhile, I have accumulated many debts: to my publishers at Cambridge University Press, in particular the Classics Editor Michael Sharp, copy-editor Christopher Jackson, and the ever patient Emma Collison, to my colleagues among the wider community of the history of science and mathematics, as well as here at the Stanford departments of Classics and the History and Philosophy of Science and Technology, and especially to two students who have helped me substantially with the detail of the book: Amy Carlow and Dr. Johannes Wietzke.

As the years have passed, my titles have evolved, and I have now metamorphosed into the Patrick Suppes Professor for Greek Astronomy and Mathematics at Stanford. I had revered Pat Suppes well before I met him, and then I was greatly privileged to co-teach with him – effectively, therefore, to be his student – in the philosophy of science. Pat Suppes always insisted, above all, that the way to make real progress in the philosophical understanding of science is through the precise understanding of the detail of a scientific practice. I am grateful to have had Pat Suppes as my inspiration as I was translating Archimedes, and I hope to share this sense, now, with new readers of Archimedes.

INTRODUCTION

I THE STRUCTURE OF ARCHIMEDES' ON SPIRALS

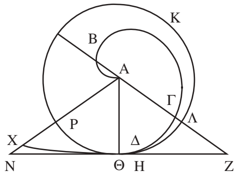
One may be forgiven for considering this, *On Spirals*,[1] to be Archimedes' finest. The figures bend and balance as the argument reaches – effortlessly, quickly, and yet, how, one cannot quite grasp – towards several magnificent results. These suggest no less than the squaring of the circle: first, a certain line (defined by a tangent to the circle) is equal to the circumference of the circle; second, a certain area is equal to the circle's third.

We are witnesses to Archimedes in action, as he engaged in a campaign of publications. At some early date, we are told in this treatise, he sent out via his mathematician friend Conon a complex geometrical challenge containing many claims. He had gradually discharged this challenge. Previously, he had sent to Dositheus the two books *On the Sphere and the Cylinder* (following on the *Quadrature of the Parabola*, which contained results independent from the original challenge sent via Conon). Now, he sends out *On Spirals*. This, once again, is sent to Dositheus. Archimedes once again proves some of the claims contained in that letter to Conon; he also reflects, briefly, on that geometrical challenge as a whole.

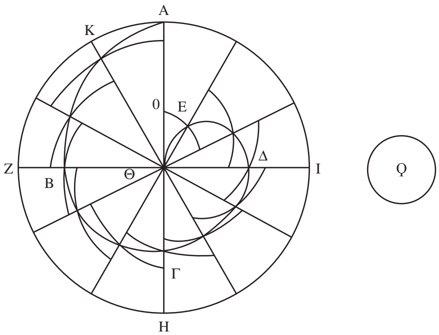
In this treatise, Archimedes promises to find not two, but four results. One of them is the result on the tangent mentioned above (being equal to the circumference of the circle). The result on the area of the spiral (being one-third the circle enclosing it) is proved and then further expanded to two extra, inherently interesting results, showing the ratios between the entire shells of spirals enclosing each other as well as the ratios of fragments of shells enclosing each other. The main results, then, are:

18. The line AZ is equal to the circumference of the circle HΘK.

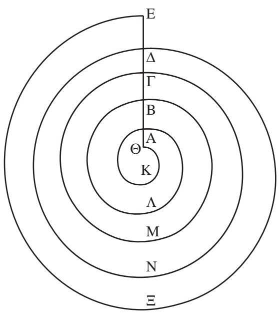
[1] *On Spirals* translates the title transmitted through the manuscript tradition, περι ελικων. A slightly expanded version, "*Spiral Lines*," is the one most often used by previous English discussions of Archimedes, and is implied by my own abbreviation to the title, SL.



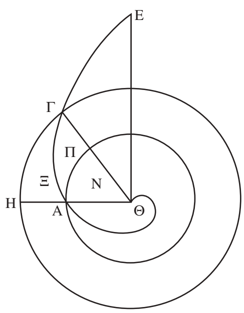
24. The spiral area ΑΒΓΔΕΘΑ is one-third the circle.



27. In the series of shells ΛΜΝΞ, Μ is twice Λ, Ν is three times Μ, Ξ is four times Ν, etc.



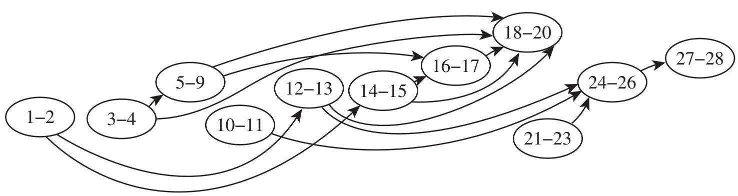
28. The shell fragment Ξ is to the shell fragment Π as $(A\Theta+\frac{2}{3}HA):(A\Theta+\frac{1}{3}HA)$.



The deductive flow of the propositions in this treatise may be summed up as a table of dependence:

Proposition	Relies on
1	
2	1
3	
4	3
5	3
6–9	
10	
11	10
12	1
13	12
14	2
15	2, 14
16	5, 14
17	5, 16
18	4, 7, 8, 13, 14, 15, 16
19	4, 7, 13, 15, 17,
20	4, 7, 13, 14, 16
21	
22	
23	
24	10, 12, 21
25	11, 12, 22
26	11, 12, 23
27	24, 25
28	26

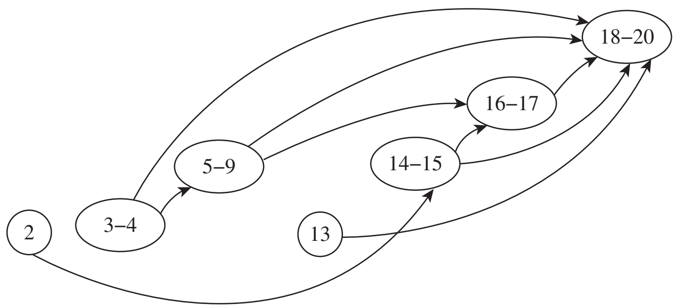
It is apparent that results cluster together in pairs and triplets, and it is perhaps best to visualize the logical flow as a chart based on such clusters:



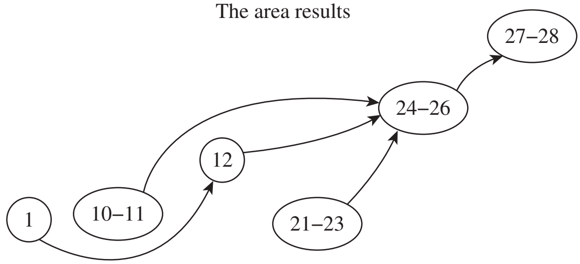
The immediate observation is how "shallow" the structure is. There is limited recursion (the top results are at level "4": 1–2 leads to 14–15 leads to 16–17 leads to 18–20, and 1–2 leads to 12–13 leads to 24–26 leads to 27–28). A substantial fraction of the treatise is at the elementary level where one directly applied widely known results (1–2, 3–4, 10–11, 21–23: notice that some of this "elementary" level is very complex). Instead of vertical recursion, we see the horizontal bringing-together of unrelated strands at two key moments of the treatise: 18–20, bringing together 3–4, 5–9, 12–13, 14–15 and 16–17; and 24–26, bringing together 10–11, 12–13 and 21–23.

Indeed, there are two such moments because there are two separate lines of reasoning. I set out the logical flow now for each strand apart (for this purpose, I distinguish 1 from 2, 12 from 13):

The tangent results



The area results



We find that the treatise cleaves nearly in half (sixteen propositions serve in the tangent results, twelve in the area results).[2] And cleave it does: the paths to the tangent results, and to the area results, are essentially independent. The one complication is the set of results 1–2, 12–13, where:

1 leads to 2

1 leads to 12 leads to 13

2, 13 are used in the tangent results

12 is used in the areas results

It is apparent that the one link shared between the two strands is proposition 12 – which is in the nature of an alternative definition of the spiral line (that lines drawn from the start on the spiral line differ from each other in the ratio of the

[2] Merely counting propositions is misleading, however, if we measure propositions by logical size – for the sake of the exercise, by the number of Steps in the proof: we find 183 Steps used in the tangent results, 195 in the area results: the area results are fewer but on the whole more complex (indeed, the tangent results appear to be slightly padded, with propositions 6, 9 seemingly unmotivated; they take 25 Steps).

angles they make with each other). Archimedes did make a choice to present this as a theorem,[3] so the two strands do hang together, if by the thinnest of threads. For indeed Archimedes also made the choice not to display the cleavability of the treatise. Adding to his bivalence of propositions 12–13, Archimedes inserted the pair 10–11 before them and, in between, inserted a passage of definitions. The result is a long passage composed of 10–11, definitions, 12–13, which cannot be read as leading at all, or strictly, to the tangent results. As for the area results, those are broken much more powerfully into two segments, 10–11 (as well as 12), and then the main sequence from 21 onwards.

Archimedes could easily have positioned proposition 12 as a definition or as a consequence obtained directly from the definitions, and then divided his treatise into two parts (two books?), one for each set of results. The complex pattern in which the two strands are brought together serves to maximize the distance between tools and results, indeed to obscure, at first reading, the very identity of the tools required for the results obtained.

This, however, somewhat misrepresents Archimedes' choice as an author. It is not as if Archimedes was provided with a pile of twenty-eight propositions which he had to arrange is some form. Rather, he was looking for interesting things to say about spirals. Considered in this way, his basic choice is seen to be saying two things or, more precisely, dual-and-more: essentially, one result for tangents; and then one result for areas, which, however, is expanded to produce further results (this is seen in the logical flow in the segment 27–28, which derives directly from 24–26). "Dual-and-more" is a repeated pattern of this treatise, seen also in the way in which almost all propositions are presented in pairs or triplets (indeed, since many propositions carry brief corollaries, even the results that come in pairs display, in fact, the structure of dual-and-more). The architecture of the treatise as a whole can be derived from these two principles: a desire to maximize the distance between tools and their applications; and a repeated pattern of "dual-and-more." Hence the elegant, combined pattern of strands within strands. Thus results which are obtained quickly and effortlessly still make one gasp with wonder: how did we get there?

2 CONVENTIONS AND GOALS OF THE TRANSLATION

The translation follows the same conventions as in the first volume, *On the Sphere and the Cylinder*, and I should repeat here the account of the conventions in Netz 2004b: 3–8. I stated there that "There are many possible barriers to the reading of a text written in a foreign language, and the purpose of a scholarly translation as I understand it is to remove all barriers having to do with the foreign language itself, leaving all other barriers intact." This entails a more-than-usual literal translation. The following conventions of my translation – and of Greek mathematics itself – should therefore be explained. (To aid further in the reading, a glossary was added to this volume, so that when less familiar

[3] Even so, it takes a mere six Steps to accomplish this result, and even these are not so much argument as explication: see the comments on the theorem.

terms are introduced for the first time, they are accounted for. A marginal note refers to the Glossary, which is located at the end of the volume.)

1. Greek word order is much freer than English word order, and so, selecting from among the wider set of options, Greek authors can choose one word order over another to emphasize a certain idea. Thus, for instance, instead of writing "A is equal to B," Greek authors might write "to B is equal A." This would stress that the main information concerns B, not A – word order would make B, not A, the focus. (For instance, we may have been told something about B, and now we are being told the extra property of B, that it is equal to A.) Generally speaking, such word order cannot be kept in the English, but I try to note it when it is of special significance, usually in a footnote.

2. The summation of objects is often done in Greek through ordinary conjunction. Thus "the squares ABGD and EZHQ" will often stand for what we may call "the square ABGD plus the square EZHQ." As an extension of this, the ordinary plural form can serve, as well, to represent summation: "the squares ABGD, EZHQ" (even without the "and" connector!) will then mean "the square ABGD plus the square EZHQ." In such cases, the sense of the expression is in itself ambiguous (the following predicate may apply to the sum of the objects, or it may apply to each individually), but such expressions are, generally speaking, easily disambiguated in context. Note also that while such "implicit" summations are very frequent, summation is often more explicit and may be represented by such connectors as "together with," "taken together" or simply "with."

3. Greek has certain pairs of particles that do not merely govern their own clause, but also attach to each other to form a single, conjoint clause out of two separate phrases. One of those conjoint particles becomes nearly technical in Greek mathematics: *te*. . . *kai*. . . (conveyed most idiomatically in English by: both. . . and. . .). Thus, in expressions such as

the area contained by: *te* the line AB, *kai* the spiral AGB

the two elements of the expression, the line and the spiral, are not merely listed in order, but instead are understood to be conjoined so as to form, together, the border of a single figure.

To express this technical, somewhat unidiomatic meaning, I translate this combination into the somewhat unidiomatic English "both. . . as well as. . ."

4. The main expression of Greek mathematics is that of proportion:

"As A is to B, so is C to D."

(A, B, C and D being some mathematical objects). This expression is often represented symbolically, in modern texts, by:

A:B::C:D

and I will use such symbolism in my footnotes and commentary. In the main text I will translate, of course, the original non-symbolic form. Note especially that this expression may become even more concise, e.g.:

"As A is to B, C to D," "As A to B, C to D."

And that it may have more complex syntax, especially:

"A has to B the same ratio as C has to D," "A has to B a greater ratio than C has to D."

The last example involves an obvious extension of proportion, to ratio-inequalities, i.e. A:B>C:D. More concisely, this may be expressed by:

"A has to B a greater ratio than C to D."

A ratio can also be "duplicate another": this means, in terms more transparent to us today, that it is its square (the ratio of 9 to 4 is duplicate the ratio of 3 to 2); "triplicate" is, in the same sense, a cube (27 to 8 is triplicate 3 to 2).

5. Greek mathematical propositions have, in many cases, the following six parts:

- *Enunciation*, in which the claim of the proposition is made, in general terms, without reference to the diagram. It is important to note that, generally speaking, the enunciation is equivalent to a conditional statement that *if* x is the case, *then* so is y.
- *Setting-out*, in which the antecedent of the claim is restated, in particular terms referring to the diagram (with the example above, x is restated in particular reference to the diagram).
- *Definition of goal*, in which the consequent of the claim is restated, as an exhortation addressed by the author to himself: "I say that...," "it is required to prove that...," again in the particular terms of the diagram (with the same example, we can say that y is restated in particular reference to the diagram).
- *Construction*, in which added mathematical objects (beyond those required by the setting-out) may be introduced.
- *Proof*, in which the particular claim is proved.
- *Conclusion*, in which the conclusion is reiterated for the general claim from the enunciation.

Some of these parts will be missing in most Archimedean propositions, but the scheme remains a useful analytic tool, and I will use it as such in my commentary. The reader should be prepared in particular for the following difficulty. It is often very difficult to follow the enunciations as they are presented. Since they do not refer to the particular diagram, they use completely general terms, and since they aspire to great precision, they may have complex qualifications and combinations of terms. I wish to exonerate myself: this is not a problem of my translation, but of Greek mathematics. Most modern readers find that they can best understand such enunciations by reading, first, the setting-out and the definition of the goal, with the aid of the diagram. Having read this, a better sense of the *dramatis personae* is gained, and the enunciation may be deciphered. In all probability the ancients did the same.

6. The main "<...>" policy: Greek mathematical proofs always refer to concrete objects, realized in the diagram. Because Greek has a definite article with a rich morphology, it can elide the reference to the objects, leaving the definite article alone. Thus the Greek may contain such expressions as

"The by the AB, BG"

whose reference is

"The <rectangle contained> by the <lines> AB, BG"

(the morphology of the word "the" determines, in the original Greek, the identity of the elided expressions, given of course the expectations created by the genre).

In this translation, most such elided expressions are added inside pointed brackets, so as to make it possible for the reader to appreciate the radical concision of the original formulation and the concreteness of reference – while allowing me to represent the considerable variability of elision (very often, expressions have only partial elision). This variability, of course, will be seen in the fluctuating positions of pointed brackets:

"The <rectangle contained> by the <lines> AB, BG," as against, e.g., "The <rectangle> contained by the <lines> AB, BG"

(Notice that I do not at all strive at consistency *inside* pointed brackets. Inside pointed brackets I put whatever seems to me, in context, most useful to the reader; the duties of consistency are limited to the translation proper, outside pointed brackets.)

The main exception to my general pointed-brackets policy concerns points and lines. These are so frequently referred to in the text that to insist, always, upon a strict representation of the original, with such expressions as

"The <point> A," "The <line> AB"

would be tedious, while serving little purpose. I thus usually write, simply,

A, AB

and, in the less common cases of a non-elliptic form,

"The point A," "The line AB"

The price paid for this is that (relatively rarely) it is necessary to stress that the objects in question are points or lines, and while the elliptic Greek expresses this through the definite article, my elliptic "A," "AB" does not. Hence I need to introduce, here and there, the expressions

"The <point> A," "The <line> AB"

but notice that these stand for precisely the same as

A, AB.

I avoid distinguishing, typically, between ευθεια and γραμμη. The precise translation of ευθεια is "straight <line>," while the precise translation of γραμμη, when a straight line is intended, is "<straight> line." Not wishing to split such hairs, I have decided to make both simply a "line." In this treatise one may often compare straight and curved lines, and it would therefore matter occasionally to distinguish between the two; hence my practice will be inconsistent, occasionally expanding such phrases.

7. The “<=..>” sign is also used, in an obvious way, to mean essentially the same as the “[SC. . ..]” abbreviation. Most often, the expression following the “=” will disambiguate pronouns which are ambiguous in English (but which, in the Greek, were unambiguous thanks to their morphology).

8. Two sequences of numbering appear inside standard brackets. The Latin alphabet sequence “(a) . . . (b) . . .” is used to mark the sequence of constructions: as each new item is added to the construction of the geometrical configuration (following the setting-out) I mark this with a letter in the sequence of the Latin alphabet. Similarly, the Arabic number sequence “(1) . . . (2) . . .” is used to mark the sequence of assertions made in the course of the proof: as each new assertion is made (what may be called “a step in the argument”), I mark this with a number. This is meant for ease of reference: the footnotes and the commentary refer to constructions and to claims according to their letters or numbers. **Note that this is purely my editorial intervention, and that the original text had nothing corresponding to such brackets.** (The same is true for punctuation in general, for which see below.) Also note that these sequences refer only to construction and proof: enunciation, setting-out and definition of goal are not marked in similar ways.

9. The “/. . ./” symbolism: for ease of reference, I find it useful to add in titles for elements of the text of Archimedes, whether general titles such as “introduction” or numbers referring to propositions. I suspect Archimedes’ original text had neither, and such titles and numbers are therefore mere aids for the reader in navigating the text.

10. Ancient Greek texts were written without spacing or punctuation: they were simply a continuous stream of letters. Thus punctuation as used in modern editions reflects, at best, the judgements of late antiquity and the middle ages, more often the judgements of the modern editor. I thus use punctuation freely, as another editorial tool designed to help the reader, but in general I try to keep Heiberg’s punctuation, in deference to his superb grasp of the Greek mathematical language, and in order to facilitate simultaneous use of my translation and Heiberg’s edition.

11. Greek diagrams can be characterized as “qualitative” rather than “quantitative.” This is very difficult to define precisely and is best understood as a warning: **do not assume that relations of size in the diagram represent relations of size in the depicted geometrical objects**. Thus two geometrical lines may be assumed equal, while their diagrammatic representation is of two unequal lines, and, even more confusingly, two geometrical lines may be assumed *un*equal, while their diagrammatic representation is of two *equal* lines. Similar considerations apply to angles etc. What the diagram most clearly *does* represent are relations of connection between the geometrical constituents of the configuration (what might be loosely termed “topological properties”).

12. I make an effort to use the same English word for the same Greek word, when the Greek word seems to be used in a formulaic context only. This is true, I believe, for such particles as οὖν and δή, which become “now” and “so” respectively. “Now,” however, is to be read without the sense of time, “so”

without the sense of consequence. Both are used in the sense of a colloquial interjection marking an emphatic transition, somewhere between a full stop and a new paragraph. Two words which I do not translate formulaically are δε and και. The first is often left without translation (as all it means in the Greek is the absence of asyndeton), sometimes with "and," "while" or "but" (but never "on the other hand"). και is "and," "as well," "also" or "even." These words are so ubiquitous that they have escaped being regimented by the Greek formulaic language, and the Greek reader does not read them as "mathematical."

There is no extant commentary on Archimedes' *On Spirals*. However, the manuscripts descended from codex A contain a small set of scholia that may go back to early Byzantine or late ancient times (they could well be roughly contemporary with Eutocius, two of whose commentaries were included in the previous volume). I include these scholia in an appendix (Appendix 2), and also translate a passage from Pappus' *Collection* (IV.21–25) (Appendix 1) that reports on another, alternative approach taken by Archimedes to the results concerning areas.

In this volume, unlike the preceding one, I do not set apart textual comments (as the text presents far fewer difficulties).[4] I generally translate the text as printed by Heiberg, and when I differ from it in non-trivial ways this is accounted for by footnotes or, rarely, the comments themselves. In Volume I, I divided the comments into smaller headings, whereas now I

[4] Since the publication of Volume I there has appeared an important study relevant to the textual history of Archimedes. A study, indeed, of a topic long neglected: the Renaissance Latin translation of the works of Archimedes by Jacob of Cremona (d'Alessandro and Napolitani 2013). The authors illuminate the circumstances of the making and dissemination of this translation. Relevant to our immediate purpose is their further suggestion that Jacob could have relied on a manuscript distinct from the codices A, B or C. Now, when considering such possibilities, base probabilities (or, in other words, common sense) should not be neglected. What is the likelihood of a manuscript, distinct from A, circulating in Italy up till the fifteenth century without leaving any trace other than Jacob's own translation? It is for a reason that both Heiberg and Clagett did not seriously entertain such a scenario. Next, d'Alessandro and Napolitani should have paused to consider the fact that Jacob's selection of works is identical to that of A, his order near-identical. (Note that all three manuscripts, A, B and C, drastically differ in selection and order. It is thus very unlikely that a manuscript independent from A would have had just the selection and order displayed by Jacob.) This alone makes it extremely likely that Jacob had A at least as his primary template. I am frankly surprised by the evidence supposedly to the contrary adduced by d'Alessandro and Napolitani. They provide a few examples where Jacob's text produces *better* mathematical sense than that of codex A (and they also note a systematically distinct diagrammatic set of principles followed by Jacob). Of course, a systematic survey showing multiple cases where Jacob provides *worse* mathematical sense than A – implying a source with its own distinctive corruptions and lacunae – would have been the evidence required for their case (while a translator, papering over lacunae and corruptions in his source material, comes not as a surprise at all).

The balance of philological probability was weighed down by a ton of prior probability against the theory of a lost Greek source to Jacob's translation. D'Alessandro and Napolitani drop a feather onto the other scale, and the balance remains unmoved.

usually present them as a continuous, discursive text. This difference has to do with a difference in the character of the commentary, on which more below.

I noted in the first volume that the comments have the character not so much of a reference work as of a monograph. Indeed, I went on to explain that I concentrated on what I perceived to be of relevance to contemporary scholarship (Netz 2004b: 4). In other words, my goal has been to introduce Archimedes' text – in a form as close as possible to the original – to a wider audience, and to show how this text can be made relevant to contemporary research in science studies. The interest is in the character of a scientific work as a text. The first volume emphasized the cognitive and verbal texture of the text: what information is allowed to be taken over directly from the diagram? Which verbal variation can be taken to be meaningful, and which is a mere notational variation? Such questions often do not involve any authorial planning – it is the genre writing, as it were, and not the author. In today's jargon, we say that such questions are "structural" and avoid "agency." This emphasis on the structural consequences of genre was easier to accomplish with the subject matter of the first volume. The first book *On the Sphere and the Cylinder* is, relatively speaking, elementary; the second book *On the Sphere and the Cylinder* is very discrete in structure, each problem standing on its own. This stands in contrast to the intricate, elegant structures of *On Spirals*, which simply cry out for interpretation in terms of authorial design. And so the comments in this volume take, on the whole, a different character. The typical question is not "Why does the Greek mathematical genre force Archimedes to write as he does?" but rather "Given the options he had, why did Archimedes choose to write as a he did?"[5]

The comments represent, of course, my own research agenda, and the first volume was close to a monograph of mine, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (Netz 1999). Some readers may be aware that I have since published *Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic* (Netz 2009), a book whose emphasis is on the aesthetic preferences of the Hellenistic era and their impact on Greek mathematics. Netz 2009, as well, emphasizes authorial design or "agency." I do mention aesthetic considerations in this volume, too. However, the comments take a somewhat different character from those of Netz 2009 (whose introduction, indeed, took *On Spirals* as paradigmatic example). The main interest in this volume is not so much in the aesthetic preferences displayed by Archimedes' authorial choices as in their epistemic consequences. Before I move on to the detail of the comments (where such observations will be largely implicit), I will say a little more at this general level.

[5] Obviously, I did pay attention to such questions of agency in the first volume as well (nor do I neglect aspects of the Greek mathematical genre in this volume): the difference is one of emphasis.

3 THE EDGE OF CONTESTABILITY

I take Geoffrey Lloyd as my starting-point. Thanks to his work, we now see ancient science within its cultural context: Greek authors engaged in theoretical mathematics as well as in such diverse projects as Aristotle's philosophy or Galen's scientific system – because such projects resonated with a cultural meaning specific to its time and place. The public, the performative, the political are crucial, and, in all of them, Lloyd emphasizes *competitiveness*. In philosophy, it sometimes goes this far (Lloyd 1990: 97):

> Once again the competitiveness characteristic of so much Greek intellectual life and culture is in evidence . . . they may be said to share . . . one recurrent preoccupation of much Greek political and legal debate, namely the demand for the justification of a point of view – except that now, in the highest level of philosophical inquiry, this was redefined as no mere matter of what was subjectively convincing, but on the contrary one of objective certainty, an incontrovertibility secured by rigorous demonstration.

Or more widely on the Greek scientific project (Lloyd 1990: 95):

> At an admittedly very general level the ambition that unites certain mathematical, philosophical and scientific investigations was . . . to secure incontrovertible conclusions by valid deductions from premises that had to be accepted.

Lloyd is surely right that Greek science takes its meaning from the nature of Greek competition, and that, starting from the effort to refute one's opponents, a premium is placed on such pursuits where refutation can be the more effective, so that this arms race gives rise to more and more refined logic. My small qualification has to do with the notion of "incontrovertibility." I present to you, as evidence, Archimedes' practice revealed in the introduction to this treatise. As we will see below, we are not sure about the reading in one crucial place, but it is clear enough that Archimedes, in his geometrical challenge, included results which he knew to be false, so as to be able to refute his opponents, who would claim to have found their proofs (without supplying, obviously, any valid proofs of that or, most likely, without supplying any proofs at all). Now, I submit, this is not quite the mindset of the one who aims to avoid controversy. For, indeed, why did Archimedes send out a challenge in the first place? Why not send out the complete results, provided in the most fully rigorous fashion? He did not do that, and instead sent out a challenge, and a misleading one at that, so as to incite *controversy*. Thus we immediately see that the Greeks could not have pursued incontrovertibility alone because, after all, incontrovertibility, strictly speaking, forecloses controversy and so makes the cultural exchange, for the Greeks, bereft of meaning. No: the goal should be to make the most impregnable statements possible while at the same time inviting polemic concerning those statements.

Lloyd mentions Archimedes in the same study (Lloyd 1990: 89–91) as an example of the price paid by the Greeks for their emphasis on incontrovertibility. He considers Archimedes' *Method*: an argument relying on mechanical assumptions and on indivisibles. Because of that, so Lloyd notes, Archimedes considered this treatise as merely heuristic. Thus the emphasis on incontrovertibility constrained the ancient mathematician and prevented him[6] from considering results which were less than incontrovertible. Similarly, idealization in the application of the exact sciences – in Archimedes and elsewhere – was massive, exactly so as to achieve incontrovertibility, so that one ended up, in such works as *On Floating Bodies*, studying not so much physical phenomena as their geometrical counterparts. In all of this Lloyd is right, and surely Archimedes does indeed aim for the most impregnable statements possible. But a point Lloyd does not emphasize enough, I believe, is that Archimedes made a *choice* to publish such works as *The Method* and *On Floating Bodies*. That is, he chose to publish works that directly flirted with invalid argumentation (*The Method*) or that relied on strong implicit assumptions concerning the physical world (*On Floating Bodies*). This was not playing it safe, and in general Archimedes did not. In his work he sought danger, not safety. The tendency I see in him is to produce such text as provokes criticism while at the same time powerfully responding to it. It is suspended right at the most exposed point, at the edge of contestability: making exactly the most contestable claim that one can still argue for with impregnable arguments.

This we will see in the detail of our reading. In one proposition after another, one Step after another, we will see Archimedes making arguments which are valid, but only on the assumption of much more argument that is not supplied by Archimedes himself – and whose supplying is often debatable. Archimedes avoids explicit statements of the grounds for claims and often avoids the explicit statements of Steps required for the argument. He structures both the treatise as a whole, as well as individual propositions, so as to create distance between the tools used, and the results such tools are used for. And he engages with very subtle foundational issues – how is motion to be analyzed geometrically? What counts as a solution to a problem? – in a manner which tends neither to solve nor to remove the difficulty. I will not anticipate the detail of our reading of such passages, but strong examples may be found in Archimedes' reliance on *neusis* constructions[7] (propositions 5–9, see especially the comments on proposition 7), in the ambiguity concerning the construction of the spiral and especially the tangent to it (see especially the comments on propositions 13, 18) and in the argument, and application, of proposition 11 within proposition 25, which are vital to the area results. But these are just the key examples: in the comments on almost every proposition we will note an Archimedean, tantalizing style – but then will also note the validity of the underlying argument.

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[6] Here, and in what follows, I will refer to the generic ancient mathematician (as to the generic ancient reader) as male. This is not precisely true, as there were ancient female mathematicians and ancient female scientists in more general (Keyser and Irby-Massie 2008: 1029 counts thirty such authors), but a more inclusive pronoun would be somewhat more misleading. The sexism of the past cannot be willed away by our own, more enlightened grammar.

[7] A neusis construction is one in which a line, equal to a given line, is set to fulfill a geometrical configuration without a proof showing how such a configuration may be constructed.

Why did Archimedes position himself at the edge of incontestability? Partly, this does make his work engage with the cultural emphasis on debate. Now, I do not mean to say this is the only possible way in which mathematics could, in principle, relate to its culture. Mathematics could also be seen – as Lloyd suggested – as the *limiting case* of debate, the place where the debate is so effectively pursued that it becomes, paradoxically, foreclosed. Arguably, such a tendency can be read, perhaps, in Euclid. There was a variety of ways in which the logical clarity of mathematics could participate in the polemical tendency of Greek culture. Archimedes' choices reflect not just a broad cultural pattern but also specific preferences, some perhaps personal, some perhaps due to his more immediate cultural setting. In Netz 2009 I looked for the specific cultural preferences of Hellenistic Alexandria and its cultural area, and the manner in which such cultural preferences could affect Hellenistic mathematics. Hellenistic poetry – perhaps reflecting its origins in the court – emphasized the surprising, the breaking of generic boundaries, the tantalizing and the self-consciously textual. The enigmatic, tantalizing tendency in Archimedes certainly fits this mold as well. And so it is perhaps possible to argue that Archimedes' mathematics is a specifically Hellenistic way in which mathematics can engage in a culture based on competition: by tantalizingly provoking, and prevailing, in such a competition.

This is a likely enough historical explanation for Archimedes' practice. What I wish to emphasize right now is not so much the historical explanation as the philosophical consequences. If Archimedes' work can indeed be characterized as seeking the edge of contestability – the most powerful results one can argue for – then it tends in Popper's direction. It aims at the least likely or the most informative claims. To seek the edge of contestability is not so different from seeking maximal falsifiability. Now, I do not argue that is a characteristic of scientific practice as a whole, or that scientific practice is useful only to the extent that it makes such claims. But this is suggestive for the way in which authors such as Archimedes can make seminal contributions in the growth of science. There was a certain peculiar historical setting, a certain peculiar temperament, that made Archimedes. He aimed not at the mainstream of incontrovertibility but at the extremes of contestability – with the epistemic pay-off of such works as *The Method, On Floating Bodies, Spirals* . . . such are the epistemic uses of aiming at the extreme, at the edge of incontestability. If my comments are rather like a monograph, then its aim may be summed up as follows: a thesis on the epistemic contribution of the scientific avant-garde.

So much for general statements, and now on to Archimedes himself.

TRANSLATION AND COMMENTARY

ON SPIRALS

Archimedes to Dositheus,[1] Greetings.

Of those theorems dispatched to Conon,[2] about which you keep sending me <letters> asking that I write down the proofs – many you have, written down in the <books> conveyed by Heracleides,[3] while some I send you, having written them down as well in this book.

Theorem: see Glossary

[1] The recipient of Archimedes' *Quadrature of the Parabola, On the Sphere and the Cylinder* I, *On the Sphere and the Cylinder* II and *On Conoids and Spheroids*, as well as this book; likely, judged by the name, of Jewish origin – whatever "Jewish" and "origin" might mean in this context (Netz 1998b). Dositheus may have been a significant scientist on his account, but nothing in the correspondence would suggest that; perhaps he made his own original contributions later. The little that is known of his achievement (see the *Encyclopedia of Ancient Natural Scientists* [*EANS*], q.v. and references there) suggests a more astronomical, rather than geometrical, career.

What did it mean to be "the recipient of . . . "? What was the reality underlying the exchange of scientific works hinted at in introductions such as this one? This, the most extensive introduction extant from Archimedes, provides us with the clearest sense of this question, and we will return to discuss this in the general comments below. In more general terms for the question of the nature of the addressee in ancient expository works, see now Wietzke 2014: ch. 2.

[2] Conon's scientific achievement is somewhat clearer than that of Dositheus (and, thanks to Callimachus' reference to him, he comes across as a fuller historical figure: Aetia fr. 110 l.7): apparently a leading scientist of his place and time (Alexandria, mid third century BC), perhaps more an astronomer than a geometer, though definitely an original geometer as well. Recently deceased, he is mentioned very favorably by Archimedes in his introductions, suggesting that he would have been the natural recipient of Archimedes' works, rather than Dositheus.

[3] A vexing prosopographic problem: (a) Eutocius in Apoll. Con. (Heiberg 1913: II, 168.7) mentions a Life of Archimedes written by Heraclius; (b) Eutocius in Arch. *DC* (Heiberg 1915: III, 228.20) mentions a life of Archimedes written by Heraclides; (c) there is an Heraclitus mentioned by Pappus (Pappus (Hultsch 1877: II, 782.5) for a certain geometrical achievement which might antedate Apollonius; (d) all of these may or may not be the same as each other and as the Heraclides mentioned in this *SL* introduction. Bernard, in his *EANS* entry s.v. Heraclitus, follows Decorps-Foulquier

Nor should you wonder why I took such a long time publishing their proofs: this came about because I wanted to allow those who busy themselves with mathematics to take up studying those <theorems> first. For how many of the theorems in geometry appear not to go along the right lines at first, to the one who eventually perfects them? Conon passed away without taking sufficient time for their study; otherwise he would have made them all clear, discovering them as well as many others, while advancing geometry a great deal: for we know that his mathematical understanding was extraordinary, his diligence unsurpassed.

With many years now having passed since Conon's death, we are not aware of even a single problem being set in motion, not by a single person.[4]

I also wish to set out each of them, one by one. For it happens that <there are> a certain two of the <theorems> in it <=the letter to Conon>, not distinguished apart but added at the end,[5] so that those who claim to find all of them, but publish none of their <=the theorems'> proofs, would be refuted by promising to find solutions to impossible theorems. So I now find it appropriate to make clear which are those problems, and which of them are

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2000: 10 n.5, against Heiberg's original index to Eutocius in Arch., as well as Knorr 1986: 294–302, in distinguishing the joint (a) and (b) (these two really must be the same author) from either (c) or (d). In truth, it would be very striking if an acquaintance of Archimedes, in the late third or early second century, would write a *Life* of his friend. But then again there are few parallels to Archimedes himself. (What comes to mind is the genre of Socratic dialogue, written by Socrates' associates soon after his death [see e.g. Clay 1990] – was Archimedes not a Socrates-like figure?) Decorps-Foulquier's plausible suggestion is that the biographer in question was Heraclides Lembus. At any rate, I note that Decorps-Foulquier's argument against identifying (a) with (d) is sound, but not compelling: she points out that Eutocius provides Apollonius' date ὡς ἱστορεῖ Ἡράκλειος, which, she argues, (i) means that the Heraclius mentioned is directly paraphrased by Eutocius (and is not just the basis of educated surmise on his part), (ii) and therefore could not be a contemporary of Apollonius (who would not provide the date of his peer!). However, (i) appears to me wrong, since Eutocius very likely relies on an intermediary source – would we not have heard much more of this biography had he possessed an actual copy? – in which case ὡς ἱστορεῖ Ἡράκλειος can well mean no more than "based, according to my intermediary source, on the authority of Heracleius": from which (ii) no longer follows.

What can we positively say? We must conclude that someone, perhaps conversant in geometry, possessing an Heracleophoric name, has traveled between Egypt and Syracuse in the second half of the third century BC. He could be the author of the solution from Pappus' *Collection* and also could have been Archimedes' biographer.

[4] Archimedes took a long time because (i) he wanted to give others a chance, (ii) he realizes it may take a long while to solve a problem well, (iii) he first gave a chance to Conon, (iv) and even following his death wanted to give a chance to everyone else.

[5] The manuscript text of this – one of the most interesting passages, historically, in all of Archimedes' works – is garbled beyond hope. I translate my own emendation, and discuss the problem below in the textual and general comments.

those whose proofs you have (which were sent to you already), and which I convey in this book.

So, the first of these problems was: given a sphere, to find a plane area equal to the surface of the sphere. This, indeed, was also the first to become clear following the publication of the book about the sphere.[6] For once it is proved that the surface of every sphere is four times the greatest circle of the <circles> in the sphere, it is obvious that it is possible to find a plane area equal to the surface of the sphere.

Greatest circle: see Glossary

Second: given a cone or a cylinder, to find a sphere equal to the cone or cylinder.

Third: to cut the given sphere with a plane, so that its segments have to each other the ratio assigned.

Segment: see Glossary

Assigned: see Glossary

Fourth: to cut the given sphere with a plane, so that the segments of the surface have to each other the ratio assigned.

Fifth: to make the given segment of a sphere similar to <another> given segment of a sphere.[7]

Given: see Glossary

Sixth: given two segments of a sphere, whether of the same <sphere> or of another, to find a certain segment of a sphere, which will itself be similar to one of the <given> segments, while it will have a surface equal to the surface of the other <given> segment.

Similar: see Glossary

Seventh: to cut off a segment from a given sphere with a plane, so that the segment has to the cone having the same base as the segment and an equal height, an assigned ratio greater than that which three has to two.

Ratio greater: see Glossary

The proofs of all these theorems mentioned, then, Heracleides conveyed. But the one positioned, separately, following them was false. It is: if a sphere is cut by a plane into unequal <segments>, the greater segment has to the smaller a ratio duplicate than the greater surface to the smaller. That this is false is clear through what was sent before, for it has been set out among those as follows: if a sphere is cut by a plane into two unequal <segments> at right <angles> to some diameter of those in the sphere, the greater segment <of the surface> shall have to the smaller the same ratio, which the greater segment of the diameter <has> to the smaller; indeed, the greater segment of the sphere has to the smaller a ratio smaller than that duplicate that which the greater surface has to the smaller, but greater than half-as-much-again.

Ratio duplicate: see Glossary

Half-as-much-again: see Glossary

And the last one separated from among the problems was a falsehood, too: that if the diameter of a certain sphere is cut so that the square on the greater segment is three times the square on the smaller segment, and through the point <where the diameter was cut> the plane drawn at right <angles> to the

The square on: see Glossary

[6] Archimedes refers to what we know as "the first book *On the Sphere and the Cylinder*" (clearly considering it as a separate publication from what we now call "the second book").

[7] Presumably what Archimedes has in mind is *SC* II.5: the problem of constructing a segment of a sphere, whose volume is that of a given segment, so that it is also similar to another given segment. (This may be thought of taking a given segment, i.e. a given volume, and reshaping it to become similar to another given segment.)

diameter cuts the sphere, the figure of this kind, viz. the greater segment of the sphere, is the greatest among all the other segments which have a surface equal <to it>. That this is false is clear through the theorems sent before: for it has been shown that the hemisphere is greatest among the segments of a sphere, contained by an equal surface.

Following those, the problems concerning the cone are these.[8]

If a section of a right-angled cone should be rotated, the diameter remaining fixed, so that the diameter is the axis,[9] let the figure drawn by rotation by the section of the right-angled cone be called a "conoid"; and if a plane touches the conoid figure, and another plane, drawn parallel to the touching plane, cuts a certain figure of the conoid, let the cutting plane be called "base" of the segment cut off, and <let> the point at which the other plane touches the conoid <be called> "vertex."

So, if the figure mentioned is cut by a plane at right <angles> to the axis, it is clear that the section shall be a circle, but it is required to prove that the segment cut off shall be half as much again of the cone having a base the same as the segment and an equal height.

And if two segments of the conoid should be cut by any planes drawn in whichever way, it is clear that the sections shall be sections of acute-angled cones if the cutting planes are not perpendicular to the axis, but it is required to prove that the segments shall have to each other the ratio which the <lines>, drawn from their vertices to the cutting planes and parallel to the axis, have to each other, in square.

The proofs of these have not yet been sent to you.

Following these, the problems offered concerning the spiral were these:

– and they are, as it were, a certain other class of problems, having nothing in common with those mentioned above; the proofs concerning which we have written for you, in this book. –

They are as follows:

If a straight line, being rotated in a plane in uniform speed, with one of its ends remaining fixed, should be returned again to where it started from, while at the same time, even as the line is rotated, a certain point is carried along the

End: see Glossary

[8] In what follows Archimedes uses the pre-Apollonian nomenclature for conic sections, where "the section of a right-angled cone" means our "parabola," "acute-angled cone" means our "ellipse" and "obtuse-angled cone" our "hyperbola" (for more on the history of the nomenclature, see Jones 1986: II.400). By rotating such conic sections around their axis we may produce solids, and Archimedes calls them, based on visual analogies, as follows: what we call the "paraboloid of revolution" (a parabola, rotated) he calls "conoid"; what we call the "ellipsoid of revolution" (an ellipse, rotated) he calls "spheroid." The results discussed here would be provided by Archimedes in a future letter sent to Dositheus now known as the book *On Conoids and Spheroids*.

[9] Perhaps meaning: so that the diameter is *called* "axis" of the resulting figure. (Archimedes did not yet introduce the *word* καλείσθω into the sequence of this particular text, but he is in a definitive mode nonetheless.)

line, in uniform speed with itself,[10] starting at the fixed end, the point shall draw a spiral in the plane. So, I claim that[11] <1> the area contained by the spiral and by the line that has been returned to where it started from is a third part of the circle drawn, with the fixed point as center and with the line traversed by the point in one rotation of the line as radius.[12] And[13] <2> if some line touches the spiral at the end of the spiral which is the last,[14] while some other line is drawn from its fixed end at right <angles> to the line rotated and returned to position, so that it meets the tangent, I claim that the line produced towards <the tangent>[15] is equal to the circumference of the circle. And[16] <3> if the rotated line and the point carried along it are carried around for many rotations and are returned to where they started from,[17] I claim that – compared to the area added

Touch: see Glossary

[10] I.e. the speed is only "uniform with" *itself* and need not be "uniform with" the previously mentioned speed of the line in rotation (whatever such being "uniform with" might mean).

[11] Proposition 24. It is important to add at this point that Archimedes' readers would not have had this footnote. It is not only that they would have been unable to find where such proofs were offered: they would have had an extraordinarily hard time even figuring out what such proofs even meant. The phrasing offered by Archimedes is opaque, which only becomes worse from one claim to the next. In the practice of reading the work, most readers would have relied on labeled diagrams and an explicit setting-out, as well as definition of goal, to work out the precise meanings – all while reading the propositions themselves. By not being able to refer at this point to the proposition itself with its diagram, then, Archimedes would have made it much more difficult to even know, in advance, what the results are that would justify the treatise. One would have to read to the end, to know the very point of what one was reading.

[12] The word *line* in the phrase "the line$_1$ traversed by the point in one rotation of the line$_2$" has two separate meanings. Line$_1$ means the finite line traversed through the rotation. Line$_2$ means the indefinite line extending out of the fixed point and rotated about it.

[13] Proposition 18.

[14] We concentrate on the scenario of a spiral rotated once, fully – the only scenario provided so far. Under this scenario, the spiral has two ends, its fixed point and then the one which is the "last" – the point reached at the end of the rotation. It is this end specified by the phrase "the end of the spiral which is the last." The term "end of the spiral" is never defined and is of course problematic (the spiral has only one real "end," its start; the other end is wherever you wish to put it; it is therefore a relative term).

[15] By "line produced towards <the tangent>" is meant the segment that the tangent cuts off from the line produced from the center, orthogonal to the start/end position of the rotating line. Archimedes is being economical here (note that "produced towards" is a single adjective in Greek, ποταχθεῖσα, and that "circle" is, this time, left undefined), so as to make his conclusion as succinct as its elegance deserves.

[16] Proposition 27.

[17] A loose expression: only the line is to be returned, while the point is to continue its outwards uniform progression – a point left moot by Archimedes' description.

by the spiral in the second rotation[18] – the <area> added in the third <rotation> shall be double, <the area added> in the fourth <shall be> triple, <the area added> in the fifth <shall be> quadruple, and always: the areas added in the latest rotations shall be multiples, according to the numbers in sequence, of the <area> added in the second rotation; while the area contained in the first rotation is a sixth part of the area added in the second rotation. And[19] <4> if two points should be taken on the spiral drawn in a single rotation, and lines are joined from them to the fixed end of the rotated line, and two circles are drawn with the fixed point as center, and the lines joined to the fixed end as radii, and the smaller of the joined <lines> is produced,[20] I claim that the area contained by: (i) the circumference of the greater circle which is on the side of the spiral between the <two> lines and (ii) the spiral itself and (iii) the produced line, has to the area contained by: (i) the circumference of the smaller circle and (ii) the spiral itself and (iii) the line joining their <=the two lines> ends, the ratio which the radius of the smaller circle, together with two thirds the excess by which the radius of the greater circle exceeds the radius of the smaller circle, has to the radius of the smaller circle, together with one third part of the mentioned excess.

Joined: see Glossary

Excess: see Glossary

So, the proofs of these and other <theorems> concerning the spiral[21] are written by me in this book, and preceding them (as is also <the case> with <any> other <books provided> in a geometrical way)[22] are the <theorems> required for their <=the four theorems stated above> proofs. And I also adopt, in these theorems, this lemma, which is also among the <lemmas in> the books sent out before: that, among unequal lines and unequal areas, the excess by which the greater exceeds the smaller, itself added onto itself, is capable of exceeding every given <magnitude>, of those which are said to be <in a ratio> to each other.[23]

Lemma: see Glossary

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[18] Area "added": i.e. we do not look at the entire area covered by the spiral through its two rotations, but only at the area which the spiral has covered through its second rotation, *excluding* what it had already covered through its first rotation.

[19] Proposition 28.

[20] "Produced" in this context means "extended beyond the smaller circle to reach the circumference of the greater one."

[21] A significant hedge, made in passing: Archimedes does not commit himself to provide only those four results (as well as those theorems required for their derivation).

[22] Literally "geometrized" (passive participle form of the verb derived from "geometry"). This is an intriguing, and unique, usage by Archimedes. The thing "geometrized" is apparently the textual-conceptual unit of a mathematical theorem or a book, and the property designated as "being geometrized" is stylistic as well as epistemic: the text is furnished in a certain way and for this reason meets a certain logical standard. The other references are: *QP* 266.2, *Meth*. 438.20, 486.7 (conceivably this could also be read in 430.24, which is the Archimedes palimpsest ARCH16rcol. 1 ll. 19–20, Netz et al. 2011: 73). In all those other cases, the contrast at hand is with proofs furnished through "mechanical" methods; the usage here is wider (as perhaps one should understand the parallel passages as well).

[23] How do we know that a straight line, multiplied a certain finite number of times, can be made to exceed a circular line? Since they cannot be made to coincide, such a claim cannot be made intuitively clear, and so the lemma is required. Whatever the original motivation of this lemma, it sets out a condition for what "being a ratio" means and as such would become entrenched in modern mathematics.

COMMENTS

As mentioned in n. 5 above, the text of a certain passage appears to have been badly garbled in codex A (our only source for the Introduction). The reading of the codex may be reconstructed as follows:

δυο τινα των εν αυτω μη κεχωρασμενα τελους δε ποτεσσομεν

This is not Greek. Two words are unrecognizable as they stand: κεχωρασμενα, ποτεσσομεν. The genitive case of τελους is unmotivated, while the significance of the δε is difficult to make out (hardly surprising, seeing that this particle is sandwiched between a word with a strange case and a non-word).

Heiberg in his first edition emended this, fairly moderately, to read

δύο τινὰ αὐτῶν ἐν αὐτοῖς μὲν κεχωρισμένα, τέλος δὲ ποθεσόμενα

which he translated into Latin as

quaedam eorum inter ea collocate sint, confici autem non possint

and which Heath renders as

two included among them which are impossible of realization

In his second edition, Heiberg returned to the same passage and revised it radically to read:

δύο τινὰ τῶν ἐμαυτῳ μήπω πεπερασμένων διὰ τέλους ποτιτεθῆμεν

which he translated into Latin as

duo quaedam eorum, quae a me ipso nondum prorsus ad finem producta sunt, insuper addita sint

and which Mugler renders as

deux de ces problèmes, que moi-même je n'étais pas encore arrivé à mener à bonne fin, ont été ajoutés à leur liste

The emendation I follow is

δύο τινὰ τῶν ἐν αὐτῷ μὴ κεχωρισμένα τέλος δὲ ποθεσόμενα

which I translate as

<there are> a certain two of the <theorems> in it <=the letter to Conon>, not distinguished apart but added at the end

Heiberg's first emendation is difficult as Greek. The position of the μεν is forced, and its supposed adversative function in the μέν–δέ pair is obscure. Nor are the meanings ascribed to κεχωρισμένα, ποθεσόμενα at all natural. It is not for nothing that Heiberg has abandoned it for the sake of a radical revision, in his second edition. (This revision, I hasten to add, is based on no significant paleographic finds: to repeat, the palimpsest is not extant for this passage.)

Heiberg's second emendation involves a radical intervention in the manuscript text, essentially rewriting it while ignoring the paleographic evidence. Worse, it makes for strange historical sense. If I understand Heiberg correctly, his view was that Archimedes did not yet solve those problems but was working on them in good faith, on the assumption that they were right. But how could that make sense for Archimedes the mathematician? For Archimedes the communicator? (So the task was sent out even before Archimedes has worked on it?) How would it make sense for the Greek, even? For the natural syntax of the continuation of the passage is that those two theorems were added *for the purpose* of refuting Archimedes' rivals.

My emendation is very light – it merely corrects an obvious slip of the pen, from κεχωρασμενα to κεχωρισμένα; and then follows Heiberg's lead in conjecturing a Doric form that would have been surprising to the Byzantine scribe, resulting in the corruption of an original ποθεσόμενα into the meaningless ποτεσσομεν; I also change the case of τέλος, from genitive to accusative (with ποτί understood – a victim, then, to the garbling of ποθεσόμενα).[24] Unlike Heiberg, I do not emend the words ἐν αυτῷ μη at all. Mostly, I differ from Heiberg in my interpretation, offering a concrete frame of reference to the relevant words. But, in truth, with a text as garbled as this one we are reduced to uncertainty.

What is Archimedes doing in this Introduction, in this correspondence as a whole? Curiously, a large part of our answer hangs on those garbled words.

So: if we follow Heiberg of 1913 and motivate the genitive case of τέλους with the preposition διά, the word "end" gains the very different meaning of "perfect," lifting us from the concrete realm of the position of different units of text to the abstract realm of different levels of mathematical achievement.

The term κεχωρισμένον is, in the Archimedean corpus, unique to this passage. But its central meaning is easy – being set apart (it is a key term of Aristotle's philosophy of mathematics, for instance, where mathematicians merely act *as if* mathematical objects were set apart from the material ones). It can be understood here in a concrete sense – being set apart in the sequence of writing – or in some metaphorical sense of "being distinguished." But it is not a bland way of saying "stated." The fact that it is used not once but three times in the continuation of this passage, always in the context of the false theorems, seems to be significant. Here is what Archimedes says. The first false theorem was "positioned, separately, following these" (μετὰ ταῦτα κεχωρισμένον); the second was "last separate, among the problems" (ἔσχατον κεχωρισμένον). It is in this context that Archimedes refers to a result established in the second book *On the Sphere and the Cylinder* as κεχώρισται ἐν αὐτοῖς – "to be found distinguished among them." In short, there is a good, concrete sense of κεχωρισμένον we can assign to two of the following usages, and we can understand the third, metaphorical usage (κεχώρισται ἐν αὐτοῖς) as motivated by the κεχωρισμένον-heavy context. Indeed, the term ἔσχατον below seems to point to a relevant sense of the word τέλους in our passage, which is also reminiscent of a passage in the introduction to *The Method* (Heiberg 1913: II,

24 Parallels for the use of *pros . . . telos* are usually in the context of a trajectory of motion whose very end is reached (e.g. Plato, *Republic* 494 a12: "walk <metaphorical for 'practice'> philosophy to the end." Perhaps the intended meaning in this passage by Archimedes is indeed "*right at the conclusion*."

430.23–24), where Archimedes promises a certain set of propositions επι τελει του βιβλιου, “at the end of the book,” as well as of a similar passage following the analysis of *On the Sphere and the Cylinder* II.4 (Netz 2004b: 204). In short, it is likely enough that the terms κεχωρισμενα, τελους, as found in codex A, could refer to the physical setting of the two false theorems within the letter sent to Conon.

This would also explain the singular dative form ἐν αυτῷ (in it). Heiberg emended this twice, first to ἐν αυτοῖς (among them) and then to ἐμαυτῷ (by myself). It is true that the text does not have explicitly stated an immediate singular neuter (or masculine) antecedent to which the αὐτῷ can easily refer. However, conceptually, the text is all about the letter sent to Conon, perhaps understood as a βίβλιον. This would not only motivate a singular, but would also provide a frame of reference for which both κεχωρισμενα, τελους, make sense: namely, the frame of a certain booklet in which Archimedes' original challenge was set.

All in all, then, I offer an emendation which is also an interpretation: Archimedes speaks of the physical way in which he structured his letter to Conon. The two false theorems were not explicitly marked in any form, distinguished, set apart; however, they were set at the end as a kind of warning.

But, as I said, the questions of the correct emendation have drifted from the technical questions of paleography and syntax to those of history and correspondence. Let us move on to consider Archimedes' frame of reference: what was he doing in this Introduction?

THE REPUBLIC OF MATHEMATICAL LETTERS AND THE BIG LETTER

Regardless of our ultimate view on the textual question: it points our attention to what is undoubtedly of central importance in this introduction – *meta-correspondence*. Archimedes, writing this letter, writes about his writing of letters: constructing, in the process, a Republic of Mathematical Letters.

Let us consider some of the history of this republic.

Archimedes sent a letter to Conon, comprised of a set of claims and tasks, asking for their proofs. A challenge, falling under three categories: spheres, conoids and spirals. The challenge was perhaps intended initially for Conon alone (or perhaps was meant to be conveyed by him to the world at large). It certainly became public property following Conon's death.

A major event, then, in the history of the Republic of Mathematical Letters: Archimedes sent out his challenge – what I would now call *The Big Letter* – to Conon.[25]

[25] Pappus, introducing a long discussion dedicated to the spiral, briefly relates an altogether different scenario (Hultsch 1876: I, 234.1–3): Conon has proposed the first problem of the spiral (showing that it encloses an area one-third of a circle). Archimedes

What happened then? According to Archimedes, nothing. He waited for responses, expecting them to fall into two possible kinds: a detailed offer of explicit proofs; or a mere statement that the tasks and claims were obtained. (For the latter case, he had available to him a ready rebuttal – that in fact two of the claims were false, so that whoever made a mere statement of having obtained the tasks and claims must have been making a vain, ignorant boast.) But he never did receive any mathematical response to this letter, not by Conon in his lifetime or by other mathematicians. Archimedes seems to suggest that Conon was honorably painstaking while other mathematicians were cowardly silent.

The introduction to *On Spirals* was not Archimedes' first letter to Dositheus, nor his first mention of *The Big Letter*. The first letter to Dositheus was the introduction to the (extant) *Quadrature of the Parabola*. There, it is stated clearly that its main theorem – the ratio of the parabolic segment to the enclosed triangle – was new.[26] This we should understand to mean that it was not in *The Big Letter*.

The first book *On the Sphere and the Cylinder* – also extant – followed, and its introduction suggested once again that its contents were new, i.e. technically speaking outside the scope of *The Big Letter*. (That is, none of the theorems in *SC* I was directly a proof of a claim, or the fulfillment of a task, made in *The Big Letter*.)[27]

Timeline of Archimedes' Letters to Conon and Dositheus

The Big Letter to Conon (challenge)
First Letter to Dositheus: *Quadrature of the Parabola* (not in *The Big Letter*)
Second Letter to Dositheus: *On the Sphere and the Cylinder*, I
Third Letter to Dositheus: *On the Sphere and the Cylinder*, II
Fourth Letter to Dositheus: *On Spirals*
Fifth Letter to Dositheus: *On Conoids and Spheroids*

has then solved this problem – though the line of reasoning reported by Pappus differs from that used in our treatise! Knorr (1978b) went on to build a theory on this foundation: that what we have is in fact Archimedes' second treatise on the spiral (the one reported by Pappus being the first one). Taking Pappus upon his word, and then following Knorr in his order of publication, would drastically interfere with our understanding of the correspondence. I will return to discuss this in greater detail while discussing proposition 14, which brings out a likely context for the emergence of the concept of the spiral. To anticipate my conclusion there, I find it likely that Conon has invented the spiral; I find it possible (but not necessary) that he proposed the first problem of the spiral; I find it likely (though unprovable) that the result provided by Pappus is ultimately due to Archimedes; and finally that I differ from Knorr in finding it *impossible* that the result provided by Pappus preceded the extant treatise of *On Spiral Lines*. Thus I wish to keep the fundamental order of correspondence delineated in this letter.

[26] Heiberg 1913: II, 262.10: "not studied before" (so, not among those already studied for the sake of *The Big Letter*).

[27] Heiberg 1910: I, 2.6–8 (Netz 2004b: 31–32): "later <than the results of QP> . . . suggested themselves to us."

The second book *On the Sphere and the Cylinder*, also extant, finally referred directly to *The Big Letter* and asserted that Archimedes now began to publish problems fulfilling tasks set out there, though leaving aside those theorems concerning the Spiral, or Conoids.[28]

Now, Archimedes' statement in the introduction to the first book *On the Sphere and the Cylinder*, as if that book was separate from *The Big Letter*, was at least disingenuous. As he took pains to clarify in both the introduction to the second book *On the Sphere and the Cylinder*, as well as in this introduction to *On Spirals*, the main result of the *First Book* is in fact sufficient to solve one of the spherical problems of *The Big Letter*. But this is typical: the entire conduct of the correspondence from the *Quadrature of the Parabola* through *SC* I and *SC* II is consistent with the picture Archimedes draws of himself: patiently waiting – *in ambush*.

The *Quadrature of the Parabola* was there as a stopgap, as it were, providing the Republic of Mathematical Letters with something to chew on while thinking on *The Big Letter*; the first book *On the Sphere and the Cylinder* was a teaser, keeping silent concerning the (rather obvious) import of this book on the first problem of *The Big Letter*. Even while producing the spherical part of *The Big Letter*, in the second book Archimedes kept silent, again concerning the (rather obvious) import of that book on the two spurious theorems. The homology should be emphasized – in the sequence of publication:

$$SC\ \text{I} \rightarrow SC\ \text{II} \rightarrow SL$$

Each book leads to key results that have a bearing on *The Big Letter*, revealed in a later book:

Book	Result	Bears on	Bearing revealed in
SC I	surface of sphere	*Big Letter*; first problem	*SC* II
SC II	two final theorems	*Big Letter*; false theorems	*SL*

Throughout, then, the picture is of Archimedes waiting for the audience to be caught in surprise: revelations are always retrospective, and the art of Archimedes thrives in the realm of the tantalizing promise – the defining characteristic of *The Big Letter* itself.[29]

[28] Heiberg 1910: I, 168.3–170.2 (Netz 2004b: 185–186).

[29] This observation further undermines the idea implicit in Heiberg's second emendation, as if Archimedes' two false theorems were sent out in error. For if so, why wait to correct the error for *after SC* II? Having said that, it should be stressed that the secret revealed retrospectively in each book is quite elementary: one would need to be rather obtuse not to see the bearing of *SC* I on the first problem, the bearing of *SC* II on the two false theorems. Archimedes reveals retrospectively, in a flourish; to reveal such things in situ would have been a bit of an anti-climax as, in context, the revelation would appear a tad *too* obvious.

This introduction to *On Spirals* is the key moment in the correspondence, as it is the moment where the central revelation is made and the correspondence's sting is removed: that *The Big Letter* contained two false theorems. This brings us back to the textual problem.

THE REPUBLIC OF MATHEMATICAL LETTERS AND THE FALSE THEOREMS

Archimedes must have experienced considerable anxiety, having just sent out his *Big Letter*. For one response he would not have welcomed would be an exposition of the error of the two false theorems. How would he be able to defend himself from the accusation of having committed an error? It seems – and here I go out on a limb – that, in this letter, Archimedes also reveals the line of defense he laid out ahead of time: he implanted a textual clue that would help to show, retrospectively, that he always did realize the peculiar character of those two theorems. Namely, they were positioned out of order: while the tasks were all arranged by the three headings of sphere, conoids and spiral, in that order, the two false theorems, although they clearly relate to the sphere, are positioned *right at the end*, following all three types. This, then, is the sense I read into Archimedes' words: "For it happens that <there are> a certain two of the <theorems> in it <=the letter to Conon>, not distinguished apart but added at the end, so that those who claim to find all of them, but publish none of their <=the theorems'> proofs, would be refuted by promising to find solutions to impossible theorems." Typically, Archimedes does not tell us, even now, that the theorems were false.[30] This he will tell us later on, in this introduction, as he comes to describe them one by one, when he says: "That this is false, is clear through what was sent before . . . " This comes as a double shock to the reader: once, that a theorem sent out by Archimedes was in fact wrong; and again, that this falsehood was in plain sight for some time already – ever since the publication of *SC* II. Thus the words "so that those who claim . . . would be refuted" does not yet assume that the reader knows that the two claims are false (and merely sets up the suspense). The "so that" clause refers to the special textual mark of extraneous position, which would allow Archimedes to hit back at his opponents, should they be sharp enough to discover the falsehood on their own.

So Archimedes is saying, effectively, as follows: I did not mark the theorems apart in any explicit way (this, after all, would have been to give the game away); but I did mark them implicitly by positioning them at the end. Thus they were in some sense μὴ κεχωρισμένα – not explicitly distinguished apart – but, in another, they were κεχωρισμένα (as he will refer to them later on) – namely, by being positioned right at the end (and not in their expected position, as part of the spherical sequence).

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30 This follows from my emendation and reading of the Greek. On Heiberg's first emendation, this passage already claims – though in Greek which is very difficult to construe! – that the claims were wrong.

We do not have a copy of *The Big Letter*. If we had one, we could see for ourselves where the two false theorems were positioned. As it is, we are reduced to guessing. My guess is based on an emendation of a garbled passage, and on the hunch that Archimedes ought to have some ready-made mechanism of getting back at his opponents. (Or perhaps there were some more hints, ammunition that Archimedes kept for the next round: one can imagine all sorts of acrostics, say, hidden in the original letter.)[31] The continuation of the introduction can be read, I believe, to support my hypothesis.

Archimedes makes several incompatible claims about the position of the false theorems:

1 "The proofs of all these theorems mentioned, then, Heracleides conveyed. But the one positioned, separately, following them [τὸ μετὰ ταῦτα κεχωρισμένον] was false." (Followed by the content of the first false theorem, and then:)

2 "And the last one separated from among the problems [τὸ ἔσχατον κεχωρισμένον τῶν προβλημάτων] was false, too." (Followed by the content of the second false theorem, and then:)

3 "Following those [μετὰ ταῦτα], the problems concerning the cone are these."

The natural reading of claim 1 is that the first of the two false theorems followed those on the sphere, but with a certain separation; this agrees well with my reading of the passage, as if the two false theorems were positioned following those on both conoids and spiral. Claim 2 seems to say unequivocally that the second false theorem was positioned right at the end of *The Big Letter* as a whole. Claim 3, however, is problematic: it seems to suggest that the two false theorems preceded the theorems on the cone (by which is meant the conoid). But this tension between claims 2 and 3 is inevitable on any interpretation of this passage. Indeed, a natural resolving of this tension is to suggest that the phrase μετὰ ταῦτα in claim 3 means what it did in claim 1: after the entire chunk of sphere-related problems. Archimedes reverts here to thinking in terms of the grand tripartition into results having to do with

[31] In the manner of Aratus, perhaps, hiding acrostics in his poem (Jacques 1960)? Of course, an acrostic fits poetry better than prose. How about an anagram? To quote a famous passage from Newton's letter to Oldenburg, November 3, 1676 (from Scriba 1963: 123): "Nevertheless – lest I seem to have said too much – inverse problems of tangents are within our power, and others more difficult than those, and to solve them I have used a twofold method of which one part is nearer, the other more general. At present I have thought fit to register them both by transposed letters, lest, through others obtaining the same result, I should be compelled to change the plan in some respects. 5accdml0effhlti413m9n6oqqrSsllt9v3x: i lab3cddl0emgl0ill4m7n6o3P3q6rSslttSvx3acm4egh 5i414m 5n8oq4r3s6t4vaadd-meeeeeiijmmnnooprrrsssssttuu." Newton's paranoia contrasts with Archimedes' pleasure of the ambush. Newton felt hunted: Archimedes was hunting. But both faced the same structural problem: how to assert something in public while keeping some part of it secret?

spheres, conoids and spirals, respectively. And then it is indeed true that, under my hypothesis as well, the results concerning the conoid were after (μετὰ) the results concerning the sphere (ταῦτα: but ταῦτα now refers to the results concerning the sphere, *excluding* the two false results).

The text is consistent with my guess concerning the structure of *The Big Letter*. The text does not prove it, however, and for a good reason: Archimedes would not be explicit about the structure of *The Big Letter*, because Dositheus would be familiar with it anyway. Hence the cryptic references to the μετα ταυτα: Dositheus would unpick those references based on his knowledge of the preceding texts. *The Big Letter* provides this introduction with its basic framework.

This letter was written for those who were part of the circle of correspondence: for the members of the Republic of Mathematical Letters.

THE REPUBLIC OF MATHEMATICAL LETTERS AND THE CHARACTER OF ARCHIMEDES' MATHEMATICS

This letter is clearly a piece of meta-correspondence, telling its reader about its relation to past letters: it fulfills the promise of a certain past letter, a promise only partly fulfilled by other past letters; it removes the sting from a previous correspondence, by pointing (I suggest) to a textual feature of the original letter. It reacts to previous letters from Dositheus, which asked again and again for Archimedes' own letters. It refers to the concrete act of conveying letters, by concrete individuals such as Heracleides.

Among other things, this is a story of Archimedes' relationship with a certain network: Conon, Dositheus, Heracleides and unnamed mathematicians. The network is fragile – Conon dies, disrupting it. It is weak – it does not respond. And it is certainly marked by tension more than by goodwill.

And this is also a story which touches upon the historical unfolding of Archimedes' mathematics. This is not a story incidental to Archimedes' mathematical career: to the contrary, this *was* his career. Throughout, we see that Archimedes publishes in the context of a particular communicative agenda within the network. He sends out a letter so as to challenge. He sends out further letters so as to reveal his mastery over the challenge. Archimedes did not just produce mathematical theories – pieces of text independent of context – which he then communicated in a certain way. A treatise by Archimedes does not merely set out the truth of a theory. It does more: divulging and withholding information, engaging with a concrete network of correspondents.

Hence the discussion of time. Archimedes tells us that he took a long time discharging the obligations of *The Big Letter*, as he still does, in fact – the conoids are not yet published. Why not publish everything at one go? Perhaps, to take the time to achieve perfection. (Is Archimedes worried of possible criticism of *imperfect* publications?) But if so, why publish the claims? Perhaps, to establish priority? (Conon, unlike the unnamed mathematicians, was a real menace in this regard.) Certainly one senses the competitive tension in words such as "nor should you wonder why I took such a long time" – Archimedes' challenge requires time; time is also a source of anxiety, as it opens Archimedes to the blame of procrastination or perhaps that of a false claim.

Hence the meaning of the false theorems: to deny the possibility of claiming priority, by claiming to have solved everything without the actual proofs to support such claims. (Was Conon in on the secret?) Anyway, we interpret this: the time-delay was functional, meant to allow certain movements in the network: first Conon, then unnamed mathematicians, are allowed an opportunity to react.

Consider Archimedes' words as he moves to the detail of the letter: "I also wish to set out each of them, one by one [because of the false theorems] . . . So I now find it appropriate to make clear which are those problems, and which of them are those whose proofs you have (which were sent to you already), and which I convey in this book." Why "wish" (βούλομαι)? Why "find it appropriate" (δοκιμάζομες)? Is it not natural that, introducing a mathematical text, one would set out its contents? But this is not some neutral table of contents. Archimedes teases apart the components of past correspondence so as to distinguish the doable from the impossible, the done already from that which is merely promised, zeroing in on the task set for the immediate book. The needs of explanation are a function of a position in the system of correspondence.

At the level of the challenge, the same is true of proof itself: the needs of proof are a function of a position in the system of correspondence. We do not have *The Big Letter* – the challenge itself. But we come closest to it with the section on the conoids, where Archimedes still does not divulge his proofs. It is worth quoting again from this passage:

> if the figure mentioned is cut by a plane at right <angles> to the axis, it is clear that the section shall be a circle, but it is required to prove that the segment cut off shall be half as much again of the cone having a base the same as the segment and an equal height.
>
> And if two segments of the conoid should be cut by any planes drawn in whichever way, it is clear that the sections shall be sections of acute-angled cones if the cutting planes are not perpendicular to the axis, but it is required to prove that the segments shall have to each other the ratio which the <lines>, drawn from their vertices to the cutting planes and parallel to the axis, have to each other, in square.

The meaning of "it is required to prove," δεῖξαι δεῖ, used twice here, is especially interesting. This is elsewhere a formulaic variation on the λέγω ὅτι – as the Greek mathematician sets out the task of a particular proof (in the "definition of goal," Proclus' διορισμός). It is thus merely a way of stating that "the claim at stake can be formulated as . . . ." Here, however, it has an added, *relative* component. The two claims are such that one is *required to prove* them, as opposed to other claims which are *clear*. The phrase "it is required to prove" is thus reinvested with meaning: "the claim at stake is not in and of itself clear, so that it is required to prove it." And it is invested with meaning precisely because the requirement is not impersonal: it is Archimedes' manner of staking out the challenge. The requirement is imposed on those who take up

Archimedes' challenge: they need not bother with that which Archimedes deems "clear," but they do need to grapple with this, the more difficult part of the task. Thus the need for proof is not some abstract, logical property: it is a concrete demand, made of humans in implicit competition.

The very axiomatic structure is imbued with this localized atmosphere, is made subordinate to the logic of the network of correspondence. Here are Archimedes' concluding remarks:

> So, the proofs of these and other <theorems> concerning the spiral are written by me in this book, and preceding them (as is also <the case> with <any> other <books provided> in a geometrical way) are the <theorems> required for their <=the four theorems stated above> proofs. And I also adopt, in these theorems, this lemma, which is also among the <lemmas in> the books sent out before: that, among unequal lines and unequal areas, the excess by which the greater exceeds the smaller, itself added onto itself, is capable of exceeding every given <magnitude>, of those which are said to be <in a ratio> to each other.

Are we right to detect a certain defensive tone? Archimedes does not commit himself to produce just the proofs pertinent to the four results. He explains that there is some preceding material given which is necessary to our understanding; that this is the thing done in geometrical books. And the use of the lemma is defended on the basis that it is among such material in previous books, it is already established within the correspondence. It would be against the rules of the game, apparently, to question it now!

"The rules of the game": because it is difficult to conceive of the correspondence other than as a competition, a tournament – one-sided as it is. Archimedes always seems to be, in some sense, arrayed *against* his audience. The entire correspondence is based on the challenge; the false theorems were there to unmask would-be takers up of that challenge. It is this sense of competition which provides the theme of surprise with its saliency. Archimedes' tournament of challenging and unmasking, waged against his contemporary mathematicians, insensibly slides into his campaign of expectations, raised and quashed, waged against us, his readers. I followed this theme of surprise in my own introduction to Netz (2009), and so I expand here on this account.

We have already pointed out the theme of retrospective surprise guiding Archimedes' correspondence: first, make a challenge; then, answer it; only then, retrospectively, clarify what your answer did to the challenge. I did mention, in this regard, the way in which (under my reconstruction) Archimedes does not even say, at first, that the two false theorems were *false*. But this, of course, is a matter of small detail and of conjectural reconstruction.

More centrally, then – and quite simply, too – Archimedes never tells Dositheus *what this letter is about*. It reads mostly like a commentary on *The Big Letter*, and for much of the reading one would be forgiven for assuming that the main point of this letter was to divulge the identity of the two false theorems. There is a mention of some new results conveyed in this letter, right at the beginning; but what those new results might be is left veiled and is probably not at the forefront of the reader's attention. Still, the reader familiar with *The Big Letter* would prick up his ears, once the transition was made to the problems "having to do with the cone" (the conoid ones). Surely, the reader will assume, those are the problems to be studied in this new letter: especially since this would mean that *The Big Letter* was tackled *in order*. It had, after all, three major parts, in this order – sphere, conoids and spiral: so the conoids, logically, should follow the sphere.

And yet, only after Archimedes lists the conoid problems does he abruptly state that "the proofs of these have not yet been sent to you."(Still leaving some hope, perhaps, that they are sent *now*?) And then simply drops the subject, moving on to the spiral. At this point – finally! – Archimedes breaks off the discussion to claim – for the first time in his correspondence, apparently – that these problems are "as it were, some different kind of problems, having nothing in common with those mentioned above" (so why were they even lumped together with them? And just how are they so different?) Never mind! The proofs, Archimedes proceeds to tell Dositheus, are now written down. This leads to a very long and complicated description of problems, one which is very difficult to unpack by a reader not already familiar with the study of spirals (little wonder that no one picked up the subject). The order of the problems, as we will note below, is not precisely that of their solution in the book (analogous to the way in which the sequence of *The Big Letter* – (1) sphere, (2) conoids, (3) spiral – was only partly reflected by the sequence of Archimedes' treatises setting out his results: (1) *SC* II, (3) SL, (2) CS).

INTRODUCING THE SPIRAL

One final surprise lies in store: the reader may well expect some introductory content setting out definitions and special lemmas required (in the manner of QP and *SC* I sent before; that *SC* II had none is less surprising, as its subject matter closely resembled that of *SC* I). Instead, Archimedes states that he requires one thing (Archimedes' Axiom, or the lemma concerning the exceedability of magnitudes in ratio), which, however, is not special to this book but used already in other books. He then moves on – as we will see – without further warning, to the sequence of theorems. Thus, we learn to our surprise, whatever introduction one needs for *On Spirals* is to be lifted out of the retelling of *The Big Letter*.

This, it appears, is reduced to the definition of a spiral (there is a surprise lying in store, for which we need to wait till after proposition 11). Let us stop and say something regarding this definition. I would not, after all, like to appear as a rabid sociologist of knowledge who sees mathematical networks everywhere and mathematical concepts nowhere. There is a concept here, and a startling one too: the spiral. It might be original to Archimedes, perhaps due to Conon himself, but, in any case, it appears to be a recent mathematical object.[32] Its definition is not a matter of rephrasing old, established knowledge; it is a matter of setting up a new domain for investigation.

[32] See n. 25 above, as well as the general comments on proposition 24.

The way in which Archimedes does this is of some interest. For the definition is neither completely general nor completely abstract. Archimedes made a choice to introduce the object in a concrete and somewhat narrow manner.

It is obvious how the definition is concrete, in that it relies upon actual motions:

> If a straight line, being rotated in a plane in uniform speed, with one of its ends remaining fixed, should be returned again to where it started from, while at the same time, even as the line is rotated, a certain point is carried along the line, in uniform speed with itself, starting at the fixed end, the point shall draw a spiral in the plane.

There is of course nothing shocking in a definition via rotation. Such is the definition of the cone, in Euclid – a rotating triangle (*Elements* XI def. 18) – as of course is that of the conoid in this very letter (and the spheroid, to come in CS: Heiberg 1910: I, 252.14–18); rotation played a major part in *SC* I, in the theorems starting from proposition 23 involving the rotation of a circle circumscribing / inscribed by a polygon, to produce a sphere circumscribing / inscribed by a sequence of segments of cones. There is, however, something quite remarkable about the kind of rotation required for the generation of the spiral. It involves the synchronization of two motions, and in this way it cannot as simply be read off as a mere shorthand for a statement of a locus. I explain. When an object is said to rotate around an axis, we can take this to define a locus of points such that each point lies on the circumference of a circle whose diameter is orthogonal to the axis, and whose radius is given by a line defined on the original rotating figure. This of course is roundabout and verbose, but it is intuitively clear that any rotation definition is in principle equivalent to such a locus definition, so that, in fact, no special reliance on motion is made when defining an object via rotation. However, once the definition involves the synchronization of two rotations, there is no obviously intuitive way of identifying the locus that the two motions produce and, instead, following the progress of the object through its combined motion becomes the only immediate intuitive way for identifying the shape of the figure. Thus one is led to consider this figure as being nothing other than a dynamic trace – and not just as a static locus whose *shorthand* is dynamic.[33]

Now, while the definition in terms of locus is not immediately apparent, it is implicit – and one thing which Archimedes deliberately eschewed was to provide such an abstract locus definition. For the spiral might, after all, be conceived as a locus, as follows. First, we define an arbitrary point of origin and an arbitrary line of origin drawn from that point. Then, each given point P in the plane determines the line drawn from that point P to the point of origin. Call this the "radius at point P." Further, this radius at point P also determines an angle: the angle between the radius at point P and the line of origin. Call this the "angle at point P." The condition of the locus of the spiral is this: that,

[33] This is a well-known observation, often made in the context of the Quadratrix, another specialized curve produced by two motions and of a more obviously ad hoc character: see e.g. Funkenstein 1986: 301ff.; an illuminating discussion which, however, needs to be revised in light of Knorr 1986: 80ff.

for any two points on the spiral taken arbitrarily such as P1, P2, the ratio between the two angles at points P1, P2 is the same as the ratio between the two radii at points P1, P2. If the angle is twice, the point should be twice out. Such is the locus definition of the spiral, translating into the language of proportion what, in Archimedes' language, is in the language of uniform speeds (this, indeed, is the content of proposition 12: would it not have made sense, then, to have turned that proposition into a definition?). Why does Archimedes avoid such a definition?

More than this: he avoids a more general definition. I have just mentioned that Archimedes' uniform speeds express, ultimately, a certain notion of proportion. But they do so by reducing speeds to a special, trivial case: in truth, a more general definition ought to have been framed in proportion terms all along. For it does not really matter that the speeds should be uniform: what really matters is that the two speeds – that of the rotating line, and that of the point progressing along the line – should be proportional. A truly general statement of the definition *in terms of motions* would be that a spiral is produced when, at any given two instants m and n, the speeds of rotation and progression to be represented as rotation_m, rotation_n, progression_m, progression_n obey:

$$\text{rotation}_m : \text{rotation}_n :: \text{progression}_m : \text{progression}_n$$

Put more simply: if the rotation revs up from (say) 20 rpm to 40 rpm, the progression should rev up at the same time from (say) 20 km/h to 40 km/h. (Archimedes' definition fulfills this proportion requirement in a trivial way, since he has all rotation speeds, as well as all progression speeds, uniform, that is, all instantaneous speeds equal to each other.)

Such a general statement is undoubtedly cumbersome, but it is definitely not beyond the capabilities of Archimedes' Greek: all one needs is the language of proportion. Nor does one need the language of acceleration (one does, however, need the notion of a speed at an instant: but would an ancient reader necessarily even feel that this was a difficulty?).

We have contrasted Archimedes' definition with two possible alternatives: one, more general, has the two motions proportional to each other, rather than each uniform. Another, more abstract, has the spiral defined as a locus satisfying a proportion between angles and line segments. Why does Archimedes prefer a less general and less abstract definition? One can devise all sorts of complicated explanations in terms of the conceptual difficulties of instantaneous speed, or of a locus (which, incidentally, would be much more difficult to define as soon as we move beyond a single rotation: for then we need the notion of angles greater than four right angles). But it appears to me much simpler to invoke simplicity itself: Archimedes' definition has the merit of being a relatively succinct, and intuitive, way of invoking the object. Read in its immediate context, the major impression is that of a growth in complexity: starting from a (relatively) simple definition, going through the (relatively) simple first problem, and growing in complexity until we reach the nearly impenetrable fourth problem – all in the spirit of Archimedes' gradual, surprising, revelations. A more sophisticated and, so, complicated, statement of the definition of the spiral would have taken away from the impression of the complexity of the results obtained concerning the spiral. And so I return full circle to the sociology of mathematics. My suggestion is that a certain piece of the mathematical concepts themselves – the choice of a particular form of definition – could be motivated by a desire to produce a certain textual order which, ultimately, could derive from Archimedes' position within a network of correspondence.

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If a certain point is carried along a certain line, moved at uniform speed with itself, and two lines are taken in it <=the original line>, the <lines> taken shall have to each other the very same ratio which the times <have to each other, =the times>, in which the point passed through[34] the lines.

For let a certain point be carried at uniform speed along the line AB, and let two lines be taken in it, <namely> ΓΔ, ΔE, and let the time, in which the point passed through the line ΓΔ be the <time> ZH, and <that> in which <it passed through> the <line> ΔE <be> the <time> HΘ. It is to be proved that they have the same ratio <to each other>: the line ΓΔ to the line ΔE, <the same as that> which the time ZH <has> to the <time> HΘ.[35]

(a) For let the lines AΔ, ΔB be composed by whatever composition out of the lines ΓΔ, ΔE in such a manner, so that AΔ exceeds ΔB,[36] (b) and as many times as the line ΓΔ is composed in the line AΔ, so many times let the time ZH be composed in the time ΛH, (c) while as many times as the line ΔE is composed in the <line> ΔE, so many times let the time ΘH be composed in the time KH. (1) Now, since the point was assumed to be carried at uniform speed along the line AB, (2) it is clear that, in as much time as it is carried through ΓΔ, in that time it is

Composed: see Glossary

[34] By "pass through" in this context we mean "pass through to the end."

[35] There is a slight solecism in the line: one expects a singular form, where the Line ΓΔ has to the line ΔE the same ratio which time ZH has to HΘ. Instead, we have a plural form, "they have," reflected by my strained translation. I believe the solecism could have been original to Archimedes, and I do not try to correct it.

[36] A few points require explanation: first, "composition" (see Glossary s.v. compose) means nearly the same as "multiplication" (one here composes a line segment Y, out of the line segment X, by attaching in line successive copies of the line segment X: we – and the Greeks, too – would say that Y is a multiple of X. See the comments). Second, "whatever" is meant to mean – all the way down to Step 8 – "whatever, *as long as the condition is fulfilled* that AΔ>ΔB." See, however, the following note to Step 9. Finally, Archimedes implies a "respectively" qualifier: AΔ is composed (=is a multiple) of ΓΔ, ΔB is composed (=is a multiple) of ΔE.

also carried through each of the <lines> equal to ΓΔ.[37] (3) Now, it is obvious that it is carried through the composed line AΔ in such a time as is the time ΛH, (4) since: as many times as the line ΓΔ is composed in the line AΔ, the time ZH, too, is composed in the time ΛH. (5) So, through the same things, the point is carried through the line BΔ, too, in as much time as is the time KH. (6) Now, since the line AΔ is greater than the <line> BΔ, (7) it is clear that the point passes through the line ΔA in a longer time than <it passes through> BΔ; (8) so that the time ΛH is greater than the time KH. (9) And it shall be similarly proved that, even if times are composed out of the times ZH, HΘ by whatever composition, so that one exceeds the other, among the <lines,> too, composed, by the same composition, out of the lines ΓΔ, ΔE, the <line> related to the exceeding time shall exceed <the other>.[38] (10) Now it is clear that ΓΔ has to ΔE the same ratio which the time ZH has to the time HΘ.

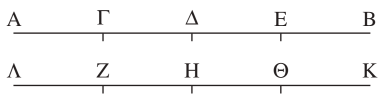

Diagram not extant for C.

COMMENTS

A treatise by Archimedes often begins with small, almost trivial propositions, with little apparent bearing on the problem at hand. This proposition appears to fit the pattern: its relevance to the study of spirals is not obvious (though we are of course aware of the central role of motion in the very *definition* of the spiral). It also has the appearance of a near-trivial observation, its claim true as a matter of tautology, or definition. Indeed, the logical moves are very frequently signposted by marks of "obviousness": Steps 2, 3, 5, 7, 8, 9, 10 are results of arguments. Of these, 2, 7 and 10 are "clear," 3 is "obvious," while 5 is "through these" and 9 is "similarly proved" (i.e. those claims are in no need of explicit argumentation, their grounds being understood on the basis

[37] This acts as Archimedes' effective definition of "uniform speed": if a point passes an arbitrary length X in a given time T, it will pass any length equal to the arbitrary length X at the same given time T. The "it is clear" may mark, perhaps, the more "intuitive" character of the claim (which, after all, is not based on any explicit definition).

[38] The force of "whichever composition" becomes more powerful at this point. It turns out that it does not matter if the composition is in fact such that AΔ>ΔB or such that AΔ<ΔB (indeed, even though Archimedes refers only implicitly to that point, AΔ=ΔB should be as good). What matters is that the following argument should be based on the assumption of the particular direction of excess. What was left out in Step a was an οὐδὲν διαφέρειν marker, a claim that this case is as good as any other, that the AΔ>ΔB is arbitrarily chosen.

of previous arguments). Step 8 is the least marked of the conclusions in this proposition: that since

(5) the point gets through BΔ in the time KH.

(7) the point gets through ΔA in a longer time than BΔ.

((**3**) Understood as background: the point gets through ΔA in time ΛH)

(8) so that the time ΛH is greater than the time KH.[39]

Now, the overall strategy of the proposition is to construct an arbitrary case, and to show the result (8) for that arbitrary case. The result is then generalized, from the arbitrary case to that of any chosen case, in Step 9; and the conclusion of that general statement for any chosen line is stated in Step 10. Thus Step 8 is in some sense the real *demonstrated* outcome of the proposition, and its relative position as a genuine "result," not a mere obvious statement, is well understood.

This, however, masks the more important structure of the proposition, where Step 8 is indeed a very easy result. The most important work of the proposition happens not there, but later: in the transition from 8 to 9 and then from 9 to 10.

The transition from 8 to 9 is fairly characterized by "it shall be similarly proved": one can indeed provide similar proofs for cases other than those where AΔ>ΔB (very obviously for the case AΔ<ΔB, since, after all, the precise direction of the sign does not matter at all for the proof; slightly more difficult for AΔ=ΔB). However, what this leaves unstated is that the very structure of the proof has shifted: the task for the labeled proof, it turns out, was not to show the result for AΔ>ΔB, but for the other case as well, or indeed more generally (which Archimedes never clarifies explicitly) for any sign of the relation.

The transition from 9 to 10 is quite unfairly characterized. To say that the transition from 9 to 10 is "clear" depends on one's conception of what grounds a proportion claim. Archimedes obviously relies on a statement equivalent to Euclid's famous definition (famously ascribed to Eudoxus) that first:second:: third:fourth when any equimultiples of the first and the third are both bigger, both smaller and both equal to any equimultiples of the second and the fourth. Or even more algebraically, the condition for the proportion a:b::c:d is that, for any arbitrary M, N:

$$\mathrm{Ma} \lesseqgtr \mathrm{Nb} \longleftrightarrow \mathrm{Mc} \lesseqgtr \mathrm{Nd}$$

[39] This is the Step least marked as "obvious." However, even here the particle used is ὥστε for "so that" (signaling a rather direct transition), not the ἄρα, "therefore," used for even moderately striking conclusions in the pair ἐπεί–ἄρα. This is indeed a theorem without a single ἄρα. One also notes the prevalence of particles marking a major transition: the ten Steps of the proof contain five οὖνs and a δή, that is, most Steps of the proof are marked as a major transition. This is not a mark of an extreme episodic structure, but rather signals the extreme "distance" in which the proof is conducted: Archimedes does not so much *argue*, as *talk about* an argument. (This is also related to the typical Archimedean stylistic feature of having the propositions directly following a general introduction somewhat more "general" and distant in character: more in the comments below on propositions 3–4).

Archimedes may of course rely on some version other than Euclid's (maybe on Eudoxus' original version?). It does not help the reader that the equimultiple relation is well hidden in Archimedes' formulation (that the antecedent line segments and times are multiplied equally is a non-transparent result of the way in which Steps b and c are set up). What Archimedes does emphasize is the arbitrary, "whichever" character of the equimultiples chosen (which, however, once again is muddied by the peculiar statement in Step a apparently limiting the discussion to the case $A\Delta > \Delta B$). In truth, the argument from 9 to 10 is understood by someone who is very attuned to the possible applications of Euclid's definition of proportion – as is the case of course with all modern scholars of ancient mathematics. Perhaps the same would have been true for Archimedes' audience, as well. Even so, the application of the definition is subtle and difficult. It is therefore striking that Archimedes makes no gesture towards the clarification of the relevant definition, and of its applicability in this case. The proposition, it turns out, is only deceptively easy. The clarity claimed by Step 10 is, so to speak, "ironic": Archimedes is using the language of simplicity to hide a deeply conceptual argument.

The conceptual difficulty may have to do with the objects themselves; it may also affect the way in which the definition of proportion is used. For Archimedes' striking avoidance of the language of "multiplication" (used in at least Euclid's version of the definition of proportion) may be related to the kind of object being multiplied.

The notion of "composition" here is quite striking. The verb σύγκειμαι is very rich in mathematical meanings (including, for instance, the synthesis in the analysis-and-synthesis pair), but its relevant meaning here is that of "addition." We have used it just above, in the statement of Archimedes' Axiom, in the expression "added itself onto itself" – the understood meaning being *indefinitely* many times, as many as are required to exceed. This already includes the idea that addition may be iterated, and this, finally, is the meaning here – a "composition" means "repeated addition." Now, Greek has a straightforward term to cover just this – multiplication – the one in use in Euclid's definition of proportion! So why not say that the objects should be multiplied by some arbitrary factor? Perhaps, time is not the kind of thing one can "multiply." It is possible to compose larger units of time out of smaller ones, but Archimedes is cautious in naming the act of composition and does not assimilate "time" to a kind of *number*. (At the level of the diagram, however, both are undifferentiated quantities, hence undifferentiated line segments.)

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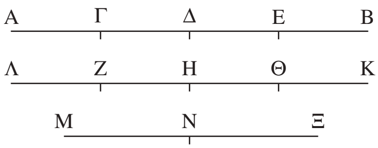
If, with each of two points being carried along a certain line – not the same – each at uniform speed with itself, two lines are taken on each of the lines,[40] of which let both the first – as well as the second – be

[40] I.e. on each of the lines one takes two lines, so that altogether one takes four lines (always understanding "line" to mean here "line segment").

traversed by the points in equal times,[41] the taken lines shall have the same ratio to each other.[42]

Let a certain point be carried along the line AB, at uniform speed with itself, and another <be carried> along KΛ <at uniform speed with itself>. And let two lines be taken in AB, <namely> ΓΔ, ΔE, and <two> in KΛ, <namely> ZH, HΘ, and let the point carried along the line AB pass through the line ΓΔ in an equal time, in which the other <point>, carried along KΛ, <passes> ZH, and, similarly, let the point pass through the line ΔE, too, in an equal <time>, in which the other <passes through> HΘ. It is to be proved that ΓΔ has to ΔE the same ratio which ZH has to HΘ.

(a) So, let the time in which the point passes through the line ΓΔ be the <time> MN. (1) In this very time the other point, too, passes through ZH. (b) So also, again: let <the time> in which the point passes through the line ΔE be the time NΞ. (2) In this very time the other point, too, passes through HΘ.[43] (3) So both have the same ratio: "ΓΔ to the line ΔE, <the same ratio> which the time MN has to the <time> NΞ"; as well as "ZH to HΘ, <the same ratio> which the time MN has to the <time> NΞ."[44] (4) Now, it is clear that they have the same ratio: ΓΔ to ΔE, <the same> which ZH <has> to HΘ.



All the codices agree except for the position of M. It is to the right of Z in EH (perhaps so in codex A?), right under Z in DG4; it is to the left of Z in codex C, so that the entire figure is completely symmetrical: I assume this was the original form.

COMMENTS

This theorem is very much a continuation of the preceding one (it thus begins a theme of *paired propositions*): it is not explicitly a corollary mostly because

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[41] The thought is that we pick the four line segments in sequence. First, we find a line segment from line X and another line segment from line Y. Both are traversed by their respective point in the same time, say an hour. Those are the "first." We then pick another line segment from line X and yet another from line Y. Both are traversed by their respective point in the same time, say a day. Those are the "second." Altogether there are two first and two seconds and four line segments in all.

[42] Namely: as first is to second in line X, so first is to second in line Y.

[43] Steps 1–2 essentially restate the setting-out in different terms, based on the Steps of the construction a, b, whose only function is to introduce a new unit, that of time.

[44] Step 3 is an application of the preceding theorem to the conditions set out by the preceding Steps a, b, 1, 2. The first part of the claim of Step 3 follows from Steps a, b; the second part of the claim follows from Steps 1, 2.

it calls for a different construction. It also maintains the same spirit of apparent ease and argumentative "distance," in this case with a mere four Steps, two of which are embedded in the construction; another is a distant "so"; while the last one is an "it is clear" (well motivated, in this case, as the argument for Step 4 is roughly that of the transitivity of proportion). It would have been feasible to prove propositions 1 and 2 in a single proposition, where proposition 1 is merely an interim result. Indeed, proposition 1 as such is not required later on in the book; it is merely a stepping-stone for proposition 2. That this was not done was probably for the sake of the appearance of simplicity itself: a single composite proposition 1+2 would have been somewhat cumbersome and difficult to follow. Archimedes preferred to have two propositions, one seemingly, and the other truly, simple; or he might have wanted pairs, as such: the architecture is overdetermined.

The goal of propositions 1+2 is also somewhat masked by their division into two parts. As it is, we follow through a system of two propositions, each with its own goal, and one has an illusion of treating the broad questions of uniform speeds and proportions, in the spirit of a general theory of mathematical/physical science. In truth, Archimedes has no interest at all in physical science in this treatise, and the function of these preliminary two propositions is precisely to sterilize the physical component of the definition: even though the spiral is defined through motions, these motions are found equivalent to geometrical proportions so that, in the actual course of the treatise, the spiral is treated as if its definition was purely geometrical. This brings us back to the problem of the definition: why define the spiral as physical in the first place? I will return to such questions when more evidence is brought in, following proposition 24.

The proposition further follows its preceding one in terms of the *character* of the diagram. Once again, times and line segments are treated on a par. However, there is a subtle addition to this equivalence of times and lines. Even though, within proposition 2, the figure for the times is structurally different from that of the figures for the lines (the lines are divided into four segments; the times are divided into two), if we compare the figures for propositions 1 and 2 we find an even deeper equivalence: the lower line, for the times, in proposition 1, is used again, essentially unchanged, in proposition 2, now however signifying lines rather than times – further establishing the implicit standardization of time as a geometrical object.

/3/

Given circles, however many in number, it is possible to take a straight line which is greater than the circumferences of the circles. (a) For with a polygon circumscribed around each of the circles (1), it is clear that the line composed of all the perimeters <=of the polygons> shall be greater than all the circumferences of the circles.[45]

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[45] *SC* I.1; though perhaps in this context it is simply taken for granted that the perimeter of a polygon is greater than the circle it circumscribes.

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Given two unequal lines, both a straight <line> and a circumference of a circle, it is possible to take a straight line smaller than the greater of the given lines, and greater than the smaller <=of the given lines>.

(1) For with the straight <line> being divided into as many equal segments – as many times such that the excess, by which the greater line exceeds the smaller, itself being added to itself, shall exceed the straight <line>,[46] the single segment shall be smaller than the excess.[47] (2) Now then, if the circumference is[48] greater than the straight <line>, with one segment[49] being added to the straight <line>, it is clear that it shall be greater than the smaller of the given <lines>, and smaller than the greater; (3) for the added <line>, too, is smaller than the excess.[50]

Added to itself: see Glossary

COMMENTS

In between Steps 2 and 3, Heiberg inserts another step – with no textual authority – which would have read as:

(3) while if it <=the circumference> is smaller, with one segment being added to the circumference, it shall similarly be greater than the smaller, and smaller than the greater

This assumes that the original text covered the two cases, circumference> straight as well as straight>circumference. The main reason to adopt this reading is the particle μέν in Step 2, which expects the δέ provided by Heiberg in Step 3. Furthermore, the loss of a hypothetical Step 3 could be accounted for by its very long homoioteleuton (though still it would be a

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[46] Guaranteed by Archimedes' Axiom. We assume the case – which will be made explicit in the next Step – where the circumference is greater than the straight line. This implicit picking of one sign out of the two possible ones is of course reminiscent of the argument of proposition 1; here it is less innocent.

[47] This Step is opaquely phrased. Call the circumference C, the straight line S. The excess is C-S. We divide the straight line into "as many times." Which "as many times"? The same as we required for multiplying the excess so it would exceed the circumference. The "as many times" is n, such that $n(C-S)>S$, when $(n-1)(C-S)<S$. Since $n(C-S)>S$, it follows that $C-S>S/n$ or, verbally, the excess is greater than the straight line divided into "as many times."

[48] The Greek has the subjunctive verb form meaning "if the circumference *should be* . . . ." This is the standard expression used in general enunciations ("if a line *should be* cut . . . ") and which I translate with a simple indicative, as it carries no more than the meaning of a general rule. Note, however, that Archimedes' language in the original would have suggested, then, that he relies on something akin to an implicit lemma (and not just an ad hoc observation).

[49] Namely, the segment resulting from dividing the straight line into "as many times" segments.

[50] Why the need to recall the claim of Step 1? One needs, as it were, to state twice a claim which is not, strictly speaking, true. See the general comments.

curiously long – and curiously "neat" – lacuna). An attractive emendation, but no more than a conjecture. Removing Heiberg's hypothetical Step 3, we have a couple of diagram-less claims each backed by a single sentence; perhaps Heiberg is right, and the second is backed by a rather complex sentence.

Are these even propositions? Should they be numbered as such?

Now will be a good time to discuss the question of "what counts as a proposition" in general terms. The following table sets out the "titles" of the various segments of the text of *On Spirals*, in Heiberg's edition, codex A and codex C. The mark "Des" for codex C represents a lacuna in that manuscript. An 'X' means that a mark is missing at that point.

Heiberg	Codex A	Codex C
1	1	Des
2	2	Des
3	3	3
4	4	Des
5	5	X
6	6	X (not reported by Heiberg)
7	7	Des
8	8	Des
9	9	Des
10	10	X
Cor	X	X
11	11	11
Cor	X	X
Defs	X	X
12	12	12
13	13	13
14	14	X (not reported by Heiberg, uncertain)
15	15	15
16	16	16
17	17	X (not reported by Heiberg)
18	18	18
[2nd figure]	19	19
19	20	20
20	21	X (not reported by Heiberg)
21	22	22
Cor	X	23 (not reported by Heiberg)
22	23	X (not reported by Heiberg)
Cor	X	24 (at middle; not reported by Heiberg)
23	24	X (not reported by Heiberg)
Cor	X	X
24	25	25
[2nd figure]	26	26
25	27	27

(cont.)

Heiberg	Codex A	Codex C
[2nd figure]	28	X (uncertain)
Cor	X	29
26	29	X
[2nd figure]	30	30
27	31	31
28	X	Des

Several observations emerge.

First, such titles as "corollary" (πόρισμα) or the unique "Definitions" (ὅροι) are not attested in medieval manuscripts. This is indeed the general rule for most such titles in ancient mathematics, an important observation made by Fowler 1999: 222 n.1.

Second, in the manuscript tradition, the numbers of "propositions" seem to be related at least in some cases not to what we call "propositions" but to what we call "diagrams" (this is reminiscent of the way in which Pappus uses diagrams to count propositions: see Netz 1999: 37 n.66). Thus, when the propositions 18, 24, 25 and 26 bifurcate into two cases, each with its own diagram, the manuscripts tend to label this with a separate number. Related to this, codex C tends to position the diagrams of its propositions immediately following the main argument and preceding the corollary, when present. As a consequence, the corollary follows upon the diagram and thus gets its own separate numbering.

Third, it appears that the scribe of codex C did not make a consistent effort to copy all proposition numbers. Of the sequence of thirty-one numbers the manuscript could have used, seven are lost because of lacunae in the parchment. Of the remaining twenty-four, seven appear to have been omitted by the scribe: about 30 percent. This is a remarkable level of lassitude on the part of the scribe who, otherwise, does not appear to stray far from the text transmitted by codex A. Clearly, however, some archetype of C had a complete set of numbers, which on its overall structure (though not in its individual detail) agreed with that of A.

The sporadic application of proposition numbers on the part of the scribe of C is perhaps best understood as a mark of a certain attitude: that such numbers were scholiastic in character and did not stand for the voice of the author. (I would assign such an understanding not to the scribe of C, but perhaps to some archetype: the scribe of C does not strike me as anyone deeply engaged with anything like the authorial voice of Archimedes.)

All the more remarkable that not one, but two minuscule manuscript traditions – those of A and of C – chose to label both proposition 3 and 4 by their own, separate numerals, and this *in the absence of diagrams*.[51] My own

[51] One should add, though, that the manuscripts agree so well for the text of *SL* that one may assume that they do in fact derive from a common archetype, one not far away: in all likelihood, *SL* stems from a single majuscule codex. The numeration represents, then, a single event in late antiquity.

impression is that this stretch of text does not make such a strong claim of propositional status. Instead, Archimedes wraps up a preliminary stage without fully fledged, complex arguments. Those are rapid remarks, making claims that are apparently even easier than the preceding, "easy" two propositions. The lack of diagrams in this case makes those claims take an "introductory" character, perhaps reminiscent of the stage where one makes general claims of a more "axiomatic" nature (and, indeed, those propositions are close to the axiomatic level of Archimedes' Axiom itself). As such, those propositions are in line with a widespread technique in Archimedes of a gradual modulation from the general introduction to the fully fledged sequence of proved theorems (Netz 2009: 103–104).

The appearance of the great ease of propositions 3–4 is even more deceptive than that of propositions 1–2. Indeed, I am not even sure we should consider proposition 4 as valid.

Recall the task. We have two unequal lines, one straight, one curved (specifically, this is described as a "circumference of a circle," though one assumes that this description is meant to cover arcs as well). We wish to squeeze a third line in between (which may be either straight or curved).

In Step 1, it is taken for granted that a line is found, equal to the difference between the two lines. This is not straightforward. To claim to be able to do this constructively is to claim to be able to find a straight line equal to any given curved line, which is tantamount to squaring the circle. Obviously then the claim is to be understood non-constructively, as a mere realist statement that such a line, equal to the difference, exists (a realist sentiment which in and of itself is unproblematic; see Knorr 1983: but in this case this realist sentiment already suggests the turn to neusis constructions, from proposition 5 on: see below).

If so, however, Step 1 is thrown further into doubt. For it is envisaged that the straight line (here assumed to be the smaller of the two given lines) be divided into the same number of equal segments, as many times as it took for the difference, added onto itself, to exceed the same, smaller, straight line. Yet, if the difference is not produced but is instead merely assumed to exist, there is no mechanism associated with its finding, or with the finding of the number of times it takes, added itself onto itself, to exceed a given line. Thus, once again, the number of times into which the line should be divided is not given but is simply assumed to subsist in some unknowable fashion. This, however, makes the entire apparatus of Steps 1–2 quite redundant. For if there is no constructive operation at all, but a mere ontological statement, why should we not assert, to begin with, that, given a difference, there is a difference smaller than it (say, its half), which we may then add onto the smaller line?

In short, because of the difficulties of an actual construction transforming straight into curved lines, there is no real sense in which the problem of proposition 4 is really solvable. From a strictly axiomatic point of view, it would have been best to treat it as a postulate. Indeed, it is most akin to Archimedes' Axiom, whose language implicitly asserts that straight and curved magnitudes, added themselves onto themselves, may exceed each other. I am not sure Archimedes is assuming that this postulate already allows him to achieve other comparisons between straight and curved lines. Perhaps he is aware of axiomatic gaps, but does not see them as crucial – it is enough for him that the problem is ontologically feasible, and the proof is not a real axiomatic reconstruction of the way in which the task is achieved, but is rather a sketch making its ontological feasibility somewhat more vivid. This cavalier attitude to problem-solving – especially in the context of the treatment of the spiral – is interesting. If Archimedes does not seek to map out the precise axiomatic underpinning of the relations of curved to straight lines, we are prevented from reading the treatise as a tool for an axiomatically sound solution to such problems as the quadrature of the circle or the trisection of the angle: more on this in the comments on propositions 7, 24 below.

Or possibly, the preference to make this claim into a proposition and not a postulate reflects no more than a stylistic decision to avoid an explicit axiomatic introduction to this treatise (once again, then, motivated by the communication strategy of the introductory letter, all organized around the surprise of the spiral being flung upon the reader). Archimedes, still, may have been wary of the claims made here, sufficiently so as to provide them with a status in between the postulate and the proposition – occupying the middle of those incommensurables – briefly argued for, but diagram-less.

What we see through the entire sequence of propositions 1–4 is apparent ease masking conceptual complexity: a duality typical of this treatise.

/5/

Given a circle and a line touching the circle, it is possible to draw a straight line from the center of the circle to the tangent, so that the straight line between the tangent and the circumference of the circle has to the radius[52] <of the circle> a smaller ratio than the circumference of the circle which is between the touching point and the <line> drawn through to the given circumference (however big)[53] of a circle.[54]

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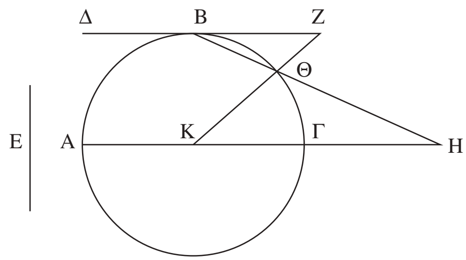
[52] Here and elsewhere in the translation, the English word "radius" stands for the Greek formulaic expression "the <line drawn> from the center" – which, however, being fixed in its formulaic form, literally carries the very same meaning, in this context, as our "radius." In this particular case I eschew literalness for the sake of a slightly less cumbersome text.

[53] "However big" means not so much that the circumference of the circle is arbitrary (which is an understood meaning of a "given," anyway) but rather that this problem, surprisingly, is not limited to a certain range of values for the given circumference (or arc). In technical terms, the problem does not possess a διορισμός.

[54] The proposition does not possess a much-needed definition of goal (see the general comments). In terms of the diagram, the task is to produce a line ZK so that ZΘ:ΘK<arcBΘ:E. It is to be noted that the given "circumference" is provided as a mere quantitative mark, which is therefore drawn – as such marks generally are – as a straight line. If a time is a straight line, why can't a circumference be so?

(a) Let there be given a circle, ΑΒΓ, its center K, (b) and let ΔZ touch the circle at the <point> B, (c) and let there also be given a circumference of a circle, however big. (1) It is possible to take a certain line greater than the given circumference,[55] (d) and let the line E be greater than the given circumference; (e) and let AH be drawn from[56] the center K parallel to ΔZ, (f) and let HΘ be set equal to E, verging towards B.[57] (g) So then: let a line joined from the center K to the <point> Θ be produced; (2) so, ΘZ has to ΘK the same ratio which BΘ <has> to ΘH.[58] (3) Therefore ZΘ has to ΘK a smaller ratio than the <ratio> which the circumference BΘ has to the given circumference, (4) because the line BΘ is smaller than the circumference BΘ, (5) while ΘH is greater <than> the given circumference. (6) Now then, ZΘ, too, has to the radius a smaller ratio than the circumference BΘ has to the given circumference.

Verging: see Glossary



The diagram is not preserved in codex C. DE have the line ΔZ extended slightly rightwards, beyond Z; perhaps so in codex A. B is unique in making the KΘ=ΘH.

COMMENTS

As we move on from the general statement to the construction in Step a, we expect a setting out and definition of goal, explaining in the concrete terms of the diagram what the difficult constructions of the enunciation actually mean. This Archimedes avoids, treating the enunciation as if it partly stood for a definition of goal; hence he moves back, in Steps 5 and 6, to speak of "the given circumference" (rather than E), and, in Step 6, to speak of "the radius" (rather than ΘK).

[55] Proposition 3.

[56] A tiny solecism, as pointed out by Heiberg: "through" would be smoother than "from." This could be an authorial solecism: Archimedes half forgets the segment AK, which, indeed, plays no role in the argument (I do not think this is a late mistake, Archimedes writing KH instead of AH; see the comments on proposition 9).

[57] To set an object "verging towards" a point P means (quite intuitively) that we set up a line segment, equal to a given value, so that, if it be extended, it would pass through the point P. This is technically known as a "neusis construction." More in the general comments here and on proposition 7.

[58] The triangles BΘZ, KΘH are similar (*Elements* I.15, I.29; Step e of the construction); then *Elements* VI.4.

This is in some sense a continuation of the preceding propositions. Proposition 1 and 2 did have an "it is required to prove" segment, but their settings-out started out, somewhat surprisingly, with such an expression as:

"Let a certain point be carried along the line AB"

The definite article on the line AB is natural only if the line has been set out already. Thus the impression is of an informal setting-out – in character with the overall informal, "distant" nature of the arguments. Propositions 3–4, following that, are no arguments at all and certainly have no formal propositional structure. Here, in proposition 5, we have hit, finally, upon some genuine geometry. It is not obvious that the task is doable, especially with its claim to avoid any διορισμός (that is, the claim is that the task is doable under any quantitative parameters). It takes some (elementary) ingenuity to achieve it – and Archimedes calls up the heavy tool of a neusis construction. Finally, even though the discourse is still marked by such particles of transition as δή (Steps g, 2) and οὖν (Step 6), we do have here – for the first time in this treatise – the argumentative particle ἄρα (Step 3). Even so, Archimedes keeps something of an informal air – which will be extended further in the sequence of propositions leading up to 9. Only proposition 10 would be the first fully formed, independent proposition in this treatise.

I wonder if this is related to the other notable feature of this proposition, namely, its neusis construction. In a neusis construction we avoid an explicit construction of a required geometrical entity, merely asking that it be placed under certain conditions: the standard case, as in here, is of a line segment of a given length placed so that its continuation would pass through a given point ("verging," the literal meaning of νεῦσις). In this case, in Step f the point H is found so that ΘH "verges" towards the point B (or, if you will, so that the lines BΘ, ΘH are on a line), while ΘH=E (namely, the given circumference). That the line "verges" as it does is crucial for the sake of the similarity of the triangles, which then allows us to transfer a proportion involving the given line, here represented by ΘH, to another proportion involving the line BΘ.

There are two difficulties here. First, there is no argument to show that the point H can be found through any mechanism whatsoever, be it ruler or compass or some more complicated tool. There is not even a hint, in the text as transmitted, that such a mechanism is called for (in the manner in which Archimedes, in *SC* II.4 [Netz 2004b: 204], draws our attention to the need for solving a specialized problem; on the other hand, Archimedes is equally silent on the need for solving the problem of finding two mean proportionals, in *SC* II.1 [Netz 2004b: 189]). Second, even given such a mechanism of finding the point H, we first need to turn the magnitude E into a linear magnitude, i.e. to square the circle.

Take the neusis first. It is clear, through considerations of continuity, that, given a straight line segment E, there is some arrangement such as ΘH which satisfies ΘH=E. This is because, with a point such as H positioned very near Γ, we can make a line such as ΘH as small as we please. On the other hand, as the point H is carried away from Γ, the line ΘH becomes progressively greater; so that it is indeed clear that at some point it would be exactly equal to E. This "sliding" operation may well be the implicit meaning of the neusis operation. However, this is not a solution of a task: this is a mere ontological observation that (given certain realist assumptions of continuity) a line solving the task exists.

But then again, why should Archimedes even aim for anything more powerful than such an ontological observation? For, after all, the other difficulty – that of the need to square the circle – makes the entire operation depend essentially on a mere ontological statement that a certain line exists (rather than that a certain line can be found). An explicit solution of the neusis construction would have pushed the entire burden of non-constructivism to the quadrature aspect, but would still not have made this proposition into a constructive one. It is thus still in sequence with propositions 3–4: a gesture towards problem-solving, still aware of the limitation imposed by the impossibility of squaring the circle – and, for this reason, among others, limiting itself to the gesture.

But can the neusis even be constructed explicitly? This was the task that Knorr set himself in 1978; I will return to discuss this in the comments on proposition 7 below.

/6/

Given a circle and, in the circle, a line smaller than the diameter, it is possible to extend a line from the center of the circle towards its circumference, cutting the line given in the circle, so that the line taken off <=from the extended line> between the circumference and the line given in the circle, has to the <line> joined from the end of the extended <line> which is on the circumference, to the other end of the line given in the circle, the ratio laid down, if the given ratio is smaller than the <ratio> which the half of the <line> given in the circle has to the perpendicular drawn on it <=the line given> from the center.[59] Perpendicular: see Glossary

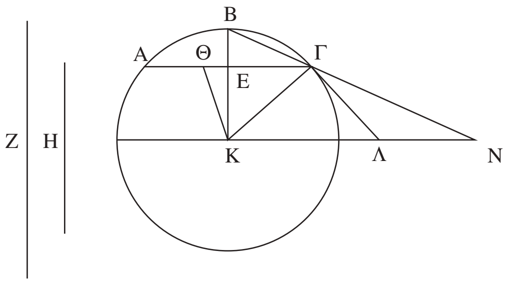
(a) Let there be given a circle, ABΓ, its center K, (b) and let there be given a line in it, smaller than the diameter, <namely> ΓA, (c) and a ratio, which Z has to H, smaller than the <ratio> which ΓΘ has to KΘ, (d) KΘ being a perpendicular, (e) and let KN be drawn from the center parallel to AΓ, (f) and <let> ΓΛ <be drawn> at a right <angle> to KΓ.[60] (1) So, the triangles ΓΘK, ΓKΛ are similar.[61] (2) Now then, it is: as ΓΘ to ΘK, so KΓ to ΓΛ;[62] (3) therefore Z has to H a smaller ratio than KΓ to ΓΛ. (g) So, that ratio which Z has to H, let KΓ have this to a <line> greater than ΓΛ.[63] (h) Let it have it to BN, and let BN be set between

[59] In the terms of the diagram: BE:BΓ is to be equal to a given ratio; the given ratio is to be smaller than ΘΓ:ΘK.

[60] Why is ΓΛ not directly drawn as a tangent? A puzzling omission.

[61] *Elements* I.29 and the construction Step e; the construction Steps d, f; *Elements* I.32.

[62] *Elements* VI.4. [63] *Elements* V.10.



DG have Z, but EH4 have Ξ instead, and so would have A. C has Z, so the mistake probably crept in only with A. Codex C has M instead of N; also, it introduces a Z at the intersection of the line KN with the circle (it also, perhaps related, has BΛ, rather than BN, as a single straight line; BN extends more rightwards). G has the line KΘ as perpendicular, KB to its right; it does not complete the diameter leftwards beyond K. GH have Z, H equal. The reading of E in codex C is uncertain.

the circumference and the line <=line KN>,[64] <passing> through Γ[65] – (4) and it is possible to cut the line in this way[66] – (5) and it shall fall outside[67] – (6) since it <=BN> is greater than ΓΛ. (7) Now, since KΓ[68] has to BN the same ratio which Z has to H, (8) EB, too, shall have to BΓ the same ratio which Z <has> to H.[69]

[64] The reference of "the line" to KN is surprising, since a "line" so far has been the given line, that is, AΓ. Clearly Archimedes is thinking with his geometrical agenda, rather than his words, in mind. The line KN is "the line" in the sense that it is the base on which the neusis-like line BN is expected to "slide."

[65] This is of the same class as a neusis construction. It is not one, technically speaking, for the trivial reason that the line does not "verge" towards a given point such as Γ, but instead passes through it. While in the previous proposition it was intuitively seen that the line can be as big or as small as we wish through considerations of continuity, as it "slides" along what was there the base, KH, so in this proposition it is intuitively clear that the line can be as big as we please, but can never become smaller than ΓΛ under the topological arrangement of the diagram, and that implied by the requirement that the line pass through (rather than "verge towards") Γ.

[66] I do not believe this refers to some independent solution for the task of the neusis-like construction (it would be a very offhand way of asserting such a major claim). Instead, I think Steps 4–6 simply emphasize the same thing in different ways: that since BN>ΓΛ, the line constructed in this way does indeed pass through Γ and extends between the circumference and the line KN, that is, they verify that the topology of the diagram holds under the neusis assumption.

[67] Outside of what? Once again Archimedes talks with his agenda in mind, not clarifying his intention to his readers. It appears that the important requirement is that B should fall "above" AΓ or (from the perspective of the center of the circle) "outside" it.

[68] Heiberg has corrected to KB; since both are radii, the equivalence between the two can also be taken tacitly. The transition into the following Step 8 definitely assumes, however, we have so transformed implicitly Step 7 to mean KB:BN::Z:H. See following note. I thank J. Wietzke for pointing out this textual observation.

[69] *Elements* VI.2 and Step e of the construction. It is a mark of the relative sophistication of this argument that, for the first time, some genuine implicit geometrical argument (carried somewhat implicitly) is at work – albeit of a very elementary character.

COMMENTS

The diagram has several surprising features, all removed by Heiberg.

First, Z is greater than H. I am not convinced that this could come about through scribal error. It is somewhat less logical than Heiberg's choice, of making Z smaller than H, since it is required that Z:H be *smaller* than a certain ratio. If the given ratio is smaller than equality (i.e., in this case, if ΓΘ<KΘ, which happens when Γ falls on the quadrant from due north-west to due north-east), clearly Z has to be smaller than H; but there are no analogous conditions when Z has to be *greater* than H. It is intriguing that codex A placed the point Γ rather high, while codex C placed it rather low, but probably not too much is to be read into it, and the sequence of two unequal lines, Z and H, does not signify a ratio with a given direction of size but a more abstract statement of a ratio which is allowed to include inequality.

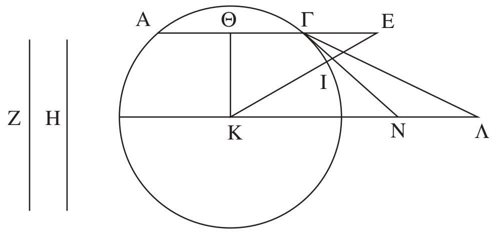
Second, this discussion of the apparent sizes of Z and H assumes that one can compare them with the apparent sizes of ΓΘ, ΘK in any realistic fashion. The diagram avoids this by drawing a series of metrically "false" arrangements. They seem to be driven by a desire to position B in the simplest position possible – the middle of the arc AΓ. From this it follows that EK appears perpendicular to the line AΓ. The "real" perpendicular, ΘK, is thus pushed aside (away from N) so that the angle at Θ is non-right, ΘΓ is no longer half AΓ, and the triangles ΘΓK, KΓΛ lose their apparent similarity. (Codex C further has a very bad right angle KΓΛ – indeed ΓΛ appears more co-linear with BΓ, in its diagram, than ΓN does: but this is probably just error on the part of C's scribe.) I do not think this is meant to "generalize" somehow the import of the proposition; it merely shows how little Archimedes appears to care for metrical accuracy.

Third, the line KN is extended all the way as an entire diameter. This is comparable to the way in which, in Step e of the previous proposition, the line AH was extended "from" K (why not *through* K? Why go all the way to A to begin with?). This is in fact a consistent feature of propositions 5–9. I am not at all sure what this should be taken to mean, but one thing it does bring out powerfully is the extent to which the entire set of five problems is understood to refer to the very same basic construction – of which see more below.

/7/

Given the same, and with the line in the circle being produced, it is possible to extend <a line> from the center towards the produced <line> so that the <line> between the circumference and the produced <line> has the ratio laid down to the <line> joined from the end of the <line> taken off inside to the end of the produced <line>, if the given ratio is greater than the <ratio> which the half of the <line> given in the circle has to the perpendicular drawn on it from the center.

(a) Let the same be given, (b) and let the line in the circle be produced, (c) and let the given ratio, which Z has to H, be greater than the <ratio> which ΓΘ has to ΘK; (1) Now then, it shall be greater than the <ratio> which KΓ has to ΓΛ.[70] (2) So, that ratio which Z has to H, KΓ shall have to a <line> smaller than ΓΛ.[71] (d) Let it have it to IN, verging towards Γ. ((3) It is possible to cut in such a way:[72] (4) and it <=IN> shall fall inside ΓΛ, (5) since it is smaller than ΓΛ.) (6) Now, since KΓ has to IN the same ratio which Z <has> to H, (7) EI, too, shall have to IΓ the same ratio, which Z <has> to H.[73]



Codex C is missing for this diagram. D has Z>H substantially, E has Z>H by a slight difference. G has N, Λ much closer; cramped together. Codex A had the label Ξ inserted at the intersection of KΛ and the circle (omitted only by B).

COMMENTS

The proposition is couched within the very same constraints as the previous one: the same circle, the same given line. The task is slightly different: in proposition 6 we have a ray extending from the center of the circle so that it cuts first the given line, and then the circumference (the sequence of the ray: K-E-B). We then connect BΓ and demand that BE:BΓ should be a given ratio. Here, in proposition 7, we have the ray extending from the center of the circle so that it cuts first the circumference, and only then the given line (K-I-Γ). We then connect IΓ and demand that EI:IΓ should be a given ratio. EI is in a sense the same as BE – it is the intercept of the ray between the given line and the circumference; BΓ is in a sense the same as IΓ – it is the line joining the cut of the ray with the circumference, and the cut of the given line with the circumference. Thus the entire difference is that of the relative positioning of given line and circumference, and the two propositions 6–7 can be seen nearly as the two cases of a single proposition. However, the different cases give rise to a fairly significant substantive difference – in proposition 6, it follows from the configuration that the task is doable only if the ratio is smaller than a certain minimum; in proposition 7, it follows from the configuration that the task is doable only if the ratio is *greater* than the same minimum. The two tasks are the same, but they are also literally "inside-out" reflections of each other: what proposition 6 accomplishes inside the circle, proposition 7 accomplishes outside it.

[70] Step 3 of preceding proposition. [71] *Elements* V.10.
[72] Cut what? Perhaps KΛ?
[73] *Elements* I.29, I.15, VI.4: same as Step 2 of proposition 5.

The language of sameness builds on the previous tendency – from proposition 5 onwards and really from the beginning of the treatise on – to avoid a fully explicit proposition. We did not have an explicit setting-out and definition of goal in propositions 5–6, and the reliance on "the same" in this proposition avoids, effectively, the construction as well. We are asked, in Step b, to have the given line produced, and, in Step c, the condition on the given ratio is reversed; but other than this the entire construction is skipped, and so we have to figure out for ourselves the new significance of the letter E – it is in fact the "same," i.e. the intersection of the ray with the given line. We also have to figure out for ourselves the identity of I – which is somehow the same as B from the preceding proposition, i.e. the intersection of the ray with the circumference,[74] but is also something else, namely the end point of the "sliding" line of the verging, or neusis, operation. As it is, we have to figure out contextually the conditions for this "verging" operation. Much of the proof is skipped as well: Step 1 carries over from the argument of the preceding proposition (hence the particle ουν and the more distant "shall be" – the result is surveyed from afar rather than directly asserted), and Step 2 no more than unpacks it, while Step 6 does no more than reiterate a construction. The one remaining substantial argument – Step 7 – does not carry over from the preceding proposition (it is indeed directly dependent upon the configuration, so that the argument would have to be different in the two propositions). However, it is exactly the same as an analogous argument in the proposition before that, number 5, a fact of which the reader is surely aware. Hence this brief proposition is really "the same," an elaborately stated task whose solution, at this stage, calls for no explicit construction argument at all.

WHY NEUSIS?

That the argument is so curtailed depends essentially on the neusis operation, inherently an argument-curtailing operation – in a sense, it substitutes the mere ontological claim, of the *existence* of a solution, for the actual task of *finding* such a solution. Now, proposition 7 is the first among propositions 5–9 to be used later on in the treatise, in the main sequence of results concerning the spiral, in proposition 18, Step c. For this reason, Knorr's fundamental study of neusis in *On Spiral Lines* (1978a) was centered around proposition 7, and it is time for us to turn there. We should study this in detail, as the question brings up the goals and nature of ancient problem-solving, and of course because of the centrality of Knorr to the study of *On Spirals*.

The question is why Archimedes uses a neusis construction. This can be further divided into two questions: (a) what were the alternatives available to Archimedes and (b) why did he not choose them?

[74] Incidentally, Heiberg does have a B in the diagram of proposition 7, as the intersection of the diameter with the circle, further from Λ – same as A in the diagram of proposition 5. Indeed, as is true for all the diagrams of 5–9, the diameter is extended through the circle; but there is no manuscript authority for this label B. (This is based on codex A alone: the diagram of codex C is not preserved.)

Until Knorr (1978a), the alternative was seen as one between a neusis (or an operation closely equivalent to it) and "stronger" construction tools, i.e. going beyond those of Euclid's *Elements*. This essentially follows Pappus (*Collection* IV.52–54: see Sefrin-Weis 2010: 301–311), who has offered a solution using conic sections (so that a complex figure involving a parabola and the opposite sections of the hyperbola, superimposed on the original circle, determines the length of the line to be inserted in the neusis). Pappus does not ascribe this solution to Archimedes: instead, he criticizes Archimedes (in a much mutilated passage, Hultsch 1876: I, 302.14–18) for offering a "solid" construction when a "plane" one is available. It is not entirely clear that this is a criticism of the neusis or of its application, or that Pappus sees his conic sections as a remedy for whatever criticism he intends. That Pappus' solution involves conic sections is a mark, to be sure, of the naturalness of conic sections to the advanced geometers of antiquity.[75]

The alternative appears to be that Archimedes relies, directly, on a neusis or some equivalent solution. It is sometimes suggested that he could rely on neusis-producing curves – in effect, the curve traced by a mechanical instrument (whether conceived hypothetically or concretely) equivalent to a sliding ruler. Perhaps the conchoid (or, more precisely, a version thereof), defined by Nicomedes and reported, in most detail, in Eutocius' commentary on Archimedes' II.1 (Netz 2004b: 298–303)? – Such was von Fritz's (1962) view.

This, in geometrical principle, is not much different from the view that Archimedes assumed a mechanical solution directly based on sliding a marked ruler.[76] This last view was endorsed by Zeuthen (1886: 261–265), but I think Knorr is somewhat unfair in saying that Dijksterhuis followed Zeuthen. What Dijksterhuis did say, in his incisive manner, is worth quoting in full (1987: 138):

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[75] There seems to be some confusion as to what Pappus wishes to achieve with the conic sections. Does he claim that their use avoids a solid solution? See Knorr 1978a: 90–91 for a statement of this puzzle (even if one need not follow Knorr's detailed resolution of this puzzle, based on hypothetical ancient sources with different definitions of a "solid problem": see also Sefrin-Weis 2010: 303–304). In truth, not too much should be read into this episode, I believe. Pappus operates under the commentator's requirement of finding *something* to say – I should know – and Archimedes' neusis is a very attractive target. Naturally, Pappus would report criticism of Archimedes (or invent it, and then politely ascribe it to unnamed others); naturally, he would offer his own solution; naturally, this would involve conic sections – the tools that come most naturally to someone like Pappus (as well as to most of his predecessors).

[76] There is a difference. The conchoid lifts the operation of the sliding ruler into the ontological level of a curve. That is, the instrument drawing the conchoids can be reconceived as a tool tracing an otherwise real curve – one whose ontological independence may be taken for granted – especially once enough is known about the properties of such a curve. To repeat: at some point, we may then feel that this curve "exists" in the sense in which triangles and circles (and conic sections, of course) "exist." So, at this point, invoking the presence of a conchoid as part of our geometrical configuration may still involve an ontological leap of faith – we ask for an object whose construction was not provided in explicit elementary terms – but no longer an intrusion of the mechanical, whose presence may be troubling for metaphysical reasons, or simply because it is felt to be an ad hoc, *deus ex machina* kind of solution.

> The commentators not infrequently speculate on the question how Archimedes could have performed the neuseis in question. The most obvious answer seems to be: as neuseis. [Neusis] does not indeed fit very well with the traditional prescript that in planimetric construction only straight lines and circles are to be used, but there is not the slightest evidence that this restriction applied in Greek mathematics to non-elementary problems.

Dijksterhuis does go on to muddle his statement by referring to practical ways one could perform such a neusis, including the insertion of a marker ruler, but this clearly is not his main point, which was instead that Archimedes' neusis meant just that – in Dijksterhuis' words from the same paragraph: "The insertion of line segments of a given length between two given curves." Knorr improved on these two alternatives. Now, it does appear that the problem, strictly speaking, is non-elementary: that is, it is impossible to produce *SL* 5–9, *as stated*, by ruler and compass alone. However, Knorr paid attention to the way in which Archimedes *applied* propositions 5–9 (concentrating, once again, on the application of proposition 7 within proposition 18). This allows one to pick a less strict problem, which is also doable by elementary techniques.[77]

In propositions 5–9 Archimedes repeatedly constructs line segments such as EI, IΓ so that a *proportion* is satisfied such as:

$$\mathrm{EI} : \mathrm{I\Gamma} :: \mathrm{Z} : \mathrm{H} \quad (\text{the ratio Z : H given})$$

This, indeed, is a non-elementary problem. However – and here is Knorr's breakthrough observation – in proposition 18, Step 3, as well as in all other applications of propositions 5–9,[78] what is really required are line segments such as EI, IΓ so that a ratio *inequality* is satisfied such as:

$$\mathrm{Z} : \mathrm{H} < \mathrm{EI} : \mathrm{I\Gamma} < \mathrm{Z} : \Theta \qquad (\text{the line segments Z, H, } \Theta \text{ given})^{79}$$

We need not a proportion but a ratio inequality. Now, to construct a ratio that fulfills an inequality is patently easier, and it is an immediate idea to implement a process of bisection so that a ratio is ultimately "squeezed" between the two given ratios. This is not at all trivial, however, as the interrelated terms EI, IΓ can be precisely "squeezed" between given boundaries, only with a

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[77] Sefrin-Weis (2010: 309) claims that Knorr's solution involves a "convergence" which strikes her as more akin to Archimedes' heuristic proofs. I do not see how this holds for Knorr's solution (1978a: 82–84), which merely uses a somewhat loose, modern language for describing a precise geometrical solution.

[78] Or, strictly speaking, propositions 5, 7–8, as Archimedes does not seem to apply 6 and 9.

[79] This is associated with Z being the radius of the circle, H being the length of a segment of the tangent to the spiral, Θ being the circumference of the circle. In proposition 18 it is shown how an impossibility follows from assuming that H≠Θ: the very possibility of squeezing a ratio in between Z:H and Z:Θ, smaller than the one and greater than the other, will lead to a contradiction. See proposition 18 below for more detail.

certain amount of trigonometry. But as Knorr proceeds to show, the trigonometry required is all available through the techniques of early Greek astronomy used by Archimedes himself in *The Sand-Reckoner*. Thus Knorr ends up doing two things (1978a: 82–84):

1 Revising the terms of the problem from a proportion to an inequality of ratios (which, however, is sufficient for the purposes of the actual application of propositions 5–9), and
2 Producing an explicit – if very complicated – solution of this problem of inequalities, through standard techniques, namely bisection and trigonometric operations already available in Archimedes' time.

Following on from this achievement, Knorr went on to ask why Archimedes did not avail himself of such an approach. And the solution he obtained is, by necessity, the same as that quoted from Dijksterhuis: Archimedes did not use elementary methods to substitute for his neusis because he thought *neusis was good enough*. Knorr goes on to suggest a very fine-grained diachronic account, as if neuses were acceptable in Archimedes' time, but already by Apollonius' time they would have been suspect (Knorr 1978a: 89). The argument appears to be that Apollonius produced a study of neuses where, apparently, various neusis problems were solved through elementary techniques, while Nicomedes produced his conchoid curve that produced a neusis. Here I begin to differ. If Knorr means to say that, up to a certain point in time, neuses were considered to be sound and then, through some décalage, they no longer were, and so, for that reason, Apollonius and Nicomedes set out to investigate how they could be replaced by other, sounder approaches, then I am not sure the evidence can support such a claim.[80]

It stands to reason that *at any time* in the history of Greek mathematics – fourth, third or second centuries BC – studies dedicated to showing how neuses can be effected, or dedicated to the properties of a curve such as the conchoid, would have been inherently interesting. On the other hand, there is no reason to believe that *at any time* in the history of Greek mathematics a neusis would have been considered exactly on a par with a more explicit solution: for, if neusis is just as good, we might as well give the game up.

As long as one has a binary, "sound" vs. "unsound" classification of solutions, one is reduced to a binary historical analysis, a décalage from the early acceptance to the later exclusion of certain techniques. But this binary division makes no historical sense: the crucial point is that neusis can be assumed to be a

[80] Of course, it is quite possible that geometers working after Apollonius and Nicomedes, while still accepting the epistemological soundness of neuses, would find their use less natural, since now one had many more techniques "on the shelf" available from the work of Apollonius and Nicomedes. But is this not beside the point? There were very few geometers active in the generations immediately following Apollonius and Nicomedes, anyway. Later Greek geometry was pursued in very different intellectual climates, where it is hard to deny that – particularly in late ancient geometry – much more interest was given to the explicit classification and "proper" allocation of problems within this classification: hence Pappus' treatment of Archimedes.

valid solution and, at the same time, replacing it by a more explicit solution can still be an interesting exercise. Instead of the binary, we need to think in terms of a gradation, or even a many-dimensional space, of solutions that are more or less "elegant," "neat," "simple" or "elementary." The history of the choices between such solutions can be equally subtle and many-dimensional: each choice would depend on a complex web of historical and textual constraints.

So let us retrace our steps, putting Archimedes' choice within its historical and textual context. Knorr's elementary solution is comparable to Archimedes' solution of problems in *On the Sphere and the Cylinder* I.3–6, where polygons are inscribed and circumscribed around circles so that their sides satisfy given ratio inequalities (Netz 2004b: 46–57). Furthermore, *SC* I.3–6 has a role in the argumentative structure of *SC* I comparable to that of *SL* 5–9 in the argumentative structure of *SL*. (The polygons in *SC* I "squeeze" a certain relation between spherical magnitudes, in the manner in which the line segments in *SL* "squeeze" a certain relation between spiral magnitudes.) Indeed, it appears that Knorr's solution is an attempt to fit the approach taken in *SC* I to the tasks of *SL*. This exacerbates the interpretative problem. It is not merely that Archimedes could in principle have conceived of Knorr's solution. Rather, he already did (and it is largely thanks to this that Knorr could come up with his own alternative solution!).

Now, why use a neusis here – *SL* 5–9 – but not there – *SC* I.3–6? Now, it would have been geometrically a non-starter to attempt a neusis construction in *SC* I.3–6.[81] The question then really is just why not follow an explicit, *SC* I.3–6-type solution, of the kind offered by Knorr, in *SL*? And for an answer, as usual, we need seriously to contemplate the counter-factual. Then, I believe, it becomes immediately obvious that such an alternative would have been, for Archimedes, *architecturally* wrong. A full set of solutions to *SL* 5–9 on the lines of Knorr's suggestion would have been an enormous addition to the structure of *On Spiral Lines*, achieving, in great, complex and indeed tedious detail, what is essentially an elementary problem. It is already clear that Archimedes wanted the introductory chapters to be *brief*. There is a clear sense of *mere* preparation.

And indeed, mere preparation this is. In the actual context of *SL*, these are tools for the proof of certain theorems. Seen from the perspective of those later theorems, there could be nothing wrong with a solution based on neusis: for the theorems assert that certain geometrical configurations have certain properties, and it is quite immaterial how we come about constructing those

[81] In *SC* I.3–6, the task is to produce equilateral polygons around the circles and the sectors set down; hence the main difficulty is not one of finding the line segments of the sides of polygons that satisfy the proportion (in itself a trivial operation), but of conducting it in such a way that the line segments involved form equilateral polygons, i.e. they subtend an angle at the center which is some integer division of the circle or the sector as a whole. To satisfy both the ratio inequality and the integer division of the angles, one really needs to keep the inequality as such – an inequality – and not transform it into a strict proportion. On the other hand, the task in *SL* 5–9 really involves just finding a single line segment satisfying the inequality, so that any line segment will do; thus one might as well make it a definite line segment and insert it through a neusis operation.

configurations. Knorr comes close to this point, only to deny it. To quote him (1978a: 84): "It has frequently been proposed . . . that Archimedes does not require the actual construction of the neusis, but only the *possibility* that it may be effected."[82] How could Knorr reject this claim? He could not suggest that an explicit construction was called for as a proof of the existence of the required object: he was, already in 1978, in possession of his mature understanding (Knorr 1983) of the Greek problem as a geometrical exercise of technique, rather than an existence proof. So, instead, he went on as follows: "there are two points weighing against this defense. First, even admitting this as Archimedes' intention, we must demand a proof of the possibility of the construction, but he does not give one" (Knorr 1978a: 84). This is a very weak argument – after all, there is much here that Archimedes elides and leaves for the reader to see intuitively – but I think Knorr errs here because he concentrates so much on proposition 7. In the following propositions 8 and 9, Steps h and g respectively, Archimedes offers an argument – tantalizingly implicit, but an argument nonetheless – for why a line satisfying the neusis construction can be found. That no such argument is offered in propositions 5–7 is a result of the extremely curtailed nature of those proofs, as well as the fact that the possibility is indeed very obvious in those propositions.[83]

The final twist, however, is this: as we read propositions 5–9, we have no way of knowing that they will be applied in a context where their mode of construction would be immaterial. For all we know, they could be tools useful for later *problems*, so that relying on a neusis here would make those later problems, too, rely on a neusis. To the extent that a neusis is a less explicit and therefore less compelling mode of problem-solving, later problems relying on 5–9 would be affected as well by the use of neusis here. In short, propositions 5–9 are open to criticism as less than compelling solutions, to the extent that they are seen as goals in themselves, or as tools for later problems. They are not open to this criticism to the extent that they are mere tools for later theorems. But this is not at all clear to the reader as he goes through the treatise: the neuses in propositions 5–9 thus demand a certain suspension of the reader's critical functions, or rather – seeing that a Greek reader would be unlikely to suspend criticism – they invite criticism, only to have such criticism allayed through a reading of the treatise as a whole. Archimedes

[82] Knorr refers to Zeuthen 1886: 265, Heath 1897: cvii–ix, Heath 1921: II, 386–388 and Tannery 1912: 314.

[83] Knorr goes on to say as follows: "Second, Archimedes does in fact introduce the actual construction of the neusis in prop. 7 and 8. His own procedure thus legitimizes the criticism raised by Pappus and raises the question, not of whether the neusis is possible, but of how the neusis is to be effected." I admit I do not quite follow Knorr's meaning here. If he intends to say that by explicitly saying that a line should be inserted as verging to a line, one raises the question of how the verging is to be effected, then this is no more than a *petitio principii*, for, exactly as Dijksterhuis put it, the natural reading of Archimedes' text is that the verging *is to be effected by a verging*. Introducing the actual construction of the neusis does not in and of itself raise the question of how it is to be effected: it *answers* this question.

seems to court criticism of the kind he knows he will be able to answer – inviting the entire sequence of detractors and defenders from Pappus on, all the way down to Knorr and myself. We have all been caught in the same snare. And, if so, the use of neusis too has a meaning within the specific communication environment set out in the introduction to *On Spiral Lines*.

/ 8 /

Given a circle and a line in the circle smaller than the diameter and another touching the circle at the end of the <line> given in the circle, it is possible to extend a certain line from the center of the circle towards the <given> line so that the <line> taken off from it <=the extended line> between the circumference of the circle and the line given in the circle has to the line taken off from the tangent the ratio laid down, if the given ratio is smaller than the <ratio> which the half of the <line> given in the circle has to the perpendicular drawn on it from the center of the circle.

(a) Let there be a given circle, <namely> the <circle> ΑΒΓΔ,[84] (b) and let a line be given in the circle, smaller than the diameter, <namely> ΓΑ, (c) and let ΞΛ touch the circle at the <point> Γ, (d) and <let there be given> a ratio, which Z has to H, smaller than the <ratio> which ΓΘ has to ΘΚ; (1) So, it shall also be smaller than the <ratio> which ΓΚ has to ΓΛ, (e) if ΚΛ is drawn parallel to ΘΓ.[85] (f) So, let ΚΓ have to ΞΓ the same ratio which Z has to H – (2) and ΞΓ is greater than ΓΛ.[86] (g) Let a circumference of a circle be drawn around K, Λ, Ξ.[87] (3) Now, since ΞΓ is greater than ΓΛ, (4) and ΚΓ, ΞΛ are at right <angles> with each other,[88] (h) it is possible to set another <line>, equal to ΜΓ, <namely> ΙΝ, verging towards K.[89] (5) So, the <rectangle> contained by the <lines>

[84] The point Δ is inert in this proposition (and redundant for defining a circle). So was B in the previous proposition. There is a gradual tendency to "pad" the alphabet so as to establish some local continuity in the reference of the major labels, doubtless related to the overall continuity between the propositions and their lack of an explicit setting-out and definition of goals (so that the particular labels take on some more "general" force).

[85] Steps 1–e reprise Step 1 of the previous proposition and 3 of the proposition before that – by now an argument so tired that even its required construction of parallel lines is hastily mentioned as an afterthought.

[86] Follows directly from Steps 1, f (*Elements* V.10).

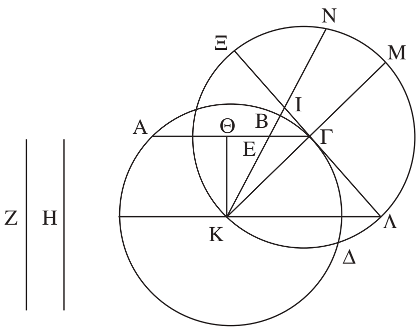
[87] Curiously, not a problem solved in the *Elements*, though easily provided by *Elements* III.1. Archimedes nearly neglects to suggest that one needs to produce a certain geometrical task in constructing this circle – a significant omission, as the existence of a unique solution depends upon there being a unique circle determined by three points; but then again, perhaps it does not really matter to Archimedes that his solution should be *unique*.

[88] Step C, *Elements* III.16.

[89] The "possibility" statement is not followed by an actual construction (so let it be so positioned . . .), but rather the possibility is taken as *tantamount* to the actual

ΞIΛ has to the <rectangle contained> by the <lines> KE, IΛ the same ratio which ΞI <has> to KE.[90] (6) and also the <rectangle contained> by KIN to the rectangle contained by KI, ΓΛ <=is the same as rect.(ΞI,IΛ): rect.(KE,IΛ)>.[91] (7) So that IN to ΓΛ, too, is as ΞI to KE.[92] (8) So that also: ΓM to ΓΛ, and ΞΓ to KΓ,[93] (9) and to KB[94] (10) is as ΞI to KE.[95] (11) And the remaining IΓ to BE[96] has the same ratio which ΞΓ <has> to ΓK.[97] (12) and which H <has> to Z.[98] (13) Now, KN fell on the tangent, and the <line> between the circumference and the line, <namely> BE, has to the line taken off from the tangent the same ratio, which Z <has> to H.

Rectangle contained: see Glossary



The diagram of codex C is not extant. H has the point Θ positioned so that it appears to be on the "upper" intersection of the two circles. G has ZH on the right-hand side. D has Z>H (slightly), E has Z>H (substantially). The diagram of B has been completely erased and redrawn by a second hand (and is not yet available in a digitally enhanced form).

construction. In this case it is evident that the point of the "possibility" is not to state that the neusis problem is solvable, but rather to point out that because ΞΓ>ΓΛ, while ΞΓK is a right angle (Steps 3–4), one can find IN=ΓM so that IN is positioned "above" Γ, ensuring the intercept of BE above the given line. A possible argument would be: any IK makes IK>KΓ (*Elements* I.19, 32), thus KN>KM. Hence KN must lie "nearer the center" than KM (*Elements* III.15), but ΞΛ is a diameter (*Elements* III.1), so clearly I is nearer the center, i.e. the bisection of ΞΛ, than Γ is: but for it to be nearer the bisection of ΞΛ, when ΞΓ>ΓΛ, is to be "above": I must be "above" Γ. (Dijksterhuis 1987: 137 has a different argument altogether, based on purely quantitative reasoning. I note incidentally that Heiberg, who apparently thinks the "it is possible" claim refers to the solvability of the neusis construction, also ignores the need for this argument.)

90 *Elements* VI.1.

91 See scholion. rect.(ΞI,IΛ) = rect.(KI,IN) (*Elements* I.35), while through *Elements* VI.2 IK:KE::IΛ:ΓΛ, hence rect.(KI,ΓΛ) = rect.(KE,IΛ). See the comments. It follows immediately of course that rect.(KI,IN):rect.(KI,ΓΛ) is also the same as ΞI:KE.

92 From the previous Step it followed implicitly that rect.(KI,IN):rect.(KI,ΓΛ)::ΞI:KE, from which, via *Elements* VI.1, we have IN:ΓΛ::ΞI:KE. See the textual comments.

93 So far the clause implicitly states ΓM:ΓΛ::ΞΓ:KΓ, based on *Elements* III.35 and analogous to previous reasoning.

94 The clause states KB=KΓ (radii of circle) and revises the clause so far to state ΓM:ΓΛ::ΞΓ:KB.

95 The force of Step 10 is ΓM:ΓΛ::ΞI:KE. This follows directly from Step 7 and the construction (Step h) IN=ΓM. The upshot of Steps 8–10 is ΞΓ:KB::ΞI:KE.

96 IΓ is what remains from ΞΓ, minus ΞI; BE is what remains from KB, minus KE.

97 We start with ΞΓ:KB::ΞI:KE (upshot of Steps 8–10). With *Elements* V.16 we have ΞΓ:ΞI::KB:KE, and with V.19 we have ("remaining") ΞΓ:IΓ::KB:BE, or ΞΓ:IΓ::ΓK:BE (radii in circle), or (*Elements* V.16 again, to close the circle) ΞΓ:ΓK::IΓ:BE.

98 Construction Step f. The upshot of Steps 11–12 is IΓ:BE::H:Z.

COMMENTS

Heiberg prints a different text for my Step 6 so that Steps 5–7 end up as:

(5) So, the <rectangle> contained by the <lines> ΞIΛ has to the <rectangle contained> by the <lines> KE, IΛ the same ratio which ΞI <has> to KE, (6) and also the <rectangle contained> by KIN has to the rectangle contained by KI, ΓΛ the same ratio which IN <has> to ΓΛ (7) so that IN to ΓΛ, too, is as ΞI to KE.

The reasoning is supposed to be as follows: since rect.(ΞI,IΛ)=rect.(KI,IN) (*Elements* III.35), while rect.(KE,IΛ)=rect.(KI,ΓΛ) (*Elements* VI.2), it follows from Steps 5–6, through *Elements* V.19, that ΞI:KE::IN:ΓΛ.

My reading is less transparent than Heiberg's. Instead of Step 6 stating the near-identity

(6) rect.(KI,IN):rect.(KI,ΓΛ)::IN:ΓΛ

I have it stating, implicitly at that, a more difficult claim (which calls for the kind of reasoning Heiberg needs for the transition from Step 6 to Step 7):

(6) rect.(KI,IN):rect.(KI,ΓΛ)::rect.(ΞI,IΛ):rect.(KE,IΛ)

My own transition to Step 7 does not call for a very complex argument, but it calls for some unpacking: one needs to understand what was stated only implicitly in Step 6, and then to manipulate that implicit expression.

Heiberg's text follows Commandino. Mine is the manuscripts' reading. While a homoioteleuton can be invoked to account for a lacuna of nine words (which is what Heiberg assumes), this is not a terribly compelling homoioteleuton, as the only shared bit is "ΓΛ" which, in the first instance, is within the longer expression KI, ΓΛ. Probably Commandino and Heiberg were struck by the abrupt, implicitly worded text of Step 6: "and also the <rectangle contained> by KIN to the rectangle contained by KI,ΓΛ." Also what, you wish to ask? This indeed has no precedent in the treatise so far. But this kind of implicit wording becomes rampant in this proposition so that, if anything, the context makes this particular, strange grammar an argument in favor of the manuscripts' reading.

THE IMPLICIT

While fully located within the set of constraints developed, by now as routine, through propositions 5–7, this proposition is of an entirely different character.

Let us quickly follow the thought underlying the construction. In proposition 6, we needed to find a certain segment intercepted between the given line and circumference (there labeled BE), so that its ratio to BΓ – the point Γ determined by the given line – should be fixed. Here, once again, we start with the line intercepted between the given line and circumference – also labeled BE – but instead of the chord BΓ we look for the segment of the tangent IΓ. BE:IΓ is to be the given ratio.

Now, because the two line segments BE, BΓ were connected in proposition 6 at the point B, it was easy to think of them as a triangle and to subsume their ratio within the simple proportions defined by triangles and parallel lines.

Here, however, BE and IN no longer share a vertex (otherwise ΞΙΓΛ would no longer be a tangent, touching at just one point!), hence the route of the simple triangle is blocked.

What can we do? Here is one way of thinking about this:

First, conceive of the two segments ΙΓ, BE as the residues of greater lines on which hopefully we may gain traction. It is clear what these should be for BE: it is very naturally conceived as the difference between BK, KE. Let us therefore conceive of ΙΓ, too, as the difference between two other lines, ΞΓ, ΞΙ (though note carefully: at this stage we do not know what Ξ is – it is just an arbitrary point).

Now, it is clear that our problem will be solved if we obtain:

ΞΓ : BK :: ΞΙ : KE :: given ratio

At which point it is very natural to think of the equality BK=ΓK from which we know we need:

ΞΓ : ΓK :: ΞΙ : KE :: given ratio

Now, the "ankle" where ΞΓ, ΓK meet each other is highly suggestive of the property of proportion with the meeting of chords in a circle: here's a way of employing the ratio ΞΓ:ΓK. There ought to be some usable circle whose circumference hosts not only Ξ and K but also M and Λ so that:

ΞΓ : ΓK :: ΓM : ΓΛ

And since the circle is so far completely undetermined (we merely require it to pass through K), we might as well make the point Λ stand on the diameter, in an arrangement more reminiscent of the preceding propositions.

Thus, at this point, we look for a circle passing through the two points K, Λ so that:

MΓ : ΛΓ :: ΞΙ : KE :: given ratio

And it is a happy thought that one can combine *Elements* III.35 (applied for another intersection of chords) together with the standard results on triangles and parallels, to derive:

NI : ΓΛ :: ΞΙ : KE :: given ratio

Which solves our problem as long as NI=MΓ.

It is apparent at this point that a construction where ΞΓ:ΓK is the given ratio, with a circle drawn through K, Λ and Ξ thus found, and IN inserted, verging towards K, equal to ΓM, would solve the problem.

The net result of this line of thought is that we spring our scaffolding for proportions outwards, to encompass a circle external to the one given in the proposition, even while keeping the old scaffolding of a set of parallel lines. This provides for a very elegant configuration – circle, parallel lines, yet another circle – and an elegant conceptual combination: *Elements* III.35 together with *Elements* VI.2. This proposition is the first elegant piece of accomplished geometry in this book.

All of this is "implicit." In technical terms, this proposition offers a synthesis, whereas my account above was more akin to an analysis. It appears that the argument prefers the implicit at other, more elementary levels of organization as well. Clearly this is the case with the two clusters of Steps:

(8) So that also: ΓM to ΓΛ, and ΞΓ to KΓ, (9) and to KB (10) is as ΞI to KE

(11) And the remaining IΓ to BE has the same ratio which ΞΓ <has> to ΓK, (12) and which H <has> to Z

As I explain in my footnotes, Steps 8–10 end up stating that ΞΓ:KB::ΞI:KE; Steps 11–12 end up stating that IΓ:BE::H:Z. The first result, of 8–10, is the key geometrical result required for the argument; the second, of 11–12, is the actual goal of the proposition (stated in particular terms). But neither is said: both have to be reconstituted from several clauses, each with its own argumentative claim. Nor are the various argumentative claims along the way fully set out: we need to work out that Step 8 actually asserts that ΓM:ΓΛ::ΞΓ:KΓ, that Step 9 refers to the equality of radii KΓ=KB, that Step 12 reminds us that ΞΓ:ΓK::H:Z. Each of those interim claims are very easy: the equality of radii is true by definition, ΓM:ΓΛ::ΞΓ:KΓ is a very central property of proportion in a circle, while ΞΓ:ΓK::H:Z is true by construction. One is therefore not asked to supply, on the fly, some complicated result; rather, avoiding a fully fledged statement of such simple claims makes them much more difficult to compute than they ought to be.

I believe the same should be understood for the sequence Steps 5–7, discussed in the textual comments above, with its elliptic statement of Step 6 and the implicit argument of Step 7: we now see how such a reading – that of the manuscripts – is in keeping with the proposition as a whole.

Going further back: the first four Steps 1–4 essentially provide the grounds for the feasibility of the neusis construction, as explained in n. 89 above. The main feasibility claim, however, is left mostly implicit (Steps 3–4 state the grounds for the possibility, but do not sketch the somewhat complicated argument required) – so much so that Heiberg felt that the feasibility claim asserted that one can solve in general such a neusis (and not that the configuration at hand ensures that I is "above" Γ).

To sum up: the proof has four main segments: 1–4, 5–7, 8–10, 11–12. The segment 1–4 belongs to the construction proper. Its main argument is implicit. The segment 5–7 sets up the tools used by the proof. It is elliptic and difficult to follow. The segment 8–10 reaches the main result, while the segment 11–12 shows how the main result accomplishes the task: both are stated through a sequence of implicit, unfinished clauses.

What is the aim of this implicit character? On the one hand, it keeps this proposition within the bounds of the previous ones. It is not evidently longer than any of the propositions 5–7. Even in the qualitative terms of the "air" of a complex argument, the proof employs no ἄρα, using instead the usual particles of observation-from-a-distance, δή (Steps 1, 5) and οὖν (Step 13), as well as the connector of a downgraded, mere-epiphenomenal-consequence result, ὥστε (Steps 7, 8). Other than this, the melding of arguments to each other allows Archimedes to attach logically separate claims as mere clauses connected via various forms of "and." The impression is as if no complex geometrical argument is at stake at all.

On the other hand, the implicit character of the proposition serves to support its elegant structure. Add in an explicit argument why I is above Γ, and an explicit unpacking of Steps 5–7, 8–10, 11–12, and one would look at an argument much longer than that of any of the preceding propositions, one that would appear like rather heavy work. The proposition as it stands is very difficult for the reader, but it does provide for a sense of effortlessness on the side of the author.

On the one hand, the implicit character of this proposition makes Archimedes appear less sophisticated than he really is (little argument!). On the other hand, it makes Archimedes appear more sophisticated (look how easy this is for him!). Those are two different models of sophistication, the first more pedagogic and explanatory, the other more adversarial and elliptic. The choices, as ever, are choices within a pattern of communication. And if Archimedes opens himself up for the criticism of the one who merely *claims* to prove – well, the next proposition will show that he is, after all, capable of making his claims explicit. Thus propositions 8–9 are the Archimedean exchange in miniature: the challenge to the reader in proposition 8; its partial lifting in 9.

Indeed, perhaps one possible key to the character of proposition 8 is to consider it in context. One obvious feature of the sequence of propositions 5–9 is its gradual transition into geometrical complexity: while there is an underlying conceptual gap separating 5–7 from 8–9, the implicit, brief character of proposition 8 makes it appear as something of a bridge leading on to the final claim in this geometrical sequence.

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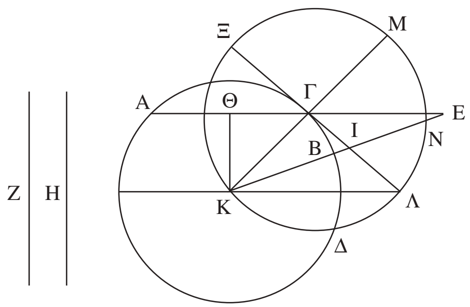
Given the same things, and with the line given in the circle being produced, it is possible to extend a line from the center of the circle towards the produced line, so that the <line> between the circumference and the extended line has to the <line> taken off from the tangent towards the touching point the ratio laid down, if the given ratio is greater than the <ratio> which the half of the line given in the circle has to the perpendicular drawn on it from the center.

(a) Let a circle be given, <namely> the <circle> ΑΒΓΔ, (b) and let a line be drawn through the circle, smaller than the diameter, <namely> ΓΑ, (c) and let ΞΓ touch the circle at the <point> Γ, (d) and <let a ratio be given>, which Z has to H, greater than the <ratio> ΓΘ has to ΘΚ; (1) so, it shall be greater, also, than the <ratio> which ΚΓ has to ΓΛ.[99] (e) Now, let ΚΓ have to ΓΞ the same ratio, which Z <has> to H. (2) Therefore that <line =ΓΞ> shall be smaller than ΓΛ.[100] (f) So, again, let a circle be drawn

[99] Same as Step 1 in the previous proposition: now, without even drawing the parallel line explicitly!

[100] *Elements* V.10.

through the points Ξ, K, Λ. (3) Now, since ΞΓ is smaller than ΓΛ, (4) and KM, ΞΓ are at right <angles> to each other,[101] (g) it is possible to set a <line>, equal to ΓM, <namely> IN, verging towards K. (5) Now, since the <rectangle contained> by ΞIΛ is to the <rectangle contained> by ΛI, KE as ΞI to KE,[102] (6) but the <rectangle contained> by ΞIΛ is equal to the <rectangle contained> by KIN[103] (7) while the <rectangle contained> by ΛI, KE is equal to the <rectangle contained> by KI, ΓΛ (8) because of its being: as KE to IK, so ΛΓ to ΛI,[104] (9) therefore also: as ΞI to KE, so the <rectangle contained> by KIN to the <rectangle contained> by KI, ΓΛ, (10) that is, as NI to ΓΛ, (11) that is, as ΓM to ΓΛ.[105] (12) And it is also: as ΓM to ΓΛ, ΞΓ to KΓ,[106] (13) that is to KB;[107] (14) therefore it is: as ΞI to KE, so ΞΓ to KB,[108] (15) and the remaining IΓ to the remaining BE is as ΞΓ to ΓK.[109] (16) And the ratio which ΞΓ has to ΓK, is that which H has to Z.[110] (17) So, KE fell on the produced <line>, and the <line> between the produced <line> and the circumference, <namely> BE, has to ΓI, the line taken off from the tangent, the same ratio, which Z <has> to H.



D has the entire figure substantially tilted clockwise, and so has E (much more subtly). In both, this may be because of space concerns, but perhaps codex A had the same tilt. D misses Δ; H misses A and positions B at the same point as Γ. D has Z>H (by much), E (by little). Codex C is badly preserved. It has Δ instead of Λ, and misses the “real” Δ. It cannot be judged if it has B, N and E. It has an extra line (similar to that of proposition 6) extending from K in a north-by-northwest direction towards the line AE. I now judge (pace Netz et al. 2011) that the line AE did extend, as it should, to the right of the upper circle. The diagram of B has been completely erased and redrawn by a second hand (and is not yet available in a digitally enhanced form).

[101] Step C, *Elements* III.16. [102] *Elements* VI.1.

[103] *Elements* III.35. I invert the original Greek case order (“to ΞIΛ is equal KIN”) to preserve the word order which, in English, carries the pragmatics of topic and comment.

[104] *Elements* VI.2, Step 8 supporting Step 7 and making explicit an essential part of the argument of the preceding proposition. For the transition from 8 to 7, see *Elements* VI.16.

[105] The upshot of Steps 9–11 is ΞI:KE::ΓM:ΓΛ (*Elements* VI.1, and construction Step g, respectively).

[106] *Elements* III.35.

[107] The upshot of 12–13 is ΓM:ΓΛ::ΞΓ:KB. Step 13 is based on KΓ, KB both being radii.

[108] One implicitly draws (*Elements* V.16) the result ΞI:ΞΓ::KE:KB.

[109] *Elements* V.17 applied on the implicit result of the previous Step 14 (see preceding note).

[110] Step e of the construction.

COMMENTS

Steps 9–11, 12–13 are technically a "meld" in the manner of 6, 8–10, 11–12 of the preceding proposition, in the sense that one cannot understand the meaning of Steps 10, 11 without reading 9 as well, or understand Step 13 without reading 12 as well.

There is a difference, though: Steps 9 and 12, the host clauses, are fully syntactic on their own: the meld is *at the edges*. Put more concretely, there is an important cognitive difference between

(i) (1) A, that is B, (2) is equal to C

and

(ii) (1) A is equal to B, (3) that is to C

The meld of (i) is internal; the meld of (ii) is at the edges. As a consequence, (i) has no explicit statement: (1), the host clause, is not a fully formed expression. As a consequence, the parasite clause (2) does not have a clear template in which to fit – it does not read as a simple call for substitution. (ii), on the other hand, does have the explicit statement (1), which now serves as host to (2) in a much more straightforward manner – we are asked to replace B by X in order to derive the implicit claim.

Indeed, the connector "that is" is, simply, a more obvious call for substitution (unlike the various enigmatic "ands" thrown about in the previous proposition). The substitutions themselves, finally, are very clear (Step 10: *Elements* VI.1, Step 11: construction Step g, Step 13: radii in circle respectively).

Most important, all of this has been said in the preceding proposition. There is simply less burden on the reader now: to the extent that the reader took anything away from the preceding proposition, he should understand the flow of the argument much more clearly.

It is clear that this proposition is much less implicit in and of itself: in particular, the underlying logic of combining *Elements* III.35 and *Elements* VI.2 is made explicit for the first time, in Step 8. One further notices the three instances of the particle αρα, "therefore," providing an air of a "normal" proof. Those "therefore"s emerge as the argument becomes more fully spelled out, but the particle is introduced even when the argument as such is unchanged from previous proofs: Step 2 of proposition 8 was connected by a paratactic "and";[111] Step 2 of this proposition, stating essentially the same claim (with a sign difference), based on the same reasoning, has "therefore." This, in and of itself, is a plausible ground for trusting the manuscripts' evidence. That is, I do not believe that the text of proposition 8 was once more expanded, or that that of proposition 9 was once more abbreviated. For, if so, why would a meddling scribe, abbreviating proposition 8, or another one, expanding 9, also change the connectors in those propositions? Not that this is a solid scenario to begin

[111] δέ in the manuscripts. Heiberg, following Torelli, emends to δή, a possible, though unnecessary, emendation – which is still a "distant" marker of transition, not a marker of argumentative consequence.

with: the syntactic structures of proposition 8 are too tight, and it is for a reason that past editors have not expanded them (other than expanding Step 6). Nor is it clear why a scholiastically minded scribe should expand 9 and not 8.

In short, I believe the structure we see is intentional: proposition 8 is almost entirely implicit, proposition 9 is more explicit. Of course, it is still not *fully* explicit. The underlying argument for why the neusis is possible at that point – i.e. why I ought to fall "beneath" Γ – is left implicit here, as was the analogous argument in the preceding proposition; the several uses of "that is" smooth considerably the argument in its final transitions; the basic arrangement of the general statement in terms of "given the same things," together with the (by now familiar) avoidance of a definition of goal, all provide the argument with a somewhat curtailed character. Most important, this is the nature of the neusis construction itself, which points to the feasibility of a construction instead of providing it explicitly. Thus the overall pattern of propositions 5–9 is maintained; they are propositions in outline. Archimedes chose to make the outline gradually more concrete.

First of all, this is because the significance of the neusis construction is changed between propositions 5–7 and propositions 8–9. In propositions 5–7, the act of neusis provides the bulk of the geometrical work, and therefore these propositions foreground their mere statement of a possibility, background the actual geometrical argument accomplished. In propositions 8–9, the act of neusis looms smaller: it is by now familiar, while a much more complex and elegant geometrical construction is called for. Thus propositions 8–9 foreground the actual geometrical argument accomplished, background the mere statement of a possibility.

The transition from 5–7 to 8–9 is a leap in the forcefulness of the argument. This leap is smoothed by having the argument of proposition 8 drastically curtailed, so as to resemble more closely those of propositions 5–7. Finally, proposition 9 is the first proposition in this book to resemble "normal geometry," and we may well expect the book to turn normal, with a sequence of fully fledged geometrical arguments.

Possibly the most puzzling feature of this treatise is the presence of propositions 6 and 9, which do not serve any function in the deductive structure of the treatise. It is possible that Archimedes saw his chain of deductions differently from us, or that he planned other parts of the treatise (which we may have lost) or that he simply wanted to add in more propositions for their own sake (as we recall, the introduction explicitly stated that the treatise may include results beyond those stated). But such suggestions appear far-fetched: Archimedes surely must have known exactly which constructions were required for propositions 18–20 (he must have started working on propositions 5–9 only because he knew he needed certain well-defined constructions, emerging from the study of 18–20). There is no reason to see the treatise as it stands as in some major way incomplete: propositions 6 and 9 are the major complication in its structure which otherwise is very carefully arranged. And propositions 6 and 9 are certainly not what we would consider extremely interesting in their own right (even though 8–9 are definitely elegant).

From Archimedes' point of view, his starting-point was a set of propositions – 5, 7 and 8 – which fit very well the pattern of paired propositions typical of the treatise as a whole (in this case, a pair because one needs to apply the result for the two cases of a double proof by contradiction or "the method of exhaustion"). He then decided to *pad* the results. The consequences of this are multiple. First, since the readers are not provided with cross-references, it becomes more difficult for them to know which results are required in 18–20 (in this way, the way in which 18–20 is put together becomes more opaque). Second, the passage of propositions 5–9 becomes more gradual. Third, the entire introductory set gains in size relative to the main set of results, later on. It seems likely that such architectural consequences are not unintended (Archimedes could easily have avoided them, and chose not to). If these consequences are intended, we imagine an Archimedes who does not wish to set out very clearly the grounds for his key claims; and who wishes to extend the introductory passage, the "development section" of his sonata, so as to intensify anticipation.

As well he might. At this point, we may well be expecting the geometrical work to start in earnest. This expectation is not to be fulfilled.

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In the footnotes to this proposition the following abbreviations are used:

SameLine	sum of same lines (A, B+I, Γ+K etc.)
DiffLine	sum of different lines (A, B, Γ etc.)
SameSquare	sum of all the squares on the same lines (on A, B+I, Γ+K, etc.)
DiffSquare	sum of all the squares on the different lines (A, B, Γ etc.)
SmallRects	sum of all rectangles such as on I,B, on K,Γ, on Λ,Δ etc.
BigRect	rectangle contained by (I, DiffLine)

If however many lines are set in order, exceeding each other by an equal <difference>, and the excess is equal to the smallest <line>, and other lines are set equal to them <=the lines exceeding each other> in number[112] while, in magnitude, each is equal to the greatest <line> <=among the lines exceeding each other>,[113] the

Magnitude: see Glossary

Equal in number: see Glossary

Equal in magnitude: see Glossary

[112] I.e. if the first sequence of unequal lines has eight lines, then the equal lines, too, should be eight in number.

[113] I.e. each of the equal lines is equal to the greatest of the *unequal* lines. In concrete numerical terms: we may have the smallest as 2, and then the sequence of the lines exceeding each other is 2, 4, 6, 8, 10, 12, 14, 16; the equal lines are all 16. (There are numerical examples accompanying the diagram in both codices A and C, but the numbers are hard to read from C: I use those of A, and I suppose those of C are the same, or some corrupt version thereof, so that this scholion of concrete numerical values is probably from late antiquity at the latest.) Note that Archimedes himself – as well as the diagram and the scholiast – observes the sequence *downwards* (even though the determining relationship is of *exceeding*).

squares[114] on the <lines> equal to the greatest <line> <=among the lines exceeding each other>, adding on both:[115] the square on the greatest <line>,[116] and the <rectangle>[117] contained by both: the smallest <line> <=among the lines exceeding each other>, and the <line> equal to all the <lines> exceeding each other by an equal <difference>[118] – shall be three times all the squares that are on the <lines> exceeding each other by an equal <difference>.[119]

Let there be, however, many lines set in order,[120] exceeding each other by an equal <difference>, <namely> the <lines> A, B, Γ, Δ, E, Z, H, Θ, and let Θ be equal to the difference, and let I, equal to Θ, be added to B; K, equal to H, <be added> to Γ; L, equal to Z, to Δ; M, equal to E, to E; N, equal to Δ, to Z; Ξ, equal to Γ, to H; O, equal to B, to Θ – (1) and the lines that come to be[121] are equal to each other and to the greatest <=of the lines exceeding each other =A>[122] – Now, it is to be proved that the squares on all the <lines> A, as well as the <lines> coming to be, adding on both: the square on A and the <rectangle> contained by both: Θ and the <line> equal to all the <lines> A, B, Γ, Δ, E, Z, H, Θ are three times all the squares on A, B, Γ, Δ, E, Z, H, Θ.

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[114] As usual, "the squares" means "the sum of the squares."

[115] To the sum of the squares we add on two magnitudes.

[116] We add on yet another square on the greatest line: if we started with n lines, there are by now n+1 squares . . .

[117] . . . and we also add a rather complex rectangle.

[118] One side of this rectangle is the smallest line – in the concrete numerical example, it is 2. The other side is much bigger: the sum of all the lines exceeding each other – in the concrete numerical example, it is 2+4+6+8+10+12+14+16.

[119] So, the above sum should be three times the following sum of squares (in the concrete numerical example): $2^2+4^2+6^2+ \ldots +16^2$.

[120] This "set in order" (or "set one after the other") is a bit curious: and what if they were not so set? This seems to rule out the possibility of some complex algebra of exceeding: say, C exceeds B which exceeds A, both by the fixed difference, but D also exceeds B by the same difference and E exceeds A (i.e. D=C, E=A). So they are to exceed each other while being *set in order*. Or this may be no more than an exercise in bookkeeping, clarifying that the sequence of the alphabet, and of the diagram, stands also for the (decreasing) sequence of magnitude.

[121] The synthetic lines produced by adding I to B, K to Γ, etc.

[122] One expects Archimedes to have two sequences of lines: one, those exceeding each other, another, those equal to the greatest; in a surprise move, he builds on the first sequence to produce the second one, in a way which even calls for a non-transparent argument. It can be seen that B+I is indeed equal to A: I is constructed equal to Θ which is the difference and so, with the lines set in order, I does fill up the difference of A from B. One then needs to follow inductively the sequence of equalities (which, however, are transparent in the diagram, which is in this instance fairly metrically correct). This intrusion of an argument happens within the setting-out: now, for a change, we get an explicit setting-out and definition of goal. The manuscripts have this intrusion as a brief "and" clause, which Heiberg, following Nizze, emends to a "so" transition (δέ to δή).

(1) So, the square on BI[123] is equal to the squares on I, B and to two <rectangles> contained by B, I,[124] (2) while the <square> on KΓ is equal to the squares on K, Γ and to two <rectangles> contained by K, Γ.[125] (3) And also similarly, the squares on the other <lines> equal to the <line> A, are equal to the squares on the segments and to two <rectangles> contained by the segments.[126] (4) Now, the <squares> on A, B, Γ, Δ, E, Z, H, Θ and the <squares> on I, K, Λ, M, N, Ξ, O, adding on the square on A, are twice the squares on A, B, Γ, Δ, E, Z, H, Θ;[127] (5) we will prove, as what remains, that twice the <rectangles> contained by the segments in each line of those equal to A, adding on the <rectangle> contained by both: Θ and the <line> equal to all the <lines> A, B, Γ, Δ, E, Z, H, Θ, is equal to the <squares> on A, B, Γ, Δ, E, Z, H, Θ.[128] (6) And since two <rectangles>, the <rectangles> contained by B, I are

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[123] BI does not refer to a line whose end points are B, I, but to a line segment composed of the two line segments B, I.

[124] *Elements* II.4. [125] *Elements* II.4.

[126] Two observations: (1) this is not a generalization, stating that καθόλου (in general) the squares – all eight of them – are equal to squares and rectangles, but an extension, referring explicitly only to the six squares not covered so far. The generalization to cover all eight squares is then understood to follow from Steps 1–3 taken together. This is an interesting vignette for the Greek treatment of generality. (2) Note that, this time, “the squares” refers not to the sum of the squares but to each square separately. How can we tell? By figuring out Archimedes’ meaning in context. Another interesting vignette on the Greek treatment of quantifiers.

[127] Directly follows from the constructed equalities I=Θ, K=H etc.

[128] Our definition of goal asked us to prove that

$$\text{Def. of Goal.}\quad A^2 + (B+I)^2 + \cdots + (\Theta + O)^2 + A^2 + (\Theta * (A + B + \cdots + \Theta) = 3 * (A^2 + B^2 + \cdots + \Theta^2)$$

or (inventing a transparent code for the expressions)

$$\text{Def. of Goal.}\quad \text{SameSquare} + A^2 + \text{BigRect} = 3 * \text{DiffSquare}$$

Now, in Steps 1–3 we have obtained an equality

$$A^2 + (B+I)^2 + \cdots + (\Theta + O)^2 = (A^2 + B^2 + \cdots + \Theta^2) + (I^2 + K^2 + \cdots + O^2) + 2 * (\text{rect.}(I, B) + \text{rect.}(K, \Gamma) + \cdots + \text{rect.}(O, \Theta)) \qquad 1\text{–}3$$

or (some more transparent code)

$$\text{SameSquare} = (A^2 + B^2 + \cdots + \Theta^2) + (I^2 + K^2 + \cdots + O^2) + 2 * \text{SmallRects} \qquad 1\text{–}3$$

so that the Definition of Goal could implicitly be revised to read

$$\text{D. o G.} + 1 - 3 \quad (A^2 + B^2 + \cdots + \Theta^2) + (I^2 + K^2 + \cdots + O^2) + 2 * \text{SmallRects} + A^2 + \text{BigRect} = 3 * \text{DiffSquare}$$

equal to two <rectangles>, the <rectangles> contained by B, Θ,[129] (7) while two <rectangles>, the <rectangles> contained by K, Γ are equal to the <rectangle> contained by both: Θ and four times Γ,[130] (8) through K's being twice Θ,[131] (9) and two <rectangles>, the <rectangles> contained <by> Δ, Λ are equal to the <rectangle> contained by Θ and six times Δ, (10) through Λ's being three times Θ, (11) and similarly also: the others, twice the <rectangles> contained by the segments are equal to the <rectangle> contained by both: Θ and the multiple, ever <ascending> according to the even numbers in sequence, of the following line.[132] (12) Now, all the <double rectangles> taken together,[133] adding on the <rectangle> contained by both: Θ and the <line> equal to all the <lines> A, B, Γ, Δ, E, Z, H, Θ, shall be equal to the <rectangle>

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which we may rearrange as follows

$$\text{D. o G.} + 1 - 3 \text{ rev.} \qquad (A^2 + B^2 + \cdots + \Theta^2) + (I^2 + K^2 + \cdots + O^2) + A^2 + 2 * \text{SmallRects} + \text{BigRect} = 3 * \text{DiffSquare}$$

But in Step 4 we are reminded that:

$$(A^2 + B^2 + \cdots + \Theta^2) + (I^2 + K^2 + \cdots + O^2) + A^2 = 2 * \text{DiffSquare} \qquad 4$$

Removing both terms from both sides of the revised and rearranged Definition of Goal, we are left with:

$$2 * \text{SmallRects} + \text{BigRect} = 1 * \text{DiffSquare} \qquad 5$$

This then is the claim of Step 5. "As what remains," charmingly, has two meanings at once: this is what remains to be proved, and this is what remains when equation 4 is subtracted from equation "D. o G. + 1–3 rev."

[129] Follows from construction in the setting-out: I=Θ.

[130] Step 6 is governed by a μέν, answered by a δέ in Step 7, which creates an expectation of parallelism between 6 and 7: thus one expects Step 7 to assert 2*K*Γ=2*Γ*H. The turn midway through Step 7, to assert 2*K*Γ equal not to 2*Γ*H but to 4*Θ*Γ is thus a lovely surprise. (One could have provided for a symmetry between Steps 6 and 7 by referring to rect.(B, Θ) in Step 6 as "the <rectangle> contained by Θ and B." Not only does Archimedes not wedge the two lines Θ, B together with an "and" to highlight the fixed term Θ, he even orders them as B, Θ. I can't believe this is not intentional.)

[131] It follows from the construction of the lines exceeding each other that the second is twice the first, the third is three times the first, etc. This consequence of the construction has not been made explicit before, but its special cases are asserted in Steps 8 and 10. From Step 8 it follows that K*Γ=2*Θ*Γ or, doubling both sides, 2*K*Γ=4*Θ*Γ, which is the claim of Step 7.

[132] As we move to a higher multiple, we also move to a "following" line – four times Γ, six times Δ, eight times E etc. We now find out why the lines are arranged from greater to smaller. To recap the meaning of Steps 6–11: they combine to show that:

$$2 * \text{SmallRects} = \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times } \Gamma + \text{six times } \Delta + \text{etc.}))$$

[133] That the reference of the "all taken together" is to the *twice* of the rectangles is understood only from the mathematical context.

contained by both: Θ, and the <line> equal to all: A, as well as three times B, and five times Γ, and the odd multiple, ever <ascending> according to the odd numbers in sequence, of the following line.[134] (13) And the squares on A, B, Γ, Δ, E, Z, H, Θ, too, are equal to the <rectangle> contained by the same lines.[135] (14) For the square on A is equal to the <rectangle> contained by both: Θ and the <line> equal to all: to both A, and the <line> equal to the remaining, of which each is equal to A[136] – (15) For they measure equally: both Θ <measuring> A, and A <measuring> all the <lines> equal to it, with A[137] – (16) so that

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[134] We have just established:

$$2 * \text{SmallRects} = \text{rect.}(\Theta,\quad (\text{twice B} + \text{four times }\Gamma + \text{six times }\Delta + \cdots))$$

Obviously, if we add on both sides rect.(Θ, (A+B+Γ+etc.) – the rectangle we earlier called BigRect – we obtain

$$2 * \text{SmallRects} + \text{rect.}(\Theta,\quad (\text{A} + \text{B} + \Gamma + \cdots) = \text{rect.}(\Theta,\quad (\text{A} + \text{three times B} + \text{five times }\Gamma + \text{seven times }\Delta + \cdots)$$

or

$$2 * \text{SmallRects} + \text{BigRect} = \text{rect.}(\Theta,\quad (\text{A} + \text{three times B} + \text{five times }\Gamma + \text{seven times }\Delta + \cdots)$$

[135] I.e. those squares (what I also call DiffSquare) are equal to "the <rectangle> contained by both: Θ, and the <line> equal to all: A, as well as three times B, and five times Γ, and the odd multiple, ever <ascending> according to the odd numbers in sequence, of the following line." Step 13 – the key step of the proof – is not a fully fledged statement, and requires Step 12 to unpack its meaning!

Why is Step 13 the key of the proof? Because we have just shown in Step 12 that:

$$2 * \text{SmallRects} + \text{BigRect} = \text{rect.}(\Theta,\quad (\text{A} + \text{three times B} + \text{five times }\Gamma + \text{seven times }\Delta + \cdots)$$

And we now assert in Step 13 that

$$\text{DiffSquare} = \text{rect.}(\Theta,\quad (\text{A} + \text{three times B} + \text{five times }\Gamma + \text{seven times }\Delta + \cdots)$$

so that, if Step 13 is true, it should follow that

$$2 * \text{SmallRects} + \text{BigRect} = \text{DiffSquare}$$

This, it was argued in Step 5, is what is required for the claim of this proposition to be true.

Steps 14–21 move on to show that Step 13 is indeed true.

[136] If we coin a new term, SameLine, to refer to the sum of the sequence of lines equal to A (including A itself), Archimedes now asserts in Step 14 that A^2=rect.(Θ, SameLine). It is still not clear why Step 13 follows. His separation of A from SameLine is in preparation for the operations from Step 16 onwards.

[137] Step 15 shows why Step 14 is true (we are still in the dark regarding Step 13). In this particular case Θ is one-eighth A, while A is one-eighth SameLine. Why? Because in the sequence of lines exceeding each other, the second is twice the smallest line, the third is three times, etc., so that the greatest line is as many times the smallest line as there are lines in the sequence; while SameLine has as

the square on A is equal to the <rectangle> contained by both: Θ and the <line> equal to A, and the double of B, Γ, Δ, E, Z, H, Θ;[138] (17) for the lines equal to A, all besides A, are doubles of B, Γ, Δ, E, Z, H, Θ.[139] (18) And similarly also: the square on B is equal to the <rectangle> contained by both: Θ and the <line> equal to both: B and the double of Γ, Δ, E, Z, H, Θ,[140] (19) and again: the square on Γ is equal to the <rectangle> contained by both: Θ and the <line> equal to both: Γ and the double of Δ, E, Z, H, Θ.[141] (20) And similarly also the squares on the other <lines> are equal to the <rectangles> contained by both: Θ and the <line> equal to both: the <line> itself <=on which the square was formed> and the double of the remaining[142] <lines>.[143] (21) Now, it is clear[144] that the

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many times the greatest line as there are lines in the sequence. This Archimedes does not explain (nor did he explicitly say that the sequence of lines exceeding each other yields a series such as 1, 2, 3, . . . n). If we have such a series where A: B::B:C (A measures B as B measures C) then rect.(A, C)=sq.(B) (*Elements* VI.17, since we are dealing with geometrical magnitudes; though "measures" is an arithmetical term).

[138] Step 16 follows from Step 14, not 15 ("so" is meant to skip Step 15). Why does it follow? It transforms Step 14, assuming rect.(SameLine, excluding A)=2*rect.(DiffLine, excluding A). See next Step.

[139] By the construction in the setting-out (or by the diagram, read metrically), it is apparent that B=O, Γ=Ξ, etc., so that SameLine as a whole, excluding A, is twice DiffLine, as a whole, excluding A.

[140] This "similarly" argument takes a little decomposition: it is not apparent that the same structure of argument is preserved as we move to squares on segments smaller than A, for, after all, did we not rely on the equality of each couple of segments such as B+I to A itself? Let us follow this:

"similarly" Steps 14–15: Θ measures B the same number of times as B measures as many times B's, as there are lines from B "downwards" (in this case – seven B's). So the square on B is equal to a rectangle contained by: Θ, and seven B's.

"similarly" Steps 16–17: But seven B's – i.e. as many B's as there are going "downwards" from B – are equal to: B, and the double of B, Γ, Δ, E, Z, H, Θ. (First we take B on its own and then, in the manner of Step 17, we see that the B's remaining from Γ onwards – six B's – are what is produced by doubling Γ, Δ, E, Z, H, Θ, through Γ+Θ=B, Δ+H=B, etc.)

[141] This Step is not in codex C (which Heiberg failed to notice). Codex C has several obvious errors in the previous Step as well, and I believe it is simply more corrupt here than A is: no change is required in Heiberg's printed text.

[142] "Remaining": going "rightwards" in the diagram, or "up" in the alphabet sequence, or in decreasing magnitude order.

[143] Once again, instead of generalizing a result, Archimedes *extends* it to cover the remaining cases, the generalization as such remaining implicit.

[144] Not immediately clear, perhaps. Let us review.

squares on all <lines>[145] are equal to the <rectangle> contained by both: Θ and the <line> equal to both: A, as well as three times B, and five times Γ, and the multiple, according to the odd numbers in sequence, of the following <line>.[146]

Sq. (A) =	rect. (Θ,	(A+	2*(DiffLine-A)	))	Step 14
Sq. (B) =	rect. (Θ,	(B+	2*(DiffLine-A-B)	))	Step 18
Sq. (Γ) =	rect. (Θ,	(Γ+	2*(DiffLine-A-B-Γ)	))	Step 19
. . .					Step 20

Hence:

DiffSquare = rect. (Θ, (X))

What is the value of X?

It is the sum:

$$A + B + \Gamma + \cdots \quad + \Theta + 2 * (\text{DiffLine} - A) + 2 * (\text{DiffLine} - A - B)$$
$$+2 * (\text{DiffLine} - A - B - \Gamma) + \cdots$$

But A+B+Γ+ . . . +Θ itself is DiffLine.

In other words, we first take DiffLine, then add to it 2*(DiffLine-A), then 2*(DiffLine-A-B), etc., until DiffLine is exhausted.

We may visualize this as follows:

First Step (DiffLine alone):

X	X	X	X	X	X	X	X
A	B	Γ	Δ	E	Z	H	Θ

Second Step (adding in 2*(DiffLine-A)):

	X	X	X	X	X	X	X
	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X
A	B	Γ	Δ	E	Z	H	Θ

Third Step (adding in also 2*(DiffLine-A-B)):

		X	X	X	X	X	X
		X	X	X	X	X	X
	X	X	X	X	X	X	X
	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X
A	B	Γ	Δ	E	Z	H	Θ

At which point it is becomes clear that X will end up being, once DiffLine is exhausted, once A, plus three times B, plus five times Γ, etc. Hence:

$$\text{DiffSquare} = \text{rect.}(\Theta, \qquad (\text{once A, three times B, five times } \Gamma, \text{etc.}))$$

Which is the claim of Step 21.

[145] Meaning what I call DiffSquare.

[146] Step 21 asserts that the claim of Step 13 is correct.

/COROLLARY/

Now, from this it is obvious that (i) all the squares on the <lines> equal to the greatest <line> are smaller than three times the squares on the <lines> exceeding each other by an equal <difference>[147] – (1) since by adding on certain <magnitudes>[148] they are three times – while (ii) they are greater than three times the remaining squares, without the <square> on the greatest <line>[149] – (2) since the added-on are smaller than three times the square on the greatest <line>.[150] And, furthermore, if similar figures are set up on all the <lines>, on both the <lines> exceeding each other by an equal <difference> as well as the <lines> equal to the greatest <line>, (iii) the figures on the <lines> equal to the greatest <line> shall be smaller than three times the figures on the <lines> exceeding each other by an equal <difference>, while being (iv) greater than three times the remaining <figures>, without the figure on the greatest <line>. (3) For similar figures have the same ratio as the squares.[151]

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[147] In my notation: $\text{SameSquare} < 3*\text{DiffSquare}$.

[148] Namely: the certain thing is $(\text{A}^2 + \text{BigRect})$. To recall: the main proposition shows:

$$\text{SameSquare} + (\text{A}^2 + \text{BigRect}) = 3 * \text{DiffSquare}$$

So obviously, indeed:

$$\text{SameSquare} < 3 * \text{DiffSquare}$$

[149] In my notation: $\text{SameSquare} > 3*(\text{DiffSquare} - \text{A}^2)$.

[150] Namely: $(\text{A}^2 + \text{BigRect}) < 3*(\text{A}^2)$. To recall, the main proposition shows that:

$$\text{SameSquare} + \text{A}^2 + \text{BigRect} = 3 * \text{DiffSquare}$$

So if

$$(\text{A}^2 + \text{BigRect}) < 3 * (\text{A}^2)$$

it would follow, subtracting from both sides, that

$$\text{SameSquare} > 3 * (\text{DiffSquare}) - 3 * (\text{A}^2)$$

or simply

$$\text{SameSquare} > 3 * (\text{DiffSquare} - \text{A}^2).$$

How do we get:

$$(\text{A}^2 + \text{BigRect}) < 3 * (\text{A}^2)?$$

Archimedes does not tell us. But this clearly amounts to

$$\text{BigRect} < 2 * (\text{A}^2)$$

which we may now expand to

$$\text{rect.}(\Theta,\ \text{DiffSquare}) < 2 * \text{A}^2$$

And we have established that:

$$\text{rect.}(\Theta,\ \text{SameSquare}) = \text{A}^2 \quad \text{Step 14}$$

so we are asked to believe that

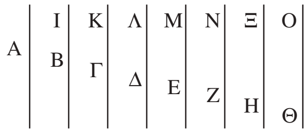
$$\text{rect.}(\Theta,\ \text{DiffSquare}) < 2 * \text{rect.}(\Theta,\ \text{SameSquare})$$

or more simply

$$\text{DiffSquare} < 2 * \text{SameSquare}$$

which is indeed obvious.

[151] Recalling *Elements* VI.20, the apparent ground for the claim. VI.20, however, discussed only rectilinear figures. The reader would not be able to guess this at this point, but what would be required eventually, when this result is applied, is a result for similar



The diagrams in both codices A and C contained numerals related to scholia which are preserved in codex A alone: see Appendix 2. The diagram of codex C is hard to make out (and is atypically set in the lower margin). It seems to position the labels as in the diagram printed. D4 have both arrays of labels descend to the right, in various unstable curves. E has the lower array descending to the right, the upper descending and then ascending. H has the lower one stable, the upper one descending; G has the lower one stable and the upper one descending and then ascending. It seems possible that the archetype of A had the same arrangement as C, and then A itself started out having the upper array descend, and then corrected this midway, with the resulting appearance of a curve descending and then ascending (E, which I take to be A by lectio difficilior).

COMMENTS

From Implicit to Opaque

Not an easy proposition. But the fundamental point is that Archimedes does not really try to make it any easier. The same atmosphere of a curtailed, suggestive argument is carried over from the previous geometrical passage. But with this much more complicated material – whose very mathematical nature is difficult to pin down – the implicit becomes opaque. It is hard to imagine the reader whose eyes do not, on first scrutiny, glaze over. It is also hard to think that an ancient reader would do better than the modern one: while the ancient reader would not share the modern's perplexity about the missing equations – he was used to reading rhetorical statements of equalities – still, he did not have the modern's training in algebraic operations on equations, so that the subtractions "from both sides" would, if anything, be harder for him.

Let me first try to make the theorem slightly less opaque.

The first thing to understand is that Archimedes has no direct interest in the claim of proposition 10. It is there purely as a stepping-stone for the corollary, to be applied in proposition 24 and more easily understood on its own terms: X is less than Y.

$$Y > X$$

What is Y? It is three times the sum of squares on a series rather like $1^2+2^2+3^2+\ldots+n^2$. In proposition 24, it will be like the series of the sectors of circles, divided by equal angles, circumscribed around a spiral line.

What is X? It is the sum of squares equal to each other, of the same number of terms as in the preceding series, rather like $n^2+n^2+n^2+\ldots+n^2$. It is even possible to think of it as n^3. In proposition 24, it will be like the series of sectors of a circle, divided by the same equal angles, constituting the circle itself.

Archimedes, then, is interested in showing that:

$$3 * (1^2 + 2^2 + 3^2 + \cdots + n^2) > n^3$$

The reader may wish at this point to verify this with numerical examples; this urge was felt by ancient or medieval scholiasts, who inserted numerical values into the diagrams. It is also easy, for a modern reader, to verify this with an argument from mathematical induction and elementary algebra (we need to think about how an arbitrary n^3 transforms as it becomes the next term in the series, $(n+1)^3$; it grows by $3n^2+3n+1$ while, at the same time, the other side of the equation grows by the addition of $3(n+1)^2$; the inequality is now seen to be much less surprising). Obviously, this was not the route that led Archimedes

curvilinear figures (specifically, we will deal with similar *sectors*). Such an extension of *Elements* VI.20 is possible, if laborious, based on *Elements* XII.2, but it is quite unclear if Archimedes considered this a gap, or what he expected his readers to make of it. For sure, here is yet another case where Archimedes does not spell out clearly the grounds for a claim.

to think of this inequality or to prove it to himself and to others. How he came to think of this, I leave for later. We now need to understand how he comes to prove it, *already in possession of the firm belief that it is true*.

So, to convince himself of the inequality, Archimedes constructs an equality. In general we will have

$$Y > X$$

if we have

$$Y = X + Z$$

And since Archimedes sought to prove

$$3 * (1^2 + 2^2 + 3^2 + \cdots + n^2) > n^3$$

he was looking for a Z that satisfies

$$3 * (1^2 + 2^2 + 3^2 + \cdots + n^2) = n^3 + Z$$

Now, let us conceive of the sides of the equation the way Archimedes did – as geometrical objects related to a series of lines.

On the right-hand side of the equation, then, we have three times the series of squares on lines forming a progression – what I call DiffSquare. On the left-hand side, we have the series of squares on the big lines equal to each other – what I call SameSquare. We need to show, then, that:

$$3 * \text{DiffSquare} = \text{SameSquare} + Z$$

To compare the two, we need first of all to turn SameSquare into the terms of Diffsquare. Now, each of the squares on the big lines equal to each other can be reconceived as a composite, through the identity (algebraic for us, geometrical for the Greeks) $(a+b)^2=a^2+2ab+b^2$. Steps 1–5 effect this reconception. Each of the squares in SameSquares becomes two squares and twice a rectangle. The two squares can be seen as "going and up and down" the series of progressing squares (but the last, biggest term is counted only once). We find, in sum, that SameSquare can be reconceived as twice DiffSquare (minus one of the big squares), together with twice those pesky rectangles. In my terms above:

$$\text{SameSquare} = 2\text{Diffsquare} + 2\text{SmallRects} - A^2$$

So now we are looking for a Z satisfying:

$$3 * \text{DiffSquare} = 2\text{Diffsquare} + 2\text{SmallRects} - A^2 + Z$$

or

$$\text{DiffSquare} = 2\text{SmallRects} - A^2 + Z$$

This is essentially achieved by Steps 1–5.

Those pesky rectangles . . . Clearly something ought to be done concerning SmallRects, and while at first glance each such rectangle seems independent of the rest so that it seems difficult to reduce them to a meaningful term, Archimedes observes that, because the lines are arranged in a progression where the difference is equal to the smallest terms, it also follows that the second line (arranged from the smallest up) is double the first, the third is three times the first, etc. Thus we may begin to conceive of all those twice-rectangles together, as a single rectangle, one of whose sides is simply the smallest line, while its other side is a complex sum: twice the first line, and four times the second line, and six times the third line, and so on . . . This is essentially achieved by Steps 6–11:

$$2 * \text{SmallRects} = \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times}\Gamma + \text{six times } \Delta + \text{etc.}))$$

So now we are looking for a Z satisfying:

$$\text{DiffSquare} = \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times } \Gamma + \text{six times } \Delta + \text{etc.})) - A^2 + Z$$

So, is there a constant Z, such that

$$\text{DiffSquare} - \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times } \Gamma + \text{six times } \Delta + \text{etc.})) + A^2 = Z?$$

This will be easy to achieve if we can show that

$$\text{DiffSquare} - \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times } \Gamma + \text{six times } \Delta + \text{etc.}))$$

is such a constant term, and Archimedes shows that it is equal to a rectangle, the smaller of whose sides is the smallest line, the larger being the sum of all the lines in the progression (what I call BigRect).

$$\text{DiffSquare} - \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times } \Gamma + \text{six times } \Delta + \text{etc.})) = \text{BigRect}$$

Or, as is found more convenient for Archimedes' proof:

$$\text{DiffSquare} = \text{rect.}(\Theta, \quad (\text{twice B} + \text{four times } \Gamma + \text{six times } \Delta + \text{etc.})) + \text{BigRect}$$

Once this is achieved, our result is obtained. Archimedes achieves this in Steps 14–21. This is achieved once again by decomposing and recomposing the components of each of the squares in the series of squares DiffSquare, similarly to the transformation in Steps 6–11 above.

None of the identities obtained here is self-evident: they involve the summation of an indefinite number of distinct identities. The last one in particular (of Steps 14–21) involves in and of itself a complex transformation. To repeat: no one would have begun to look for any of those identities, without firmly believing, to begin with, that there ought to be such a Z so that:

$$3 * (1^2 + 2^2 + 3^2 + \cdots + n^2) = n^3 + Z$$

Why Archimedes came to believe in this, I will try to explain in my comments on proposition 24. For now, I concentrate on a separate question: how did it come about that this (admittedly difficult) result ended up being couched in such opaque terms, never clarified in the manner I have attempted above?

Let us follow some layers of the opacity.

1. The diagram is the smallest possible.[152] It does not draw even all the terms explicitly called for in the setting-out: in the stingy move of a provincial

[152] Not in the number of lines in the sequence picked as a random representative. Here, Archimedes picks a sequence with eight lines, more than the minimum – a *lot more* than the minimum (in general, one wants to have more than the absolute minimum to ensure the result is not true for some special cases: one is reminded of the use of a pentagon – not a triangle – for a "general polygon" in *SC* I.1). I guess that this is because he does wish, for the arguments 1–3, 6–11, to have two examples followed by a meaningful "etc." clause. Argument 1–3 does not use A, while argument 6–11 does not fully use A or B, but really starts with Γ. Thus, in 6–11, the two genuine examples are Γ, Δ. There are thus four lines even before the "etc." clause, and so for the "etc." clause to be felt as meaningfully extending over a range, one has four lines beyond Δ. At any

theater, the two roles of "different" and "equal" lines are taken on by the same cast, BI serving once, as B, as an unequal line, then again, as the full BI, as an equal line. This is of course useful: one does see a certain relation in this way. But one could also first construct the separate equal lines, and then show how they are equal to given sums of unequal lines. Why does Archimedes not do so? Partly because the troupe is in a sense too big: one has eight lines in the sequence (even though fewer would do, as remarked in the footnote). Thus Archimedes is squeezed out of the alphabet. Still, he has considerable room he does not utilize. In my footnotes I explain to the reader, and to myself, what goes on by concentrating on six key terms, as explained in the box on p. 68: SameLine, DiffLine, SameSquare, DiffSquare, SmallRects, BigRect. Archimedes could have made the computation of the argument easier on his readers, too, by constructing magnitudes labeled as such (a mere six letters of the alphabet!). Even more generous, one could construct, say, a line with three labels, such as ΠΡΣ, so that ΠΡ equal A and ΠΣ equals DiffLine, and so ΡΣ equals DiffLine-A – a construction useful for clarifying, say, Step 14. Indeed, one could have had a more complicated line in which all of DiffLine is set out side-by-side, line-by-line, allowing a much easier view of the argument from Step 14 onwards.

2. Too many labels? But this is because we demand the proposition to do so much. One other way of clarifying the argument would have been to subdivide it. One could have a theorem going through the argument of Steps 6–12

$$2 * \text{SmallRects} + \text{BigRect} = \text{rect.}(\Theta,\ (\text{A} + \text{three times B} + \text{five times } \Gamma + \text{seven times } \Delta + \text{etc.})$$

and then another theorem going through the argument of Steps 14–20

$$\text{DiffSquare} = \text{rect.}(\Theta,\ (\text{A} + \text{three times B} + \text{five times } \Gamma + \text{seven times } \Delta + \text{etc.})$$

from which an easy corollary would be

$$2 * \text{SmallRects} + \text{BigRect} = \text{DiffSquare}$$

Based on this, one could easily derive the main result through the argument of 1–5.

It is not just that Archimedes made the choice to do everything at once: the manner in which everything is done at once becomes blurred. That the main claim is equivalent to Step 5 is a very difficult argument, asking the readers to engage in a very powerful computation based on Steps 1–4 and the Definition of Goal. That the result of Step 5 follows from the proposition as a whole is asserted in Step 20, based on combining Steps 12 and 13 with Step 5 – an argument that combines complex mental calculation with the need to synthesize widely separate chunks of the argument. There is an elliptic character to the combination of the arguments, which is perhaps inevitable, with each interim result being so complicated: but the elliptic is not an accident, it is a basic feature of this proof.

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rate, this plethora of lines does not help in making the argument any clearer: it merely involves the mind in a more complicated set of terms to contemplate – and clogs up the alphabet.

3. Archimedes extends practices he has followed for the last nine propositions: statements are elliptic and tantalizing, the overall structure is “at a distance.” As I point out in note 133, the reader has to make out for himself that the reference of the “all taken together,” in Step 12, is to the *twice* of the rectangles – that one talks about 2*SmallRects and not just SmallRects. Immediately following that, Step 13 merely asserts that DiffSquare is equal to “the same rectangle” – an ellipsis which the reader has to go back and fill in for himself, once again, to mean “rect.(Θ, (A+three times B+five times Γ+seven times Δ+etc.).” The net result is that each of the Steps 12–13 – which, taken together with Step 5, contain the argument of the proposition – is elliptic. At the point where the key to the argument is supplied, it is supplied only in part; the reader is asked to supply the difference. But this is of course done again and again. Step 11 is an obvious “similarly” type extension only if we have already established that the sequence I, K, Λ . . . is a series of integer multiples of Θ. A version of this fundamental result is also required for Step 15 (which – implicitly! – shows that Θ:A::A:SameSquare). This fundamental result is never stated and is left for the reader to compute. Further, everything in this proposition depends on the extension of results across a series. You would expect Archimedes to make this as evident as possible, by phrasing the results for all terms in a precisely repeatable way. But he goes out of his way to create a jarring, surprising transition between Steps 6 and 7 in the sequence

$$\text{rect. contained by segments} = 2\,n * \Theta * (\text{lower segment})$$

for which see note 130 above. He also makes it very difficult to follow the “similarly” transition from Step 16 to 18. All in all, one can say that the proposition demands that we combine several lines of thought: 1–5, 6–12, 13–20. Each is very difficult, as is their combination. None are set out clearly.

4. And indeed, once again, this is presented in an atmosphere of surveying a sequence of claims from a distance, rather than presenting one in argumentative detail. What was natural, with the very elementary claims at the beginning of the treatise, is almost incredible now: there is no ἄρα in this proposition. It begins with a δή, in Step 1, the major steps from then on marked by οὖν (Steps 4, 12, 21). Key claims such as 5, 6, 13 are introduced by a mere δέ, perhaps to be translated as “and.” The main difference from previous propositions is that there is much more use of γάρ, in Steps 14, 15 and 17: relying on this particle, Archimedes can avoid an explicit argument for the very complex claim of Step 13, presenting it instead as a fiat which is then briefly argued for retrospectively.

Of course, all of this does not yet mention a final consideration, which makes the proposition even more opaque: its strange, difficult subject matter. The point is that Archimedes is not trying to make a difficult subject matter more accessible through a transparent exposition: he strives, on the contrary, to maintain the sense of an opaque passage.

And yet, just what is the kind of object studied in this proposition? This is not easy to answer – in itself, a mark of the opacity of the proposition. We will return to this question, having taken on board the second, and final, treatment of this subject matter.

/11/

In the footnotes to this proposition the following abbreviations are used:

Series One	The terms such as ΞN, MΛ, . . . AB
Series Two	The terms such as ΞY, MT, . . . ΔO
UnequalSquareSum	The sum of squares on the terms of Series One
EqualSquareSum	The sum of squares on the terms of Series Two
GreatSquare	Square on a line such as AB (any of the terms of Series Two)
GreatSmallSquare	Square equal to the sum of: Rectangle contained by lines such as (AB, BΔ) + One-third the square on the difference between AB, BΔ

If however many lines are set in order, exceeding each other by an equal <difference>; and other lines are set in multitude smaller by one than the <lines> exceeding each other by an equal <difference>,[153] each being equal in magnitude to the greatest <=among the original lines>:[154] all the squares on the <lines> equal to the greatest <line> have to the squares on the <lines> exceeding each other by an equal <difference> (without the smallest), a ratio smaller than: the square on the greatest <line>, to the <square>[155] equal to both: the <rectangle> contained by the greatest <line> and the smallest <lines>, as well as a third part of the <square> on the difference,[156] by which the greatest <line> exceeds the smallest; but, to the squares on the <lines> exceeding each other by an equal <difference> (without the square on the greatest) <the squares on the lines equal to the greatest have a ratio> greater than the same ratio <=the ratio of the square on the greatest

Multitude: see Glossary

[153] So, this time: if the original series has, e.g., seven lines/terms, the second series has six lines/terms . . .

[154] . . . all six are equal to the greatest among the first series. Algebraically: Series One is b, b+a, b+2a, b+3a . . . b+ma. Series Two has m terms, all equal to b+ma. Notice that this time we are not explicitly told the difference is to be equal to the smallest line. See the general comments.

[155] The reference is in fact ambiguous: a square is equal to an unspecified X (neuter gender) equal to a rectangle plus the third of a square. What is that X? A square? A rectangle? An area? A *thing*? Is it useful that we may ask that question – and that Greek readers probably did not? Such ambiguities sustain the essential ontological ambiguity of Greek mathematics: is it about concrete geometrical objects or about generalized, abstract magnitudes?

[156] The Greek word for "difference" is the nominal form of the verb "exceed" used here. A more literal translation would have been "excess" (I find the word stylistically awkward, but this may be my own idiosyncratic bias). So, we refer to the square on the *difference* of this arithmetical progression.

line, to the square equal to the rectangle and one-third a square specified above>.[157]

For let there be however many lines, exceeding each other by an equal <difference>, set in order, AB greater than ΓΔ, ΓΔ than EZ, EZ than HΘ, HΘ than IK, IK than ΛM, ΛM than NΞ, and let ΓO, equal to one difference, be added to ΓΔ, and <let> EΠ, equal to two differences, <be added> to EZ, and <let> HP, equal to three differences, <be added> to HΘ, and to the others in the same manner. So, the resulting <lines>

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[157] Proposition 11 is more complicated than 10, primarily in stating a proportion inequality rather than a simple equality.

We take one sum, which is the sum of all the squares on the lines/terms of Series Two – six lines/terms in the example we choose following Archimedes. So it is the sum of six equal squares. Call this EqualSquareSum (algebraically: $m*(b+ma)^2$). We may then compare this with two other sums. One is the sum of the squares on the lines/terms of Series One, but *excluding the smallest line/term* there – so as to make sure we have, once again in our example, seven lines/terms, or in general an equal number of terms as in Series Two (algebraically: $(b+a)^2+(b+2a)^2+\ldots+(b+ma)^2$). Another is the same sum of the squares on the lines/terms of Series One, but this time *excluding the greatest line/term* (algebraically: $b^2+(b+a)^2+\ldots+(b+(m-1)a)^2$). Thus we may compare EqualSquareSum to two versions of UnequalSquareSum: either UnequalSquareSum minus smallest, or UnequalSquareSum minus greatest. (Clearly, "UnequalSquareSum minus smallest" is greater than "UnequalSquareSum minus greatest.")

So we consider EqualSquareSum:UnequalSquareSum minus smallest, (Ratio A)

As well as EqualSquaresum:UnequalSquareSum minus greatest (Ratio B).

Ratio A is the ratio of X to a greater magnitude, Ratio B is the ratio of the same X to a smaller magnitude. Thus it is natural that Ratio A is smaller than Ratio B.

Specifically, we are told that Ratio A is smaller than a certain Ratio C, but that Ratio B is greater than the same Ratio C.

What is Ratio C? It is the ratio involving not entire series but single entities (thus a simpler ratio, reducing the dimension of the degree of freedom of the number of lines in the series). The antecedent term is the square on the greatest, or GreatSquare (algebraically: $(b+ma)^2$). The consequent term is the (square equal to the) sum of the rectangle contained by (greatest, smallest), together with one-third the square on the difference between greatest and smallest (algebraically: $((b+ma)*b)+\frac{1}{3}(ma)^2$). Call this GreatSmallSquare.

And so Ratio C is GreatSquare:GreatSmallSquare.

The theorem sets out to prove that:

(i) EqualSquareSum:UnequalSquareSum minus smallest < GreatSquare: GreatSmallSquare

(ii) EqualSquaresum:UnequalSquareSum minus greatest > GreatSquare: GreatSmallSquare

Or, if you prefer a single statement, closer in spirit to Archimedes formulation:

EqualSquaresum:UnequalSquareSum minus greatest > GreatSquare:GreatSmallSquare > EqualSquareSum:UnequalSquareSum minus smallest

I put the last statement in smaller type, so as to fit a single line. Of course, it cannot be read this way, which hardly matters, which must be part of Archimedes' point. More in the notes to follow.

shall be equal to each other and, each equal to the greatest <line>. Now, it is to be proved that the squares on all the arising <lines>[158] have to all the squares on all the <lines> exceeding each other by an equal <difference> (without the square on NΞ) a ratio smaller than the square on AB to the <area> equal to both: the <rectangle> contained by AB, NΞ, as well as the third part of the square on NY,[159] but that to the squares on the same <lines> (without the square on AB) they have a greater ratio than the same ratio.

(a) Let a <line> equal to the difference be taken away from each of the <lines> exceeding each other by an equal <difference>;[160] (1) So, the ratio which the <square> on AB has to both taken together: the <rectangle> contained by AB, ΦB, as well as the third part of the square on AΦ –<all the following> have this ratio: the square on OΔ to the <rectangle> contained by OΔ, ΔX, as well as the third part of the square on XO; as well as the square on ΠZ to: the <rectangle> contained by ΠZ, ΨZ, as well as the third part of the square on ΨΠ; and the squares on the other <lines> to the similarly taken areas.[161]

[158] That is: the lines arising from adding ΓO to ΓΔ etc. "all the arising lines" are the same as "all the lines equal to the greatest." It is very interesting to see how, in the course of a definition of goal, one reverts to a general, letter-less reference resembling an enunciation. Letters are avoided, though, not to obtain generality but to avoid even more clutter. Whichever route one takes – general statement or lettered references – references are very difficult to establish.

[159] Remarkably, already within the definition of goal (i.e. even prior to a formal "construction" stage) we encounter a letter nowhere established by the text and merely provided by the diagram (the letter Y). This is due to the manner in which the operation of adding on lines was merely generally referred to as being "continued in the same manner." In fact, all the letters beyond P are unspecified in this proposition. So far, three letters are unspecified: Σ, T, Y.

[160] The effect of this brief "construction" is to set up the point series from Φ to Ϙ. In this sense it is required prior to Step 1. None of the letters is explicitly specified. In the diagram the series appears equal to the segment NΞ (the points from Φ to Ϙ appear at the same height as the point N). This is the only indication that NΞ=difference. See the general comments below.

[161] This statement appears somewhat opaque, because of its length as well as its reliance on letters whose reference was not explicitly specified. The content, however, is absolutely straightforward: the various ratios which are asserted to be the same with each other are all the very same ratios, merely transposed visually along the series of lines: AB is exactly the same as OΔ; AB, ΦB is exactly the same as OΔ, XΔ; AΦ is exactly the same OX. Thus sq.(AB):rect.(AB,ΦB) + $\frac{1}{3}$sq.(AΦ)::sq.(OΔ):rect.(OΔ,XΔ) + $\frac{1}{3}$sq.(OX), for no other reason than sq.(AB):rect.(AB,ΦB) + $\frac{1}{3}$sq.(AΦ)::sq.(AB):rect.(AB,ΦB) + $\frac{1}{3}$sq.(AΦ). Notice at this stage that the complicated object "rect.(AB,ΦB) + $\frac{1}{3}$sq.(AΦ)" is the same as what we called "GreatSmallSquare" in the context of the general enunciation.

What is truly curious is that you begin Step 1 with the expectation that there is some specific geometrical manipulation embedded in the configuration of squares and rectangles it refers to, establishing a particular proportion statement; but then you

(2) So, also: all the <squares>, on all the <lines> OΔ, ΠZ, PΘ, ΣK, TM, YΞ <have this ratio> to all the <rectangles> contained by both: NΞ, as well as the <line> equal to all the mentioned lines; as well as the one-third part of the squares on OX, ΠΨ, PΩ, ΣϠ, TϘ, YN shall have the same ratio which the square on AB <has> to both taken together: the <rectangle> contained AB, ΦB as well as the third part of the square on ΦA.[162] (3) Now, if the <rectangle> contained by NΞ, as well as <by> the <line> equal to all the <lines> OΔ, ΠZ, PΘ, ΣK, TM, YΞ, as well as the third parts of the squares on OX, ΠΨ, PΩ, ΣϠ, TϘ, YN, should be proved to be smaller than the squares on AB, ΓΔ, EZ, HΘ, IK, ΛM, but greater than the squares on ΓΔ, EZ, HΘ, IK, ΛM, NΞ, the claim shall be proved.[163]

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recognize that the proportion statement would have been valid for any geometrical configuration dreamed up arbitrarily by Archimedes.

[162] *Elements* V.12. We had a series of trivial proportions A_1:B_1::A_2:B_2::A_3:B_3:: . . . :: A_8:B_8 (trivial because A_1=A_2=A_3 etc., B_1=B_2=B_3 etc.). We now compress it all to a single proportion: A_1+A_2+A_3 . . . + A_8: B_1+B_2+B_3+ . . . + B_8:: A_1:B_1

There is only one slight transformation: whereas in the statement of the individual ratios in Step 1 above, each rectangle was defined by two sides analogous to AΦ, ΦB, (OX, XΔ; ΠΨ, ΨZ; etc.), Step 2 exploits the equality of the analogous constituents and transforms the smaller side of each rectangle into the same side NΞ. (It is now assumed that NΞ is indeed equal to the difference; as Heiberg notes, the argument does not really require this assumption. See the general comments.) Thus we have one big rectangle (instead of the summation of seven separate rectangles) whose one smaller side is NΞ, while its greater side is the summation of the lines AΦ, OX, ΠΨ etc.

The outcome, in terms of the general enunciation, is to state:

EqualSquareSum:(A Certain New Combination)::GreatSquare:GreatSmallSquare

Other than A Certain New Combination, all the terms are established already by the general enunciation. This, however, is not transparent, because the general enunciation was never specified in complete terms.

[163] Step 2 has established that:

EqualSquareSum:(A Certain New Combination)::GreatSquare:GreatSmallSquare

We need in the general enunciation to prove that:

(i) EqualSquareSum:UnequalSquareSum minus smallest < GreatSquare: GreatSmallSquare
(ii) EqualSquaresum:UnequalSquareSum minus greatest > GreatSquare: GreatSmallSquare

Thus Archimedes can state quite straightforwardly (through *Elements* V.8) that all he needs to show is that:

(i) A Certain New Combination < UnequalSquareSum minus smallest
(ii) A Certain New Combination > UnequalSquareSum minus greatest

In the following we first set out to study comparison (ii) A Certain New Combination is to be compared with UnequalSquareSum minus greatest.

(4) So, the <rectangle> contained by: NΞ, as well as <by> the <line> equal to all the <lines> OΔ, ΠZ, PΘ, ΣK, TM, YΞ, as well as the third parts of the squares on OX, ΠΨ, PΩ, ΣϠ, TϘ, YN, are equal to the squares on XΔ, ΨZ, ΩΘ, ϠK, ϘM, NΞ and the <rectangle> contained by: NΞ, as well as the <line> equal to all the <lines> OX, ΠΨ, PΩ, ΣϠ, TϘ, YN, and the third part of the squares on OX, ΠΨ, PΩ, ΣϠ, TϘ, YN;[164] (5) while the squares on AB, ΓΔ, EZ, HΘ, IK, ΛM are equal to the squares on BΦ, XΔ, ΨZ, ΩΘ, ϠK, ϘM and the <squares> on AΦ, ΓX, EΨ, HΩ, IϠ, ΛϘ, and the <rectangle> contained by BΦ and the double of AΦ, ΓX, EΨ, HΩ, IϠ, ΛϘ.[165] (6) Now then, the squares on the <line> equal to NΞ are common to each,[166] (7) while

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This, however, would be clear only under a precise reading of Step 2, which translates it (as I did in n. 162 above) into the terms of the general enunciation. The reader was not supplied such a translation, and therefore the claim of Step 3 appears as quite a surprise.

[164] The Step asserts effectively:

$$\text{rect.}(N\Xi, O\Delta + \Pi Z + P\Theta + \Sigma K + TM + Y\Xi) =$$
$$\text{sq.}(X\Delta) + \text{sq.}(\Psi Z) + \cdots + \text{sq.}(N\Xi) + \cdots + \text{rect.}(N\Xi, OX + \Pi\Psi + P\Omega + \Sigma\text{Ϡ}$$
$$+T\text{Ϙ} + YN)$$

This is obvious if we consider that rect.(NΞ,OΔ) = sq.(XΔ) + rect.(NΞ, OX) (in the rectangle NΞ, OΔ, we subdivide the long side OΔ into its constituent parts XΔ, OX, and then exploit the equality NΞ=XΔ).

What the Step obtains is a reformulation of A Certain New Combination.

[165] *Elements* II.4 applied successively to each of the unequal lines; BΦ is taken (in the rectangle) as equal to each of the smaller sides. What this Step obtains is a reformulation of UnequalSquareSum minus smallest.

[166] "To each" – namely, to the reformulated version of both A Certain New Combination as well as UnequalSquareSum minus smallest. The first, A Certain New Combination, became in Step 4:

$$\underline{\text{sq.}(X\Delta) + \text{sq.}(\Psi Z)} + \cdots + \underline{\text{sq.}(N\Xi) + \text{rect.}(N\Xi,\ OX + \Pi\Psi) + \cdots + YN)}$$
$$+\underline{1/3\text{sq.}(OX) + \text{sq.}(\Pi\Psi) + \cdots + \text{sq.}(YN)}$$

The latter, UnequalSquareSum minus smallest, became in Step 5:

$$\underline{\text{sq.}(B\Phi) + \text{sq.}(X\Delta) + \cdots + \text{sq.}(\text{Ϙ}M)} + \underline{\text{sq.}(A\Phi) + \text{sq.}(\Gamma X) \pm \cdots + \text{sq.}(\Lambda\text{Ϙ})}$$
$$+\underline{\text{rect.}(B\Phi,\ 2 * A\Phi + \Gamma X + \cdots + \Lambda\text{Ϙ})}$$

The implicit upshot of Steps 3–5 is that we need to compare the terms above.

The terms are now each composed of three components. The first component in both is a series of squares, in magnitude all equal to NΞ, in multitude all equal to the number of lines in the diagrams minus one. Intriguingly, the "minus one" is different in each of the terms (the first misses out the "first" square, on BΦ; the latter misses out the "last" square, on NΞ). This makes the claim of Step 6 much more difficult to perceive. Once we do perceive it, we recognize that our goal is to compare now the following two terms, each with two components (that is, we can "cancel out" the first components):

the <rectangle> contained by NΞ and the <lines> equal to OX, ΠΨ, ΩP, ϡΣ, ϘT, YN is smaller than the <rectangle> contained by BΦ, and the double of AΦ, ΓX, EΨ, HΩ, Iϡ, ΛϘ[167] (8) through the lines now mentioned <=AΦ, ΓX, . . ., ΛϘ> being equal to the <lines> ΓO, EΠ, PH, IΣ, ΛT, YN,[168] but greater than their remainders;[169] (9) and the squares on AΦ, ΓX, EΨ, HΩ, Iϡ, ΛϘ <are greater than the third part of the squares on OX, ΠΨ, PΩ, Σϡ, TϘ, YN>.[170] (10) For this has been proved in the above.[171] (11) Therefore the said areas are smaller than the squares on AB, ΓΔ, EZ, HΘ, IK, ΛM.[172]

$$\text{rect.}(\text{N}\Xi, \text{OX} + \Pi\Psi) + \cdots + \text{YN}) + 1/3\text{sq.}(\text{OX}) + \text{sq.}(\Pi\Psi) + \text{sq.}(\text{YN})$$

and

$$\text{sq.}(\text{A}\Phi) + \text{sq.}(\Gamma\text{X}) + \cdots + \text{sq.}(\Lambda\text{Ϙ}) + \text{rect.}(\text{B}\Phi,\ 2 * \text{A}\Phi + \Gamma\text{X} + \cdots + \Lambda\text{Ϙ})$$

167 The claim now is rect.(NΞ, OX+ΠΨ)+ . . . +YN)<rect.(BΦ, 2*AΦ+ΓX+ . . . +ΛϘ). This involves yet another of the components of the terms compared in the (implicit) outcome of Steps 3–5. What remains in need of comparison are the terms $\frac{1}{3}$sq.(OX)+sq.(ΠΨ)+sq.(YN) | sq.(AΦ)+sq.(ΓX)+ . . . +sq.(ΛϘ).

168 The series AΦ . . . ΛϘ is the same as the series NY . . . ΓO, “rotated 180 degrees.”

169 The “remainder” of ΓO is ΓX, the “remainder” of ΠE is EΨ, and so too TΛ (whose remainder is ΛϘ), and YN (whose remainder is nothing). Thus the “remainders” of the entire series ΓO, EΠ, PH, IΣ, ΛT, YN are the series ΓX, EΨ, HΩ, Iϡ, ΛϘ which is smaller by one line than the series AΦ, ΓX, EΨ, HΩ, Iϡ, ΛϘ.

Now, if we wish to compare the double of A and B, and B is subdivided into B_1, B_2, (“lines” and “remainders”), of which A equals B_1 but is greater than B_2, then surely the double of A is greater than B. Thus Step 8 indeed justifies Step 7.

170 The manuscripts omit the last clause, almost certainly a scribal error (corrected by editors from Commandino on). It is a marvel that more such scribal errors were not made in the transmission of this heavy, lumbering piece of text.

The claim now is:

$$\text{sq.}(\text{A}\Phi) + \text{sq.}(\Gamma\text{X}) + \cdots \quad + \text{sq.}(\Lambda) > 1/3\text{sq.}(\text{OX}) + \text{sq.}(\Pi\Psi) + \cdots \quad + \text{sq.}(\text{YN})$$

This Step 9 completes the comparison of the two three-component terms established by Steps 3–5. In Step 6, we learn that a certain component in the first term is equal to a certain component in the second one. In Step 7 we learned that another component in the first term is smaller than another component in the second one; we now learn in Step 9 that the remaining component in the first term is also smaller than the remaining component. One of the component pairs is equal, the other two are smaller, and thus the term as a whole is smaller.

171 Step 9 is indeed exactly claimed in proposition 10 (as restated in the corollary).

172 This reverts to the statement of Step 3, and we learn that “the said areas” refers not to the combination set up in the reformulation of Step 3 within Steps 4–5, but to the original set referred to in Step 3, much more salient in the proof as a whole and called above “A Certain New combination”:

$$\text{rect.}(\text{N}\Xi, \text{O}\Delta + \Pi\text{Z} + \cdots \quad + \text{Y}\Xi) + 1/3(\text{sq.}(\text{OX}) + \text{sq.}(\Pi\Psi) + \cdots \quad + \text{sq.}(\text{YN})$$

And we will prove what remains, that they <=the said areas> are greater than the squares on ΓΔ, EZ, HΘ, IK, ΛM, NΞ.[173] (12) So, again, the squares on ΓΔ, EZ, HΘ, IK, ΛM, NΞ are equal to the <squares> on XΓ, EΨ, HΩ, IϠ, ΛϘ and the <squares on XΔ, ΨZ, ΩΘ, ϠK, ϘM, NΞ>[174] and the <rectangle> contained by: NΞ, and the double of all the <lines> ΓX, EΨ, HΩ, IϠ, ΛϘ.[175] (13) And the <squares> on XΔ, ΨZ, ΩΘ, ϠK, ϘM, NΞ are common[176] (14) while the <rectangle> contained by: NΞ, and the <line> equal to all the <lines> OX, ΠΨ, PΩ, ΣϠ, TϘ, YN, is greater than the <rectangle> contained by NΞ and the double of all the <lines> ΓX, EΨ, HΩ, IϠ, ΛϘ.[177] (15) And also: the squares on XO, ΨΠ,

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[173] We now wish to show

A Certain New Combination > UnequalSquareSum minus smallest

or, in more explicit terms,

$$\text{rect.}(N\Xi, O\Delta + \Pi Z + \cdots + Y\Xi) + 1/3(\text{sq.}(OX) + \text{sq.}(\Pi\Psi) + \cdots + \text{sq.}(YN) > \text{sq.}(\Gamma\Delta) + \text{sq.}(EZ) + \cdots + \text{sq.}(N\Xi)$$

[174] A lacuna in the text, emended by Commandino.

[175] As the "again" suggests, this Step 12 exactly repeats the argument of Step 5 above, reapplying *Elements* II.4.

We have once again reformulated sq.(ΓΔ)+sq.(EZ)+ … +sq.(NΞ) into a three-component term:

$$\underline{\text{sq.}(X\Gamma) + \text{sq.}(E\Psi) + \cdots + \text{sq.}(\Lambda\text{Ϙ})} + \underline{\text{sq.}(X\Delta) + \text{sq.}(\Psi Z) + \cdots + \text{sq.}(N\Xi)} + \underline{\text{rect.}(N\Xi, 2 * \Gamma X + E\Psi + \cdots + \Lambda\text{Ϙ})}$$

[176] Without making this explicit, Archimedes in what follows will assume the same decomposition of A Certain New Combination as in Step 4, so that this also becomes a three-component term:

$$\underline{\text{sq.}(X\Delta) + \text{sq.}(\Psi Z) + \cdots + \text{sq.}(N\Xi)} + \underline{\text{rect.}(N\Xi, OX + \Pi\Psi + \cdots + YN)} + \underline{1/3(\text{sq.}(OX) + \text{sq.}(\Pi\Psi) + \cdots + \text{sq.}(YN))}$$

Remember, however, the three-component term established just above:

$$\underline{\text{sq.}(X\Gamma) + \text{sq.}(E\Psi) + \cdots + \text{sq.}(\Lambda\text{Ϙ})} + \underline{\text{sq.}(X\Delta) + \text{sq.}(\Psi Z) + \cdots + \text{sq.}(N\Xi)} + \underline{\text{rect.}(N\Xi,\ 2 * \Gamma X + E\Psi + \cdots + \Lambda\text{Ϙ})}$$

What we need to do is to show that the first three-component term is greater than the second. We are reminded now that one of the components is indeed common to both terms, so it can effectively be ignored. We are left comparing

$$\underline{\text{rect.}(N\Xi, OX + \Pi\Psi) + \cdots + YN)} + \underline{1/3\text{sq.}(OX) + \text{sq.}(\Pi\Psi) + \cdots + \text{sq.}(YN)}$$

and

$$\underline{\text{sq.}(X\Gamma) + \text{sq.}(E\Psi) + \cdots + \text{sq.}(\Lambda\text{Ϙ})} + \underline{\text{rect.}(N\Xi,\ 2 * \Gamma X + E\Psi + \cdots + \Lambda\text{Ϙ})}$$

[177] Another component is dealt with now:

$$\underline{\text{rect.}(N\Xi, OX + \Pi\Psi) + \cdots + YN)} > \underline{\text{rect.}(N\Xi, 2 * \Gamma X + E\Psi) + \cdots + \Lambda\text{Ϙ})}$$

ΩP, ϡΣ, ϘT, YN are greater than three times the squares on ΓX, EΨ, HΩ, Iϡ, ΛϘ.[178] (16) For this, too, has been proved.[179] (17) Therefore the said areas are greater than the <squares> on ΓΔ, EZ, HΘ, IK, ΛM, NΞ.

/COROLLARY/

And furthermore, if similar <figures> are set up on all the <lines>, on both the <lines> exceeding each other by an equal <difference> as well as the <lines> equal to the greatest <line>, all the <figures> on the <lines> equal to the greatest shall have to the <figures> on the <lines> exceeding each other by an equal <difference> (without the smallest figure) a smaller ratio than the square on the greatest <line> to the <square> equal to both: the <rectangle> contained by the greatest <line>, and then the smallest <line>, as well as a third part of the <square> on the difference, but, to the figures on them <= the lines exceeding each other> (without the <figure> on the greatest), <the figures on the lines equal to the greatest have a ratio> greater than the same ratio <=the ratio of the square on the greatest line, to the square equal to the rectangle and $\frac{1}{3}$ a square specified above>. (1) For similar figures shall have the same ratio as the squares.[180]

A has the label series Φ, X, Ψ etc. on a slight descending trajectory, but not codex C, whose flat line I reproduce without conviction. DE4 as well as C read Y for Ϙ, and so did probably the common archetype to AC. EH has T for ϡ, 4 has an ambiguous reading between ϡ and T. I suspect A had the correct reading ϡ and that the errors are independent (otherwise I would expect a T in 4, too). D has Λ for A.

COMMENTS

A Less Difficult Proof

Proposition 11 has two parts – Steps 4–11, 12–17 – which are exactly analogous, and for this reason the relative brevity of the second part, 12–17,

The argument is analogous to that of Step 8 (double the second series, ΓX+EΨ+ . . . +ΛϘ, is the first series, OX+ΠΨ)+ . . . +YN, *but without YN*; so, clearly, double the second series is smaller than the first series). This time, however, the argument is left implicit.

It now remains to compare:

$\underline{1/3\text{sq.}(OX) + \text{sq.}(\Pi\Psi) + \cdots + \text{sq.}(YN)}$ and $\underline{\text{sq.}(X\Gamma) + \text{sq.}(E\Psi) + \cdots + \text{sq.}(\Lambda Ϙ)}$

[178] The claim now is sq.(OX)+sq.(ΠΨ)+ . . . +sq.(YN) > 3*(sq.(XΓ)+sq.(EΨ)+ . . . +sq.(ΛϘ)). This completes the comparison (albeit, confusingly, with the transformation of a "one-third" formulation to a "three-times" one): the first term is greater than the second.

[179] This is the other part of the corollary to proposition 10 above, completing the application of the preceding proposition in full.

[180] *Elements* VI.20.

does not count as a genuine ellipsis: a reader who has followed the argument of 4–11 is well prepared indeed to fill in the gaps in 12–17. (In proposition 10, by contrast, a large part of the difficulty of the argument had to do with the way in which the task was subdivided into smaller tasks whose overall arrangement was only incompletely specified by the text itself.) There are complications. Thus the main task, that of the definition of goal, is reinterpreted explicitly at Step 3 and then only implicitly in Steps 4–5: when the reinterpreted task of Steps 4–5 is obtained in Steps 6, 7, 9, the reader has to piece them together by himself *and* then see that, because they apply to the implicitly reformulated goal of Steps 4–5, it also applies to the explicit goal of Step 3. This is in the spirit of the complicated demands put on the reader in proposition 10, but of course in a much milder form.

It is not that Archimedes makes an effort to present the proposition in a more accessible way (the surface appearance of proposition 11 is, if anything, more complex: see below); rather, the nature of the task of this proposition is simpler. The proposition appears to deal with a more complicated subject matter – that of a proportion statement instead of an equality – but, as Steps 1–3 clarify, the proportion reduces to an equality statement. And while the terms used appear convoluted, they are indeed contrived in a precise sense. What makes them appear especially complex – the reference to a third of a set of squares – is in fact calculated as a direct application of proposition 10 so that the very complexity of the formulation simplifies the solution, saving us the trouble of a specialized proof (required in the preceding proposition). It is almost better to think of proposition 11 as a kind of corollary to proposition 10: a twist on the way in which the result of proposition 10 can be presented (the terms of proposition 10 are added onto other terms and then inserted into a proportion statement).

The major difference between propositions 10 and 11 is that proposition 10 demands, while proposition 11 does not, that the smallest term be equal to the difference. Of course, proposition 10 is applied inside proposition 11, but this is because its application is limited to a subset of the lines: the set of lines in proposition 11 to which proposition 10 is applied is the series ΓX, EΨ, . . ., ΛϘ, where the difference is indeed equal to the smallest, since ΛϘ is indeed the smallest difference, not the smallest term – that is, we are not guaranteed explicitly by the proposition that ΛϘ=NΞ. This equality is indeed assumed by the proposition, in the manner in which the construction move (a) spells itself out in the diagram (i.e. there is no extra point above or below N; instead, the point on the line ΥΞ, homologous in function to the points Φ, X etc., is assumed to be the same as N)[181] and then in Steps 2–3, which directly assume line NΞ as equal to a difference. This is essentially a matter of economy:

[181] In what sense is it established by the diagram that NΞ=ϘM? Probably not in the sense that the two appear equal (i.e. N appears to be of the same height as Ϙ). In the manuscript evidence, N appears somewhat higher (than Ϙ, which may be mislabeled as Υ), and the entire row is somewhat "undulating" in appearance. Perhaps Archimedes' original drawing was as sharp as Heiberg's, but nothing in the practices of Greek mathematics as a whole makes us think the diagram was designed to represent such metrical facts directly. Indeed, one is at first struck to see that, in the diagram of

instead of having the line YΞ divided at two points, one standing for "the first term of the progression," the other standing for "the length equal to the difference of the progression," both are represented by the same line NΞ. However, the functions are kept separate: the manipulations of the proof that operate on NΞ as the difference never use its equality with the smallest term. Nothing in the argument would differ if, instead of referring to NΞ, one would refer to some other line, let us say \$Ξ (\$ then positioned parallel to Ϙ, ϡ and above or below N). Such a diagram (and text) would have set out the argument in a clean way. As it happens, the reader must establish its validity for himself and verify, step by step, that the separation of functions is followed.

It is hard to account for all of that. Had the enunciation of proposition 11 lost its reference to the equality of the difference with the smallest line? Almost certainly not, since then its application in proposition 25 would be false. It is also hard to imagine a textual corruption whereby some kind of "\$," a distinction in the diagram itself between the two functions of "N," was lost – for then we would face not only a false diagram but also a text which is repeatedly false in its references to one of the functions of N. None of these is promising; and so we are led to the conclusion that the proposition states a general claim, and then its proof appears to work with a more narrow case, its validity verifiable only through a step-by-step process of distinguishing two functions of the same point.

The most fundamental point is that, now, the reader would not bother, since (a) the future application of proposition 11 is nowhere suggested, and therefore the reader is not made aware of the significance of the general application of the proposition; and (b) the reader is conditioned by the preceding proposition to assume that the reference is indeed to the case where the smallest term equals the difference. It would be egregiously unlikely for any reader, at this stage, to bother to verify the general validity of the proof, instead of taking it automatically to refer to the narrow case (and, if anything, to consider the absence of explicit restriction in the general enunciation to be a mere oversight, perhaps intended to be completed based on the preceding proposition). In short, the proposition sets itself up to be read as stating less than it actually does. More on this when we get to the application of the proof in proposition 25. But the reader may well notice already: in the case of propositions 5, 7, 8, Archimedes masked their future application by "padding" them with propositions 6 and 9; in this case, the future application is masked with a misleading enunciation and diagram.

Be that as it may, it does remain striking that the identity of the terms seems to rely so powerfully on the diagram: which brings us to the question of the overall character of the argument and its subject matter.

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proposition 11, the smallest term is not the same as the difference. But, on a closer look, one notices that the difference is not very sharply maintained as the same; and that, more remarkably, the smallest term does not appear equal to the difference in proposition 10 either. Metrical equality is not represented by measured equality in the diagram. Instead, the diagram conveys structural information: the entire row from Φ to N, undulating as it is, is clearly homologous; N is unique on its line YΞ.

The Subject Matter?

A further difference between propositions 10 and 11 has to do with the main source of validity. Most of the argument of proposition 10 is mediated through claims of computation: adding up terms, one sees that they build up a series of odd numbers. Proposition 11 has nothing of this kind but instead relies, again and again, on the diagram – whose functioning in the proof is much more in the spirit of Greek geometrical proofs.

I explain: in a move quite untypical of Greek geometry, the diagram of proposition 10 had two line segments, adjacent (sharing a point) referred to not by the labels on the points of their extremes (one of which is shared) but by two labels directly on the line segments themselves. This means that such a term as BI refers not to a line segment whose ends are B, I, but to a composite made of B, I: something rather like B+I. Now, typically, the relations denoted by diagrammatic labels are of a topological character: lines overlap or intersect. In proposition 10 the relations denoted by diagrammatic labels took a more "algebraic" significance: two values could be added together. Indeed, expressions such as BI are in fact quite rare even in proposition 10: almost always – with two crucial exceptions in Steps 1–2 – the proposition refers to segments in isolation.[182] Which further serves to highlight the strangeness of proposition 10: it involved no spatial interaction of the constituents (beyond the treatment, in Steps 1–2 and those following from them, of two line segments as a base for a square on the segment composed on both, as well as being bases for individual squares and an individual rectangle).

Proposition 11 is much more "normal" in this regard. Labels refer to points, and so the two-letter term (the only kind of term used in this proposition) refers to a line segment. The spatial composition and decomposition of such line segments may be represented by the diagram and its labeling, and is done so repeatedly in the proof: Steps 4, 5, 8 are based on such decompositions and underlie practically all of the argument from Steps 4 to 11 (together with the application of proposition 10). Analogously, Steps 12, 14 rely on similar decompositions (and while the analogue to Step 4 remains implicit in the

[182] By "isolation" I mean the absence of immediate concatenation of such labels as BI. The proposition does frequently refer to an aggregate of such terms as A, B, Γ, Δ, E, Z, H, Θ. Now, in a modern edition the difference is clear: immediate concatenation is marked as a single string such as BI; an aggregate is marked by commas inserted inside the aggregate. Such distinctions are not typically respected in the medieval manuscripts (where, if anything, long series of terms are often presented as a sequence of concatenated pairs, so that A, B, Γ, Δ, E, Z, H, Θ, for instance, may well be laid out as AB ΓΔ EZ, HΘ), and could well have been altogether ignored in the ancient papyrus tradition. I take the distinction as meaningful, at the level of mathematical contents: what modern editions typically represent by immediate concatenation is naturally seen as a single mathematical object; what modern editions typically represent through a series of commas is more naturally seen as a plurality of mathematical objects, considered together for some purpose. To be "naturally seen" is of course to some extent a matter of judgement, so that the definition offered here is subjective and somewhat fuzzy.

sequence 12–17, this makes its reliance upon the diagram even more crucial). Furthermore, consider Step 1, setting up the starting-point for the proof: it relies on nothing else than the pointing-out of a homology running through the diagram. This is similar to the manner in which the various small sides of the rectangles are collapsed into a single small side NΞ, or BΦ (Steps 3, 5): the equality is textually based but is certainly mediated, in the mind of its reader, via the diagram. (More complex is the "180 degree rotation" required by Step 6.) Indeed, as pointed out above, the very identity of the segment NΞ as simultaneously the smallest term and the difference is based on the diagram alone. And, most simply, the identity of many of the letters is not given by the text (which provides some of the construction elliptically) but by the diagram.

This diagrammatic activity carries a price: the complexity of the decompositions explains the need for labeling via points – and for more variety in the many-term composites. In proposition 10, the composites almost always involved the banal series A, B, Γ, Δ, etc. In proposition 11, we have a rich variety, with terms starting from, say, OΔ; or from its part OX; or from its part XΓ. The heavy use of long lists of composites (of which more below), constantly changing their contents, is what underlies the first impression that proposition 11 is impenetrably difficult.

Here, then, is an opposition: proposition 10 is more computational; proposition 11 is more diagrammatic. But be careful: proposition 11 is not necessarily "spatial," or "geometrical." This is a more precise way of stating its apparent impenetrability. It uses the tools of a geometrical argument, without taking benefit of visual, geometrical intuitions. There is no space outside that of the diagram used in this proof. Claims which cannot be visualized through the diagram (e.g. that certain sets of squares are common to two different combinations of terms) are not visualized at all and are instead mediated verbally. Commenting on the previous proposition, I noted how useful it would have been to denote six key combinations by special terms, and to refer to them as such. In the previous proposition, such labeling shortcuts would have made the argument simpler. In this proposition, they would have made the argument more *visual*. But as it is, the argument is not entirely visual. It is visual only as long as it refers to its lattice, which in and of itself does not carry a set of geometrical intuitions.[183]

[183] What I think we need to clarify in our minds is the distinction between the "diagrammatic" and the "geometrical." An argument can be "diagrammatic" to the extent that it relies on diagrams (and in Greek mathematics, whose diagrams take a very precise form, this has a very precise meaning: use linear systems marked by letters). And an argument can be "geometrical" to the extent that it evokes a certain order of reality consisting of geometrical figures (and in Greek mathematics, where this order takes a very precise form, this has a very precise meaning: refer to the universe characterized by Euclid's main theorems). I suggest, then, that the argument of propositions 10–11 is only minimally geometrical (it does, after all, essentially rely on *Elements* II.4); proposition 10 is not even deeply diagrammatic (it is, instead, mostly computational). Proposition 11 presents the very interesting case of a proposition which is minimally geometrical and yet heavily diagrammatic.

What do I mean by saying that the lattice does not carry a set of geometrical intuitions? I mean, essentially, that there is no toolbox of established geometrical results whose reference is a lattice of punctuated lines. It is this which ultimately explains the "strangeness" of this pair of propositions. The decision by Archimedes not to break up proposition 10 into its constituents was decisive. Had we had a series of four or five propositions reaching the conclusions of propositions 10–11, we would have had a mini-theory setting up a universe of mathematical knowledge surrounding the manipulation of certain sequences. But Archimedes never allows such a universe any chance, as it were, to "congeal." The lattice of punctuated lines – a geometrical, Greek analogue to the modern study of series – was presented by Archimedes not as a newly founded theory, but as a strange, incongruous stump.

But why should Archimedes linger in such a universe, create such a theory? Indeed – and this must be stressed again and again – none of this is of any interest to Archimedes apart from the study of spiral lines. He would most likely not have come to conceive of any of those theorems, and most certainly would not have come to present them, had they not been required for propositions 24–25; and he came to conceive of the claims, in those propositions, only because he realized that they emerge from the theory of the spiral. In other words, to understand propositions 10–11 we need to understand 24–25, and I will return to explain the origins of those theorems, then, in my comments on those propositions below. For now, we are with Archimedes, pressing on with the spiral.

/DEFINITIONS/

/1/ If a straight line is joined in a plane and, being rotated at uniform speed however many times, with one of its ends remaining fixed, is returned again to where it started from, while at the same time, even as the line is rotated, a certain point is carried along the line, at uniform speed with itself, starting at the fixed end, the point shall draw a spiral in the plane. /2/ Now, let the fixed end of the line – <a line> which is itself being moved around – be called the *start of the spiral*, /3/ and the position of the line, from which the straight line started to rotate – the *start of the rotation*.

/4/ Let a straight line,[184] through which the point carried along the line[185] passes during the first rotation, be called *first*;[186] and <a line>, which the same point completes during the second rotation, *second*; and let the other <lines>, similarly to these, be called by the same name <=number> as the rotations; /5/ and let the area taken by both the spiral drawn during the first rotation, as well

[184] Meaning "a line segment."

[185] The word "line" here does not refer to the same line segment defined here. Rather, the expression "the point carried along the line" is there to establish, taken as a whole, the identity of the moving point of the spiral.

[186] Somewhat obscured by my use of italics is the fact that definition 4 defines not the terms "first," "second" but rather "first line," "second line" etc.; and similarly in the following, the term defined is "first area," "second area," etc.

as by the line, which is first, be called *first*, and <the area> taken by both the spiral drawn during the second rotation as well as by the second line, *second*, and let the others be called in this manner in sequence.

/6/ And if from the point, which is the start of the spiral, some straight line is drawn, let those things which are at the same side of that line, at which the rotation is made, be called *preceding*, and those at the other side, *following*.[187]

/7/ Let the circle drawn with the point, which is the start of the spiral, as center, and the line, which is first, as radius, be called *first*, and <let> the <circle> drawn with the same center and with the double line as radius <be called> *second*, and <let> the other <circles be called> in sequence with these in the same manner.

COMMENTS

It is certainly startling for the definition of the *spiral* to appear all of a sudden now. But it is not new: Archimedes did employ essentially the same definition in his introductory letter (see pp. 20–21), and the likeliest reading of that was that he was quoting (whether verbatim or not) from the original letter to Conon. Archimedes has been describing the production of spiral lines for quite some time.

Still, we did not expect a definition so belated. Indeed, because of the theorematic appearance of the first definition (and because of the lack of any marking or numbering of this set of definitions), one could well imagine that the words "If a straight line joined in a plane . . . " are the beginning of yet another theorem, rather than a definition. The surprise may well be intended and forms part of the overall trend of the prose of this treatise (indeed, of Archimedean prose in general): as the text proceeds from one theme to the next, the transitions are made intentionally rough. This segmentation of the text – no less than that of the spiral – is left unmotivated. And so we entered, unexpectedly and without explanation, the strange world of propositions 10–11– to leave it again, with no more explanation, just as unexpectedly. We do have a sense, though, that the main character – the spiral – has finally appeared upon the scene and that the main action of the treatise is about to unfold; previous textual segments now appear, in retrospect, as so much preparation. The interlude of definitions is a powerful textual marker.

There is a more specific effect to the appearance of the spiral, in this form, in the context of this particular book. As stated above, Archimedes could have made a choice, here, to quote the letter to Conon explicitly in providing the definition of the spiral, once again, in the introductory letter to this treatise; he made a choice to repeat the definition of the spiral twice in this book – in the introductory letter and in this interlude of definitions. Thus this textual marker

[187] Imagine that we draw a line Λ from the center of the clock, pointing to the hour 4. Imagine that the rotation drawing the spiral starts pointing at 12 and moves clockwise. Then the area from 12 to 4 "precedes" the line Λ, while the area from 4 to 12 "follows" the line Λ.

serves not merely within the text to separate the introductory material from the main geometrical action. It also functions intertextually. In the middle of the treatise, Archimedes brings to the reader's mind once again the letter he has sent to Conon. It is as if he was saying that "now, at last, the promise will be fulfilled . . . " – the introductory material is introductory for a certain purpose, which emerges not out of an abstract mathematical reality but out of a particular, material correspondence. Here, one final time, is what I wrote to Conon; and here, at last, is what I did not.

The statement, as noted above, reads like a theorem – a conditional, subjunctive in the antecedent, indicative in the consequent: *if* a drawing is made in a certain manner, it *does draw*, in fact, a spiral line. The identity of the term "spiral" is perhaps then established previously (this, after all, is a natural Greek word);[188] the validity of the claim is perhaps left for the reader's visual intuition. (How otherwise? The meaning of the word in natural Greek does not have a pre-existing mathematical definition, so that there is no way to prove that *just this construction* constructs *just this object*.) Archimedes simultaneously *asserts* that a natural way of bringing about a "coil" is through the drawing specified here; and also *postulates* that he shall be using the word "spiral" to refer only to the line arising from this construction.

Could the line between definitions and theorems be blurred? There is at least one more suspect for that: the last definition "7" asks us that to call "the <circle> drawn with the same center [the start of the spiral] and with **the double line** as radius *second*." I emphasize the words **the double line**. Clearly, in context, we expect Archimedes to define the second circle by its having, as radius, **the second line** (the first circle was defined, just now, by having, as radius, the first line). Archimedes moves from the expected expression **the second line** to another expression, **the double line**, but surely this is not felt as a genuine difference. There is no intention, say, that the definition of the second circle should be different in character from that of the first circle; and as Archimedes proceeds to ask us to generalize the definition to following circles, clearly he does not ask us to choose this or that definition (the **n line** or **the n-times line**). It is taken as if the alternatives are obviously equivalent, because it is indeed a very easy theorem (through the definition of the spiral

[188] It means "coiled" (adjective) or "coil" (noun). The nominative singular form ἕλιξ has a word count of 225 instances in the TLG, most of which are in a non-mathematical sense. (As is often true for Greek mathematical words, a great many of the occurrences of the mathematical sense are in the corpus of philosophical commentary, where the object sometimes appears as a token figure or as an example making some point about the linear and the curved.) I can't seem to trace, myself, archaic occurrences (the poetic form εἶλιξ, too, is not attested prior to Nicander); however, Attic dramatists already use the word (Liddell and Scott cite Aristophanes and Euripides, and through the Thesaurus Linguae Graecae I find Sophocles, as well), and there is no reason to think it was in any sense perceived as "technical" for the readers of Archimedes. It derives rather transparently from the truly ubiquitous verb (attested from Homer onwards) ἑλίσσω, "turn around."

and perhaps proposition 1) that the n-line is n-times the first line. Thus we find that proposition 7 contains – implicitly and unproblematically – a theorem.

The expression "start of the spiral" encapsulates a significant geometrical intuition: the spiral *essentially* has a starting-point – and this is an essential feature of its geometrical existence (in this, after all, the spiral differs from both its parents – the line and the circle).[189] As for the remaining definitions, they imply, taken together, the role that integer numbers play in the making of the geometry of the spiral. We will see much more of that below.

/12/

If however many lines, drawn from the start of the spiral, fall on the spiral during a single rotation, making the angles equal to each other, they <=the falling lines> exceed each other by an equal <difference>.

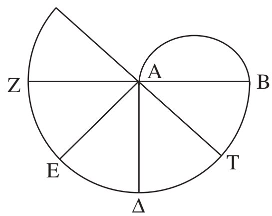
Fall: see Glossary

Let there be a spiral, on which <are> the lines AB, AΓ, AΔ, AE, AZ making equal angles to each other. It is to be proved that AΓ exceeds AB, and AΔ <exceeds> AΓ by an equal <difference>, and the others similarly.

(1) For, <in the time> in which the rotated line reaches from AB to AΓ, in that time the point being carried along the straight <line> passes through the difference, by which ΓA exceeds AB, (2) and, in which time <it reaches> from AΓ to AΔ, in that <time the point> passes through the difference, by which AΔ exceeds AΓ. (3) And the rotated line reaches both from AB to AΓ and from AΓ to AΔ in an equal time, (4) since the angles are equal.[190] (5) Therefore the point carried along the straight <line> passes through the difference, by which ΓA exceeds AB, and through the difference, by which AΔ exceeds AΓ, in an equal time. (6) Therefore AΓ <exceeds> AB and AΔ <exceeds> AΓ by an equal <difference>, as well as the rest <of the lines, accordingly>.

[189] There is an important verbal choice here, of mathematical consequences. Archimedes chose to have the same word for the beginning point as well as the beginning line of the spiral, hence the term "start." Otherwise, it would be quite natural to refer to the "start of the spiral" as the "*center* of the spiral." Avoiding this term tends to de-emphasize the relationship between the spiral and the circle, and to emphasize more the dynamic character of the spiral. It is not altogether clear, however, that such terminology is original to Archimedes in this treatise (clearly the spiral was discussed in some detail between him and Conon; see p. 33); we cannot be certain that such verbal decisions imply a deliberate Archimedean choice. For whatever reason, then, let us note that the spiral, as developed here, has a *start* rather than a *center*.

[190] This seems like an implicit definition of "being rotated at uniform speed with itself."



Codex D has a different diagram altogether, as in the thumbnail.

COMMENTS

The diagram displays the equal angles quite explicitly, and measurably: half a right angle each. (It is for this reason, to be sure, that we have exactly four lines drawn, to divide a straight line into four equal angles.) There is an inert extra line at the top, mysteriously labeled by the point H by Heiberg. While the angles are metrically persuasive, the spiral itself is not (though the illusion might be good enough). AB is a small semi-circle; BΓΔEZ trace another semicircle, double in radius (which is then inertly continued upwards, a further eighth of a rotation). The claim of the proposition, then, is most decidedly not metrically expressed by its diagram: while the angles do exceed each other by an equal amount, the lines are all equal. (Remarkably, this was noted and corrected by the scribe of codex D; Heiberg's diagram, of course, is indeed metrically correct.)

The significance of this theorem is in evoking the language of propositions 10–11: as we hear now that lines are to exceed each other by an equal difference, we expect such lines to display, soon, the properties of the preceding propositions, in this way beginning to tie together the various themes developed so far in the treatise. However, this is not exactly Archimedes' plan, and the following proposition does not take up the theme of proposition 12, leading instead in yet a new direction.

/13/

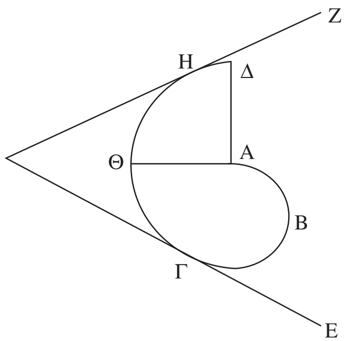
If a straight line touches the spiral, it will touch it at one point only.

Let there be a spiral, on which are <the points> A, B, Γ, Δ, let the point A be the start of the spiral, and let the line AΔ be the start of the rotation, and let a certain line, ZE, touch the spiral. So, I say that it touches it at one point only.

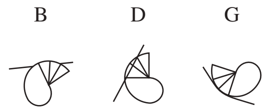
(a) For let it touch, if possible, at two points: Γ, H (b) and let AΓ, AH be joined, (c) and let the angle contained by AH, AΓ be bisected,[191] (d) and let Θ be the point at which the <line> bisecting the angle falls on the spiral. (1) So, both AH exceeds AΘ and AΘ <exceeds> AΓ by an

[191] The literal Greek is "cut in two," meaning in this technical context "cut in two equal parts."

equal <difference> (2) since they contain equal angles to each other;[192] (3) so that AH, AΓ are twice AΘ. (4) But they are also greater than double the <line> in the triangle, AΘ, bisecting the angle.[193] (5) Now, it is clear that the point at which AΘ meets the straight <line> ΓH is between the points Θ, A.[194] (6) Therefore EZ cuts the spiral, (7) since some <part> of the points in the <line> GQH is inside the spiral; (8) but it was assumed to be tangent; (9) therefore EZ touches the spiral at one point only.



B, D and G (as well as a second hand in E) each have a distinct diagram (see thumbnails). Codex C certainly missed Z, and probably E as well.



COMMENTS

This proposition is somewhat related to *Elements* III.2, the first theorem provided by Euclid for the circle: that the straight line joining any two points on the circle passes wholly inside the circle. Why? Because *any* straight line falling inside the triangle whose two sides are the two radii of the circle, is smaller than the radii themselves (this in turn is true for very elementary

[192] Proposition 12.

[193] This – the one genuine geometrical action of the theorem – is left implicit. It can be obtained as a very elementary result, e.g. as follows: in the triangle ABC, AD bisects the angle CAB. Have the side ABAC, and find the point E on AB such that (angle AED)=(angle ADC) (certain to be found, since (angle ABD) <(angle ADC)). From the similarity of the triangles AED, ADC we have rect. (AE, AC)=sq.(AD), and a fortiori rect. (AB,AC)>sq.(AD). Since, when any rect. (P,Q)=sq.(R) we have 2R<P+Q, a fortiori when rect.(P.Q)>sq.(R) 2R<P+Q. It follows that 2*AD<AC+AB.

[194] If we have a genuine triangle AΓH and a genuine angle bisector along the line AΘ, it has to be smaller than AΘ by the property stated in Step 4 above. Notice that we did not yet exit the proof-by-contradiction stage, and we are still working under the assumption of a touching at two points; see the comments.

considerations of triangles). In this case, *Spirals* 13, we have a more limited claim: that one line, namely the midway line between the two lines of the spiral, is similarly smaller (and this for a somewhat more complicated reason having to do with considerations of triangles). Not only is this passage similar to *Elements* III.2, it would also not feel out of place within the middle section of *Conics* I, propositions 17–36, dedicated mostly to results of touching and cutting between curves and straight lines, and mediated mostly through proofs by contradiction of a similar character. In short, the sense is that we embark on a project comparable to book III of the *Elements* or to Apollonius' *Conics* I. In such an imaginary treatise – "*The Elements of Spiral Lines*" – one could have developed at leisure some basic theorems, and then constructions (say, given certain points on the spiral, to find its start). The function of this proposition within the program set in the introduction is not at all obvious, and so the reader, once again, looks for a different program altogether. The effect of proposition 12, suggesting how propositions 10–11 could be developed for the sake of the main task of the treatise, has been subverted.

If we are led to think of *The Elements of Spiral Lines*, then this theorem appears deficient. What is most striking is a reliance on intuition that goes well beyond the axiomatic apparatus available.

We wish to consider a straight line, EZ, and a spiral line, ΑΓΘΗ. We define a certain point on the spiral, Θ, which gives rise to a straight line ΑΘ. We then identify the point at which the straight line EZ cuts the straight line ΑΘ, and we find that this point (not labeled by this proposition) is "inside" the line segment ΑΘ. This is the argument down to Step 5, and it is straightforward. The trouble is the conclusion we wish to draw from it. After all, there is no contradiction yet: a line is allowed to cut another line where it pleases. What does Archimedes find objectionable about this arrangement? Step 6 goes on to assert that, under this arrangement, the line EZ cuts the spiral ΑΓΘΗ, which is explicated in Step 7 by the observation that some parts of the line EZ fall inside the spiral. This Step 6, finally, is supposed to provide for a contradiction, by Step 8, since the line was assumed to be a tangent.

We find several things.

First, we are supposed to know what's "inside" a spiral. There is no definition of that, and apparently the term is supposed to gain its significance, and verification, through visual intuition.

Second, we are told that a line "cuts" the spiral simply because it has a single point inside it (and two points coinciding with it). As a minimum, one would expect to be told that the line also has some points outside the spiral (even that would still allow room for more axiomatic foundations than those available here, assuring that a line being "outside" and "inside" a curve must cut it; but let us not insist on that). How do we know that the straight line has certain segments "outside" the spiral? Do we not judge this purely based on the diagram?

Third, it appears that the two terms "touch" and "cut" are assumed, directly, to rule each other out: a straight line is not allowed to cut a curve at one point and touch it at others. This, then, would be a direct application of the definition of "touching" from the *Elements* (Definition III.2), where a line "touches" a circle when "making contact, and produced, it does not cut it." Such, indeed, appears to be the underlying logic of Steps 6, 8, 9 which simply juxtapose the verbs "touch," "cut" to produce a contradiction. But this is patently false for many curves, *including the spiral itself*! For, obviously, a straight line touching the spiral during the first rotation will cut it during all higher rotations. Clearly, we are still within the terms of the preceding proposition 12 (which indeed is even applied in this proposition), calling for a single rotation.

But even this does not really solve the problem. To some extent I merely pointed out the usual Greek problems with the definition of "tangent" (or, really, the lack thereof for any curves other than the circle), so that any treatment of tangency seems to contain a visual–intuitive component. But there is more than this: the theorem truly is deficient as a matter of logic. For surely we can envisage the following kind of curve: one where the same line is a tangent at two points and, in between, cuts through the curve. So one could imagine the straight line EZ touching the spiral at the points Γ, H, continuing outside the curve for a while and then "lacing in" somehow to pass through the point where EΓHZ, AΘ cut each other. It would be hard not to argue, then, that the same straight line was a tangent to the spiral at two points, even though the results of Step 6 and 8 still applied: one could both cut and touch a curve in such a way. Surely Archimedes' intuition is not that this is an invalid application of the term "tangent" (in which case all he does is to apply, mechanically, a *verbal* form from Euclid), but rather the substantial point that, even though some curves could both cut and touch, *the spiral is not like this*. But to believe that the spiral is not like this is precisely to assume what we need to prove in this proposition.

This problem would be mostly resolved, had Archimedes shown that all points between Γ, H are inside the spiral (and not just one point). The difficulties with the vague definition of "touching" would remain, but the argument would be made clear, and certainly free of a petitio principii. This stronger claim is in fact true, and while I cannot see an easy way there, it is certainly within the means available to Archimedes.[195] I can only imagine that

[195] We need to show that, in an arbitrary triangle with an arbitrary internal line such as ABXΔ (AB>AΔ>AX), a spiral drawn through A,B,X will cut AΔ further away from Δ at a point such as E. We concentrate just on the case where the angle BΔA is right (which loses no generality, as, with an acute angle instead, we have the internal lines in the triangle becoming at first *smaller*, so, obviously, inside the spiral – until, that is, they hit the right angle and begin catching up with the spiral again; and so, in general, other cases can be extended from the case of the right angle as a fortiori). We need to show, in other words, that at this point AX<AE, that is (AX-AΔ)<(AE-AΔ), that is (AB-AΔ):(AX-AΔ) >(AB-AΔ):(AE-AΔ), that is

$$(\mathrm{AB} - \mathrm{A\Delta}) : (\mathrm{AX} - \mathrm{A\Delta}) > (\text{angle } \mathrm{BA\Delta}) : (\text{angle } \mathrm{XA\Delta})$$
$$(\text{proposition } 14,\ \text{independent of proposition } 13)$$

Now, we know from a result stated, without proof, in *Aren.* (Heiberg 1913: I, 232.3–10) that:

Archimedes was aware of the truth of the stronger claim and believed in the validity of proposition 13 based on this awareness; and that he chose to present proposition 13 as he did for strategic reasons: perhaps not to burden the rather trivial claim with a very heavy proof; perhaps to use for this purpose proposition 12 (to create the impression that this, in fact, was the entire purpose of proposition 12). At any rate, we see that Archimedes was quite ready, in the support of a claim he knew to be true, to deploy an argument which he knew to be deficient. This is in line with the attitude displayed again and again so far in the book: brisk to the point of flirting with invalidity. (So brisk, indeed, that the one piece of geometrical reasoning Archimedes does employ – Step 4 – is left implicit, elementary as it is.)

Surely the proposition falls short as a theorem in *The Elements of Spirals*. But then again, it is not. Being brisk to the point of flirting with invalidity would not do for a Euclid, or for the Apollonius of *Conics* I. But it seems to be an effective strategy for Archimedes' purpose, in this treatise. We still need to follow a few propositions to find out better what this strategy is about.

We have repeatedly noted the role of visual intuition in this brisk proof. Note, finally, that the diagram is *not* designed to support such intuitions. True to the spirit of the figures of proof by contradiction, it presents impossibility via a patently impossible visual structure: in this case, the straight line

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$$B\Delta : X\Delta > (\text{angle } BA\Delta) : (\text{angle } XA\Delta).$$

(This is the kind of trigonometric result Archimedes uses in his astronomical calculations.)

So all we need to show is

$$(AB - A\Delta) : (AX - A\Delta) > B\Delta : X\Delta$$

(from which we would a fortiori get our desired result)

or

$$(AB - A\Delta) : B\Delta > (AX - A\Delta) : X\Delta$$

or, with the circle drawn in the triangle cutting Φ,Γ

$$B\Gamma : B\Delta > X\Phi : X\Delta$$

Extend BΦ, have BH parallel to XΦ. Now:

$$BH : B\Delta = X\Phi : X\Delta$$

So all we need to show is:

$$B\Gamma > BH$$

But since ΔΦ crosses through the segment BΓ, while the angle BHΔ is obtuse, this is obvious.

To be sure, Archimedes would find a more elegant argument (I suspect mine is extremely inelegant). But the result is very obviously within Archimedes' means; it also refers, quite clearly, to the type of argument from proportion inequalities one uses in the trigonometry of astronomical calculations.

EZ is broken sharply into two parts, inflecting where the point Θ would have cut it.[196] The choices made in drawing this figure speak to the logic of the argument. This manuscript's diagram is clearer than Heiberg's in one respect, it makes the function of Θ unambiguous (as a point on the spiral). Heiberg, who allows the line EZ on the segment ΓH to coincide with the spiral (so as to make it as line-like as possible) makes the point Θ ambiguous between a point on the spiral and a point on the line EZ; but the entire point of the argument is that the line EZ will not, in fact, meet the line AΘ on the spiral. Thus Heiberg's diagram is truly confusing. However, the manuscripts' diagram is no less problematic: it depicts the point where the line AΘ is to meet with EZ as lying outside the spiral, indeed at the outermost point of the broken line EZ. This is exactly not what the proof of the impossible case requires: Step 5 – the actual claim obtained within the reductio phase of the argument – explicitly makes the lines AΘ, EZ cut each other *inside* the spiral, giving rise to the impossibility. For indeed, the diagram of the manuscripts has the patent impossibility of a broken line, but it does not have the line *cutting* the spiral! It is as if the argument proceeded by showing that a line, touching the spiral at two points, would have to be broken: an impossibility. But this was not Archimedes' line of reasoning: it was that a line touching the spiral at two points would have to cut through it somewhere in between. Now, the scribe of manuscript D, perhaps aware of this difficulty, offered his own revised diagram with the line EZ passing wholly inside Γ,H. But this clearly won't do either, for now the diagram does not display the case of the contradiction at all, but rather displays the state of affairs the proposition shows to be true: we are no longer even in the reductio phase.

The most direct way of drawing a diagram to fit the requirements of this particular reductio, within the standards of representation for Greek mathematics, would be to have a line broken three times: first passing as a tangent through the point Γ; then lacing its way into the spiral to reach the point Θ; then emerging out so as to be able to become a tangent, once more, at point H. This is precisely the arrangement required by the statement of the reductio argument. Why would Archimedes not draw such a figure? Perhaps because this is the case he wishes to ignore in this theorem: he merely takes it for granted that a spiral and a straight line would not be allowed to display such a lacing pattern, and so it suits him best to draw the case of the reductio in a more abstract way, one that merely suggests the contours of the problem at hand. The diagram, like the argument, is brisk, produced at a certain distance.

[196] The diagram makes a decision, left unspecified by the proposition, that the straight line EZ should, in between the points Γ, H, be wholly outside the spiral (and not coincide with it). Manuscripts BG, followed by Heiberg, made the milder choice to display the straight line as coinciding with the spiral through the length ΓH; manuscript D made the choice to display the straight line as wholly inside the spiral through this length.

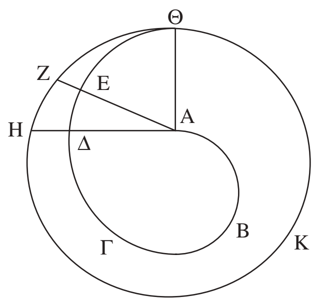
/14/

If, from the point that is <the> start of the spiral, two lines fall on the spiral drawn during the first rotation, and are produced to the circumference of the first circle, the lines falling on the spiral shall have to each other the same ratio that the circumferences of the circle between the end of the spiral and the ends of the lines which were produced so as to come to be on the circumferences <have to each other> (the circumferences being taken in the preceding direction, from the end of the spiral).

Let there be a spiral drawn in the first rotation, ΑΒΓΔΕΘ, and let the point Α be <the> start of the spiral, and let the line ΘΑ be <the> start of the rotation, and let ΘΚΗ be the first circle, with the lines ΑΕ, ΑΔ falling on the spiral from the point Α, and further falling on the circumference of the circle on the <points> Ζ, Η. It is to be proved that they have the same ratio: ΑΕ to ΑΔ, <the same> which the circumference ΘΚΖ <has> to the circumference ΘΚΗ.

(a) For, the line ΑΘ being rotated, (1) it is clear that the point Θ is carried at a uniform speed along the circumference of the circle ΘΚΗ while Α, being carried along the line, passes through the line ΑΘ, (2) and the point Θ, being carried along the circumference of the circle, <passes through> the circumference ΘΚΖ, while the point Α <passes through> the line ΑΕ, (3) and again both the point Α <passes through> the line ΑΔ and Θ <through> the circumference ΘΚΗ, each being carried itself at uniform speed with itself. (4) Now, it is clear that they have the same ratio: ΑΕ to ΑΔ, <the same> which the circumference ΘΚΖ <has> to the circumference ΘΚΗ. [(5) For this has been proved outside, in the first <propositions>.]

(6) And similarly it shall be proved that even if one of the falling <lines> should fall on the end of the spiral, the same thing happens.



In codices AC the spiral was drawn as a combination of two semicircles. Codices BDG redrew it to appear like a spiral (BD further positioned the point Α to be at the center of the circle ΖΘΚ). G has the point Θ slightly to the left of Α. H, and perhaps C, have lost the point Γ.

COMMENTS

This theorem could have been positioned anywhere following proposition 2. In particular, it is independent of the two preceding theorems concerning the spiral, and so the impression does accumulate that we proceed now "horizontally," in a kind of *Elements of Spiral Lines*.

How do we even know we rely on proposition 2? Step 5 makes this explicit, but Heiberg suspects, reasonably, that this is a late scholion. Otherwise, the verbal clues are minimal. What could provide such verbal clues? The key requirements for the application of proposition 2 would have been that the points move "in an equal time" and that each moves "in uniform speed with itself." As it is, Archimedes never mentions "equal time" (using instead certain connectors to express this idea: *both* X happens *as well as* Y, the understood meaning being that the two happen *simultaneously*). "Uniform speed with itself" is alluded to in the language of Step 1 and is explicitly quoted only in the language of Step 3. The conclusion of Step 4 does not really appear to follow from the application of proposition 2: rather, the connector "it is clear that . . . " suggests that the derivation of Step 4 from Steps 2–3 is designed to be intuitive in and of itself, which of course it is. Proposition 2, as well, has the same feel of a distant, brief argument, merely presenting the grounds for claims that are supposed to be self-evident. The spiral has led us back full circle.

Step 6 is extraordinary. Nothing in the argument relied on the point Z being distinct from Θ: there was no reason why the greater circumference should not be identical with an entire circumference of a circle. It is not often that a Greek geometrical text invites us explicitly to generalize the argument to cases not apparent in the diagram. Rather, the expectation is that such generalizations will be carried automatically. What this Step suggests perhaps is that the term "circumference of a circle," unqualified, is understood in ordinary circumstances to mean "less than an entire circle," and so, for this reason, one needs the further explication of Step 6 to generalize not so much beyond one's diagram as beyond one's language. More of this problem of generalization and segmentation of the logical space to come below.

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And if lines fall from the start of the spiral on the spiral drawn during the second rotation, the lines shall have to each other the same ratio, which the said circumferences together with an entire circumference of a circle taken <have to each other>.

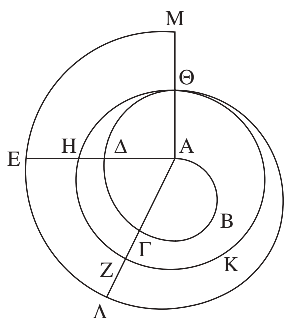
Let there be a spiral, on it <the circumferences> ΑΒΓΔΘΛΕΜ,[197] ΑΒΓΔΘ drawn in the first rotation, and ΘΛΕΜ in the second, and let the lines ΑΕ, ΑΛ fall <on them>. It is to be proved that ΑΛ has to ΑΕ the same ratio which the circumference ΘΚΖ together with an entire

[197] Codex A has α ΑΒΓΔΘ, C has αι ΑΒΓΔΘ. Heiberg suggested one should perhaps read τὰ ΑΒΓΔΘ, but printed the reading of codex A, which is cautious but in a very difficult sense. Torelli (without the benefit of C) emended drastically to α ΑΒΓΔΘΛΕΜ. I find myself leaning towards Torelli's heavy intervention, largely because I'm supposed to produce a translation and so I'd better have some sense in my text.

circumference of the circle <has> to ΘKH together with an entire circumference of the circle.

(1) For in as much time[198] as the point A, carried along the line <AΛ>, passes through the line AΛ; the point Θ, carried along the circumference of the circle, also passes through both the circumference of the circle entirely and yet again the circumference ΘKZ, (2) and again: <in as much time as> the point A <passes through> the line AE, Θ also <passes through> both the circumference of the circle entirely and yet again the circumference ΘKH, each carried at uniform speed itself with itself. (3) Now, it is clear that the line AΛ has the same ratio to AE, which the circumference ΘKZ together with an entire circumference of the circle <has> to the circumference ΘKH together with an entire circumference of a circle.

(4) It shall be proved in the same manner that, even if lines should fall on the spiral drawn during the third rotation, they have the same ratio to each other which the said circumference, together with the entire circumference of the circle taken twice, <has to the other circumference with the circle taken twice>; (5) and similarly, it is proved that the lines falling on the other spirals, too, have the same ratio which the said circumference together with the entire circumference of the circle taken as many times, as the number, smaller by one, of the rotations, <has to the other circumference with the circle taken that many times>; (6) even if one falling line should fall on the end of the spiral.



Codices AC drew the spiral as the combination of four semicircles. Codices BDG redrew it as a spiral, with codices BD moving the point A to the center of the circle. Codex G added an external circle with A as center, AM as radius; it also has M, Θ to the northwest of A. Codex C may have lost the label M (the reading is very difficult at that point).

COMMENTS

Propositions 14–15 are reminiscent of propositions 1–2. In both cases, we have a single theorem divided into two textual segments, because two separate diagrams are required for two steps of the same claim. Indeed, in the case of propositions 14–15, one hardly needed any separation. One could simply have

[198] Translating ὅσῳ, the reading of the manuscripts, as against ἴσῳ, Heiberg's emendation following codex B. I thank J. Wietzke for pointing out this textual observation.

stated the claim of proposition 15, Step 5, suitably arranged, as a single general statement of which propositions 14–15 provide a single proof. The sequence we do have, instead, appears to be rather muddled. Propositions 14–15, taken together, make six separate claims:

Lines falling on the spiral, and the circumferences associated with them, are in the same ratio:

1. when both circumferences are smaller than a circle (14, Steps 1–5)
2. when one circumference is equal to a circle, the other smaller than a circle (14, Step 8)
3. when both circumferences are more than one circle but fewer than two (15, Steps 1–3)
4. when both circumferences are more than two circles but fewer than three (15, Step 4)
5. when both circumferences are more than any number (n) of circles (n>2) but fewer than (n+1) (15, Step 5)
6. when one circumference is equal to exactly (n+1) circles, the other smaller than that (but greater than (n) circles) (15, Step 6).

Propositions 14–15 make the rather arbitrary choice to make an explicit (though very minimal) argument for claims 1, 3, and to let claims 2, 4–6 follow by analogy. But why do that? Why not let claim 3, equally, follow by analogy from claim 1? (Certainly the "it shall be similarly proved" operator can go as far as *that*). And again, why not prove all the various cross-integer results which appear equally valid (say, that with two lines, one falling on the first rotation, the other on the second, the ratio shall be as the associated circumferences)?

We notice several things. First, we find that Archimedes strongly wishes to maintain the sense of easy claims and proofs. A single general statement, based on proposition 15, Step 5, would have appeared unwieldy: complex and heavily relying on numerical values (a bit like propositions 10–11 above, or some later propositions such as 28 below). Perhaps, the overall composition requires a certain lull now, before the main action of the treatise begins in earnest.

Second, whereas a straight line, being extended, remains qualitatively the same thing, a spiral is different: there appears to be a qualitative difference between a spiral less than a single rotation, a spiral of a complete single rotation, a spiral that goes beyond that, etc. etc. The "inflexion points," where a spiral completes a rotation, strike one, at first glance, as constituting qualitatively different categories of the spiral. Archimedes, who possesses much more understanding of the spiral, must already know that this first impression is largely wrong and that the spiral, for most purposes of this treatise, is qualitatively homogeneous along its rotations. He never tries, however, to convey the homogeneity of the spiral.

Third, when faced by an essentially open choice about how to segment the logical space opened by the many cases, Archimedes opts for a *pair* of cases, ignoring some (cross-integer) results and relegating some of the results to a "corollary" status (it was merely Heiberg's choice not to label 14, Step 6, and 15, Steps 4–6, as "corollaries"); an architecture of "twice over – and then some." We will see more of this below.

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If a straight line touches the spiral drawn in the first rotation, and a line should be joined from the touching point to the point which is <the> start of the spiral, the angles which the tangent makes with the joined <line> will be unequal, and that in the preceding <lines will be> obtuse, while in the following <lines> acute.

Let there be a spiral on which ΑΒΓΔΘ, drawn in the first rotation, and let the point Α be <the> start of the spiral, and the line ΑΘ <be the> start of the rotation, and the <circle> ΘΚΗ <be the> first circle, and let some straight line ΔΕΖ[199] touch the spiral at Δ, and let ΔΑ be joined from Δ to Α. It is to be proved that ΔΖ makes an obtuse angle with ΑΔ.

(a) Let a circle, ΔΤΝ, be drawn, with Α as center and ΑΔ as radius. (1) So, it is necessary that the circumference of the circle in the preceding <lines> falls inside the spiral, but in the following <lines> outside, (2) through the fact that, among the lines falling on the spiral from Α, the <lines> in the preceding are greater than ΑΔ, but those in the following are smaller. (3) Now, that the angle contained by the <lines> ΑΔΖ is not acute, is clear, (4) since it is greater than the <angle> of a semicircle.[200]

But it is to be proved as follows that it is not a right <angle>:

(5) For let it be, if possible, a right <angle>; (6) therefore ΕΔΖ touches the circle ΔΤΝ.[201] (7) So, it is possible to insert a line from Α to the tangent, so that the line between the tangent and the circumference of the circle has to the radius of the circle a smaller ratio than the circumference between the touching point and the falling <line> has to the given circumference.[202] (b) So, let it fall <as> the line ΑΙ. (8) So, it cuts the spiral at Λ, (9) and the circumference of the circle ΔΝΤ at Ρ. (c) And let the line ΡΙ have to the

[199] Heiberg corrects the manuscripts' reading into ΕΔΖ simply because it breaks from the lettering rule that lines are lettered in spatial sequence (preferring an alphabetic sequence to a spatial one). I would rather see here an exception to the rule than go against the reading shared by both A and C.

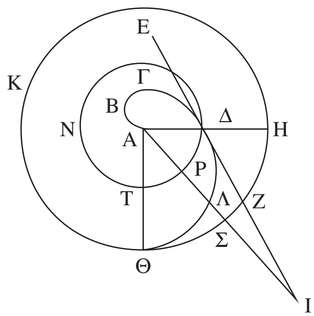
[200] "The angle of the semicircle" is the angle contained by the radius and the *circumference* of the circle (it is naturally visualized as the angle at the "base" of a semicircle, hence the name); in other words, an angle whose one side is a curved line. The angle of the semicircle is proved by Euclid, *Elements* III.16, to be greater than any acute angle, so Step 3 indeed follows as an a fortiori from Step 4; Step 4 in turn follows from Step 1, or perhaps from the diagram as interpreted by that Step. Once again: the entire sequence of propositions 12 onwards is steeped with the concerns, and claims, of Euclid's *Elements* III.

[201] *Elements* III.16 – the same theorem as the one stating that the angle of the semicircle is greater than any acute angle.

[202] Proposition 5 – recalled with many verbal changes. See the comments.

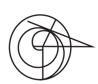
<line> AP a smaller ratio than the <ratio> which the circumference ΔP has to the circumference ΔNT.[203] (10) Therefore IA in its entirety, too, has to AP a smaller ratio than the circumference PΔNT to the circumference ΔNT,[204] (11) that is, <than the ratio> which the circumference ΣHKΘ has to the circumference HKΘ.[205] (12) But that <ratio> which the circumference ΣHKΘ has to the circumference HKΘ, the line AΛ has to AΔ; (13) for this has been proved.[206] (14) Therefore AI has to AP a smaller ratio than, indeed, ΛA <has> to AΔ, (15) which indeed is impossible. (16) For PA is equal to AΔ.[207] (17) Therefore the <angle> contained by the <lines> AΔZ is not a right <angle>. (18) And it was proved that neither is it acute. (19) Therefore it is obtuse. (20) Thus the remainder is acute.

(21) And it shall be proved similarly that even if the tangent touches the spiral at the end, the same thing shall happen.



Codices AC drew the spiral as a sequence of two semicircles. Codices DG redrew it as a spiral and also produced a different orientation as in the thumbnails. Codex C misses H, K.

Codex D



Codex G

[203] Step c explicates and completes the statement of Step a: the line of AI is to fall, fulfilling the condition of Step 7, so that the given circumference defining the construction is specified as the circumference ΔNT.

[204] Extension to inequalities of *Elements* V.18.

[205] Step 11 claims PΔNT:ΔNT::ΣHKΘ:HKΘ (can be argued based on *Elements* VI.33, though is perhaps best seen as directly intuitive). It follows implicitly from Steps 10–11 that IA:AP<ΣHKΘ:HKΘ.

[206] Proposition 14. The implicit result obtained above, together with Step 12, gives rise to Step 14.

[207] Radii in a circle. Step 16 transforms Step 14, implicitly, into the inequality: IA: AΔ<ΛA:AΔ or (*Elements* V.8) IA<ΛA, the impossibility of which derives from the diagram as interpreted by Step 8.

COMMENTS

At one level this proposition fits the character of propositions 12–15: it is an elementary result about the spiral, reminiscent of the fundamental Euclidean theorems on the circle. This proposition is closest in spirit to *Elements* III.16, showing that the tangent is at right angles to the radius. *On Spirals* 16 invokes a language specific to Euclid's proposition ("angle of the semicircle"), and in its main argument it follows a similar line of thought: trying to interpose a line between the tangent and the circle, we find the absurdity of a line being smaller than its part.[208] (One wonders if the entire line of thought giving rise to proposition 5 and its application here may have come from this kernel-idea.) The relation between the two results, on the spiral and the circle, is pleasing and somehow goes towards explicating what kind of a creature the spiral is: it is like a circle that keeps "popping out."

At another level this proposition goes back to the earlier stage of propositions 5–9 in its explicit reference to proposition 5 but also in the overall scale and texture of the argument: it is brief and yet geometrically interesting.[209] Remarkably, Archimedes evokes the texture of propositions 5–9 by the deliberate deployment in the diagram of the letter I – a very marked choice within Greek geometry. This is especially remarkable, because the letter I was not used in proposition 5, which this proposition requires. Instead, it was used in the later propositions 7–9. The sense is therefore that Archimedes relies not so much on the individual claim of proposition 5 but rather on proposition 5 as a segment among several results obtained before (this is perhaps further underlined by the remarkably imprecise quotation of the *text* of proposition 5, which makes the reference appear not to be a direct application of the particular claim of the textual segment "proposition 5"). This fits a certain tendency of this "elementary" stage of the argument, already from proposition 12 onwards, to reprise the themes of the preceding "introductory" stage. Proposition 12 reprises themes from propositions 10–11 (it referred to lines exceeding each other by an equal amount; proposition 13 then followed upon the result of proposition 12). Propositions 14–15 heavily relied on proposition 2, the stage of the argument bringing together time and proportions. And now this proposition 16 very directly continues the line of thought from propositions 5–9, tangents drawn to precise specifications. These were the three key moments of the introductory stage: times and proportions; tangents; lines

[208] In the diagram of *Elements* III.16, AE is the tangent and ΑΓΦ is hypothetically drawn between AE and the circumference; Γ is where angle ΔΓΑ is right (by hypothesis, lines ΑΦ, AE are distinct, and so they make right angles with point A on distinct points); ΔΓ cuts the circle at H. From angle ΔΓΑ's being right, it follows that ΔΑ>ΔΓ or, impossibly, ΔΗ>ΔΓ.

[209] The geometrical interest arises from the combination of the proposition. Step 11 is a direct application of proposition 5 (which is not about the spiral at all); Step 12 follows from proposition 14, a very elementary result, almost true by definition, for the spiral. The combination of these simple yet very distinct results elegantly gives rise to the impossibility of Step 14.

exceeding each other. Archimedes reprises them, in perturbed sequence: first the third, then the first, now the second. The complex zigzag of the introductory stage is re-zigzagged or, more precisely, re-zagzigged.

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And furthermore, if the line touches the spiral drawn in the second rotation, the same thing shall happen.

(a) For let the line EZ touch the spiral drawn in the second rotation at Δ, (b) and let the rest be constructed the same as before. (1) So, similarly, the <parts> of the circumference of the circle PNΔ in the preceding <parts> of the spiral shall fall inside, while the <parts> in the following <fall> outside.

Now, the angle <contained> by the <lines> AΔZ is not a right <angle>, but obtuse.[210] For let it be, if possible, a right <angle>; (2) so, EZ will touch the circle PNΔ at Δ. (c) So, again, let AI be drawn with the tangent and let it cut the spiral at X, (d) and the circumference of the circle PNΔ at P. (e) And let PI have to PA a smaller ratio than the <ratio> which the circumference ΔP has to the circumference of the circle ΔPN in its entirety and [to] ΔNT. (3) For this has been proved possible.[211] (4) Therefore IA in its entirety has to AP a smaller ratio than the circumference PΔNT together with an entire circumference of the circle to the circumference ΔNT together with an entire circumference of the circle.[212] (5) But the ratio which the circumference PΔNT together with the circumference of the circle ΔNTP in its entirety has to the circumference ΔNT together with the circumference of the circle ΔNTP in its entirety – that ratio the circumference ΣHKΘ together with the circumference of the circle ΘΣHK in its entirety has to the circumference HKΘ together with the circumference of the circle ΘΣHK in its entirety, (6) and the ratio which the last mentioned circumferences have – that ratio the line XA has to the line AΔ; (7) for this has been proved.[213] (8) Therefore IA has to AP a smaller ratio than AX to AΔ;[214] (9) which is

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[210] The argument that it is not acute is taken over implicitly from the preceding proposition, Steps 1–3.

[211] Proposition 5, again; the one difference is that the given circumference is more than a circumference of a circle.

[212] The argument of Steps 8–10 of the preceding proposition, telescoped.

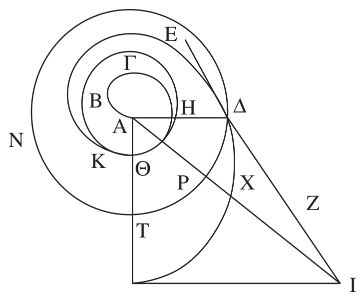
[213] Proposition 15 (and not, unlike Step 13 of the preceding proposition, proposition 14).

[214] The sequence of argument in Steps 4–8 is: (4) IA:AP is smaller than a certain ratio of circumferences derived from circle ΔPNT (based on the construction, from proposition 5). (5) The same ratio of circumferences is transferred to one derived from the circle ΣHKΘ. (6) The last ratio is transformed (from proposition 15) to a ratio of lines of the spiral: XA:AΔ. We end up with: IA:AP<XA:AΔ. All this follows exactly the argument

indeed impossible [(10) for PA is equal to AΔ,[215] (11) while IA is greater than AX].[216] (12) Now, it is clear that the angle contained by the <lines> AΔZ is obtuse; (13) thus the remaining <angle> is acute.

(14) And the same things shall happen, even if the tangent should touch at the end of the spiral.

(15) And it shall be proved similarly, that if some line should touch the spiral drawn in however many rotations, even if at its end, it shall make the angles with the line joined from the touching point to the start of the spiral unequal: the <angle> in the preceding <parts> obtuse, and in the following, acute.



Codices BD have the point Δ higher; codices EGH4 have the point I lower, and thus probably codex A. Codex C did not produce a continuous spiral at all: see thumbnail. Codex A had O instead of Θ (corrected by BE), K instead of X (corrected by BD), Ϙ instead of I (corrected by D; B is hidden in the gutter at this point). Codex C missed Θ. H has Π for Γ (so, perhaps, E, that also has Π for T); it also positions H nearer the point Δ.

COMMENTS

Heiberg dislikes Steps 10–11. He attributes them to a scholiast and argues that the proper place for this, more expanded argument would be in proposition 16, Step 16 (where Archimedes merely asserts the equality of the radii as in Step 10 here, and does not mention the inequality of the line segments lying on a single line, as in Step 11 here). Indeed, Heiberg wonders if proposition 16, Step 16, too, could not be an interpolation, Archimedes merely stating in both propositions, without explanation, that an impossibility was reached.

These are questions of consistency. Would Archimedes be implicit concerning some complex geometrical arguments – as we see him throughout the

of Steps 8–14 in the preceding proposition, with the difference that we refer to proposition 15, not 14 (which are no more than different cases of the same proof).

[215] Radii of the circle.

[216] The construction, or the diagram; as well as Step 1. See the comments on the textual question concerning Steps 10–11.

treatise – and then be quite explicit concerning quite trivial claims such as the equality of the radii? Could Archimedes be more explicit here, less explicit there, concerning the very same argument? Could he even – against our intuitions – move in the direction of making his arguments *more* explicit?

Indeed, the more common trajectory is that of growing implicitness. Most notably, this proposition 17 simply takes for granted that the angle cannot be acute, which required an interesting, if brief, argument in the preceding proposition. I wonder if Archimedes even noted how precisely explicit he was concerning this trivial claim of proposition 16, Step 16 / proposition 17, Steps 10–11. The claim quickly explicates information packed into the diagram. Perhaps, with the diagram powerfully present in one's mind, one "sees" the same text, encoded in a dual visual–verbal format, in both proposition 16, Step 16 and proposition 17, Steps 10–11. Such speculations aside, we definitely need to assume that, if Archimedes was indeed the author of both claims, then he probably did not proofread his text carefully for consistency. This in and of itself would not be surprising: we may compare this with the very distant, almost distorted way in which the text of proposition 5 is quoted in its application by those two propositions. It is not necessarily that Archimedes did not bother with such textual details; it is quite possible that he preferred the richer texture of imprecise repetition, preferring *variatio* to consistency.

This textual question is an example of the main theme of this proposition, which we see repeated throughout the treatise: a binary repetition. More precisely, the theme is that of "dual – and more." A claim is asserted, reasserted and opened up (once or both times) further beyond the dual repetition. In the pair of propositions 16–17 this takes the form:

1st rotation (further: end of same) / 2nd rotation (further: end of same) (further: any rotation)

Both propositions 16–17 contain a brief extension to the end of the rotation; proposition 17 contains another extension, to any rotation. This is almost the same as the structure of propositions 14–15:

1st rotation (further: end of same) / 2nd rotation (further: 3rd rotation) (further: any rotation) (further: end of any rotation)

In a different way, propositions 10–11 also display a similar structure:

A complex equality with squares (further: inequality) (further: other similar figures) / A complex ratio inequality with squares (further: other similar figures)

Propositions 14–15, 16–17 differ from propositions 10–11 mostly in the manner of their deductive dependence: in 14–15, 16–17 this takes the form of repetition (proposition 15 follows the same deductive structure as 14; proposition 17 follows the same deductive structure as 16), while in 10–11 this takes the form of application (the most difficult claim of proposition 11 derives from proposition 10). All three, however, display deductive duality: two propositions strongly related in their logical structure to each other, and only rather weakly, in comparison, related to other results in this treatise or elsewhere. The same is true also of the pair of propositions 12–13 which, while not engaging in the corollaries of "furthermore" are also two propositions strongly related in their logic (13 does almost nothing, geometrically, beyond applying 12), and only more weakly related to other results (the result that they do quote – proposition 1 – is very trivial). The sequence of propositions 10–17 is made of four consecutive islands each consisting of two propositions.

Looking back, we see the same structure in propositions 1–2: very similar claims, with proposition 2 applying nothing beyond proposition 1; as well as in propositions 3–4, if indeed they are to count as such. In short – with the exception of the sequence of problems 5–9 – everything in this treatise so far is made of dual islands. In the more significant pairs 10–11, 14–15, 16–17 this takes the specific form of "dual – and more."

This of course reflects to some extent the structure of the spiral itself. The spiral rotates, repeating the same things, and then rotates some more, all the while appearing, at first glance, to change its qualitative nature. Archimedes, then, reflects this with claims that repeat and then open themselves up for further, open-ended repetition. But note that this geometrical necessity does not truly constrain Archimedes' structuring of his dual islands. Instead, textual duality is a choice. We noted already, in propositions 14–15, that the choice to prove the claim separately for these two cases, and then to generalize without proof for other cases, is in fact not geometrically motivated: it would make perfect sense, geometrically, to have a single argument showing the validity of both 14, 15, as well as their corollaries. The same is true for propositions 16–17 – which becomes obvious once we note that the only logical difference between 16 and 17 is that 16 relies on 14, while 17 relies on 15. The geometrically unmotivated distinction between 14 and 15 is imported into a geometrically unmotivated distinction between 16 and 17.

Propositions 14–15 and 16–17 are a single claim, exploded into a "dual – and more" structure. Propositions 10–11 and 12–13 are unnecessarily restricted to the dual. As pointed out above (p. 79), proposition 10 could become much clearer by being broken into several smaller arguments, turning the overall structure of propositions 10–11 into an entire passage with four to five propositions, comparable to the preceding passage with propositions 5–9. Further, propositions 10–11 do not absolutely require the "further" structure of their corollaries (though admittedly this is a natural enough structure). One could also make the claims, from the start, at the level of the generality of similar plane figures, and then, in the course of the proof, transform the claims for similar plane figures to claims for squares. Once again: this is perhaps less elegant, but the point remains that the geometry, in and of itself, does not force the "dual – and more" structure.

As for propositions 12–13, the reliance of proposition 13 on the proof of proposition 12 is very much a geometrical error (which Archimedes surely recognized). The alternative would be to base proposition 13 on a more complex, trigonometric result, so that if one still wishes to provide proposition 12 (which is indeed required *independently* by the treatise), the duality is lost: most natural perhaps would be to have proposition 12 stand alone, and have proposition 13 expanded (rather than bifurcated) to include the complex trigonometric argument. Instead of a dual island, one would have two separate monads, one much bigger than the other.

"Dual – and more" is a procrustean bed to which Archimedes fits, deliberately, much of this treatise. And if this structure mirrors the geometrical structure of the spiral, then the relation of geometrical and textual structure is not that of geometry forcing an author, unwittingly, to follow a certain textual pattern. Rather, we have here an author patterning his text, perhaps as an elaborate meta-textual statement, or perhaps, more simply, because of an interest in creating patterns.

The pattern has been rich, almost kaleidoscopic. A series of zigzags in the introduction (1–2, 5–9, 10–11, to which we may add the minimal pair 3–4); answered by a related, but differently patterned, zigzag in the intermediate stage (12–13, 14–15, 16–17). Almost everything – but with one important exception – is composed of dual islands. The structure of "dual – and more" is paramount in the pairs 10–11, 14–15, 16–17, which therefore crosses the border from the introductory to the intermediate. This structure also gradually becomes more and more central to the treatise, perhaps to suggest how we get near to the main theme of finding results about this "dual – and more" object which is the spiral.

Which we are about to do. We have reached the middle of the treatise measured in its bulk, and the next proposition, to our shock, will bring us our first substantial result and the squaring of the circle.

/18/

If a straight line should touch the spiral drawn in the first rotation at the end of the spiral, and a certain <line> is drawn from the point, which is <the> start of the spiral, at right <angles> to the start of the rotation, the drawn <line> shall meet the touching <line>,[217] and the line between the tangent and the start of the spiral shall be equal to the circumference of the first circle.

Let there be a spiral ABΓΔΘ, let the point A be <the> start of the spiral, the line ΘA <the> start of the rotation, and the circle ΘHK the first <circle>. And let some line, <viz.> ΘZ, touch the spiral at Θ, and let AZ be drawn from A at right <angles> to ΘA. (1) So, that <line> shall meet ΘZ, (2) since ZΘ, ΘA contain an acute angle.[218] Let it meet the <point> Z. It is to be proved that ZA is equal to the circumference of the circle ΘKH.

(3) For if not, it is either greater or smaller. (a) Let it first be, if possible, greater. (b) So, I took a certain line, ΛA,[219] smaller than the

[217] I usually use simply "the tangent" for what I translate here as "the touching <line>"; I use this more complex form to highlight the elegant similarity of this with the preceding "the drawn <line>." Later in my translation I will return to using "tangent."

[218] Proposition 16 (as well as the parallels postulate).

[219] The diagram positions Λ on the circumference of the circle, a typical economy of ancient diagrams; in fact the position of Λ on AZ cannot be determined.

line ZA but greater than the circumference of the circle ΘHK.[220] (4) So, there is a certain circle, ΘHK, and a line in the circle, smaller than the diameter, ΘH,[221] and a ratio, which ΘA has to AΛ, greater than the <ratio> which the half of HΘ has to the perpendicular to it <=to HΘ> drawn from A (5) because <it is> also <smaller than> the <ratio> which ΘA has to AZ.[222] (c) Now, it is possible to extend <a line> from A towards the produced <line> AN, so that the <line> between the circumference and the produced <line>, NP, has to ΘP[223] the same ratio which ΘA <has> to AΛ.[224] (6) Now, NP shall have to PA a ratio which the line ΘP has to AΛ.[225] (7) And ΘP has to AΛ a smaller ratio than the circumference ΘP to the circumference of the circle ΘHK; (8) for the line ΘP is smaller than the circumference ΘP,[226] (9) while the line AΛ is greater than the circumference of the circle ΘHK.[227] (10) Now, NP, too, shall have to PA a smaller ratio than the circumference ΘP to the circumference of the circle ΘHK.[228] (11) Now, NA, in its entirety,[229] too, has to AP a smaller ratio than, indeed, the circumference ΘP together with the circumference of the circle in its entirety to the circumference of the circle

[220] Proposition 4. The verb form is extraordinary; it seems to suggest Archimedes' personal presence, filling in for an impossible geometrical act (so that's how *I* squared the circle – the precise meaning of finding AΛ).

[221] While this is not immediately obvious in the diagram, H is understood to be the point where the tangent to the spiral ZΘ cuts through the circle ΘHK. Heiberg points out that, to be equal to the diameter, ΘH would have to pass through the center of the circle A (*Elements* III.7), that is, through the start of the spiral, impossible for a tangent (proposition 13). In all likelihood the claim is just taken to be visually obvious. Notice, finally, that all that the point H does in this proposition is to underwrite the applicability of proposition 7, that is, to locate the position of the line, extended, on which N is to be found.

[222] If we join the perpendicular from the point A to the line HΘ (or ZΘ), it is then a perpendicular inside the right-angled triangle ΘAZ, with the similarities of internal triangles following (*Elements* VI.8). In particular we have: (half HΘ):(perpendicular):: ΘA:AZ. (Bear in mind that the perpendicular bisects the chord HΘ: *Elements* III.3.) Step 4 points out that ΘA:AΛ<ΘA:AZ (AZ>AΛ by construction (*Elements* V.8). From this, together with the geometrical consideration above (left implicit by Archimedes), Step 4 follows.

[223] Line, that is, and not circumference. [224] Proposition 7.

[225] Step 5, *Elements* V.16, gives NP:ΘA::ΘP:AΛ; ΘA=PA (radii in circle; this transition between radii is a very typical move in the spiral lines: one dials along the circle until one hits the spirally significant line).

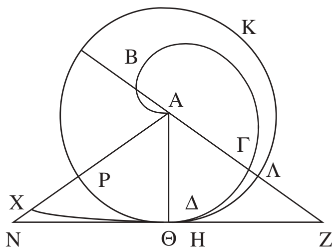
[226] In a sense follows from Archimedes' treatment of concavity in *On the Sphere and the Cylinder* I; probably generally taken for granted in Greek geometry.

[227] By hypothesis: Step a. Step 7 follows from 8–9 via *Elements* V.8, with V.16.

[228] Steps 6–7; extension to inequality of *Elements* V.11.

[229] The phrase "in its entirety" is inert here, merely anticipating its application to a "circumference in its entirety" which in turn acts as an allusive reference to proposition 15.

ΘHK.[230] (12) And the ratio which the circumference ΘP with the circumference of the circle ΘHK in its entirety has to the circumference of the circle ΘHK – that ratio XA has to AΘ;[231] (13) for this has been proved; (14) therefore NA has to AP a smaller ratio than, indeed, XA <has> to AΘ; (15) which indeed is impossible; (16) for NA is greater than AX[232] (17) while AP is equal to ΘA.[233] (18) Therefore ZA is not greater than the circumference of the circle ΘHK.



BDG have the point Z higher, at the level of A; H4 have ZΘN flat. I suspect codex A (preserved only by E) had Z slightly higher than Θ, as does codex C and as I print it. The segment of the spiral ΘX is probably missing from codex C, and was a straight line (as I print it), in codex A, "corrected" to a curved line by codex D alone. BDG have removed, in a sense "correctly," the line segment of ZA, extended to the circle to the northwest of A. Codex A had H instead of K (corrected by G, missed by D; B has K correctly but differently positioned, at "10 o'clock" of the circle). Codex C may have missed the letter altogether (the text is difficult to read).

(d) So, again, let ZA be, if possible, smaller than the circumference of the circle ΘHK. (e) So, I took a certain line, again, AΛ, greater than AZ but smaller than the circumference of the circle ΘHK,[234] (f) and, from Θ, I draw ΘM parallel to AZ. (18) Now, again, there is a circle, ΘHK, and in it a line smaller than the diameter, ΘH, and another <line>, touching the circle at Θ, and a ratio, which AΘ has to ΘΛ, smaller than the <ratio> which the half of HΘ has to the perpendicular to it <=HΘ> drawn from A (19) since it is also smaller <than> the <ratio> which ΘA has to AZ.[235] (20) Now, it is possible to draw the <line> AΠ from A to the tangent, so that PN, the <line> between the line in the circle and the circumference, has to ΘΠ, the <line> taken off from the tangent, the ratio which ΘA has to AΛ.[236] (21) So, AΠ shall cut the circle at P, (22) and the spiral at X.[237] (23) And also alternately: NP shall have to PA the same ratio which ΘΠ <has> to

[230] Step 10 yields Step 11 via an extension to inequalities of *Elements* V.18:

$$NP : PA < (\text{circumf.}\Theta P) : (\text{circumf.}\Theta HK) \text{ gives rise to}$$
$$NP + PA : PA < (\text{circumf.}\Theta P) + (\text{circumf.}\Theta HK) : (\text{circumf.}\Theta HK)$$

[231] We now identify point X; the result is a direct application of proposition 15.

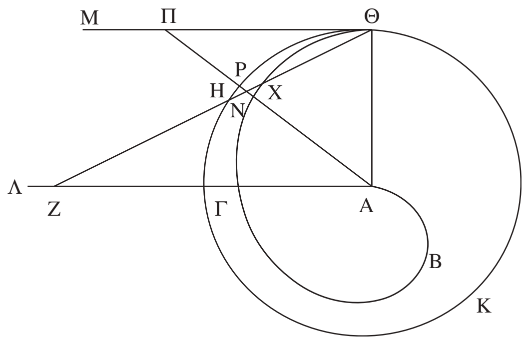
[232] Inclusion; the key observation once again is proposition 13 guaranteeing that the point N lies outside the spiral.

[233] Radii in a circle. [234] Proposition 4.

[235] Exactly the same as Step 4 above. [236] Proposition 8.

[237] The parallel statement is not made in the first part of the proposition (so that the X appears there as a purely diagrammatically defined point). Quite possibly this is a textual lacuna. But then again, is it not interesting that the phrasing of Step 19 is more full than that of the parallel Step 4?

AΛ.[238] (24) But ΘΠ has to AΛ a greater ratio than the circumference ΘP <has> to the circumference of the circle ΘHK; (25) for the line ΘΠ is greater than the circumference ΘP,[239] (26) while AΛ <is> smaller than the circumference of the circle ΘHK.[240] (27) Therefore NP has a greater ratio to AP than the circumference ΘP <has> to the circumference of the circle ΘHK;[241] (28) so that PA, too, has to AN a greater ratio than the circumference of the circle ΘHK <has> to the circumference ΘKP.[242] (29) But the ratio which the circumference of the circle ΘHK has to the circumference ΘKP – that <ratio> the line ΘA has to AX; (30) for this has been proved.[243] (31) Therefore PA has to AN a greater ratio than ΘA to AX; (32) which indeed is impossible.[244] (33) Therefore ZA is neither greater nor smaller than the circumference of the circle ΘHK; (34) therefore <it is> equal.



I print the points N and P close to each other on the circle. In both codices A and C, the points coalesced (I don't believe this was authorial). In codex A, point H then became positioned on the line ZP between Z and P, while N and P were positioned on the same point. In codex C, all three points N, P and H were positioned on the same point. Furthermore, codex C made the line ZH into a tangent of the circle, cutting the line MΘ between Θ and Π. Codices BDG have corrected the diagram, with BG pushing P slightly away from H (as I suspect was in the original) and D pushing it into the bisection of HΘ (as Heiberg eventually did as well). The diagrams thus lost the separate line AH; I do not dare introduce it without manuscript authority (D drew it, BG did not). Codex A had Z well to the left of M; codex C had it slightly to its right.

C has X for K and may have lost Λ (the text is difficult to read at this part of the page; I am now sure, however, it did not lose H). Codex A had the label N squeezed into the space between the spiral and the circle, above the line AΠ. To fit it better, it was rotated counter-clockwise and so was mistaken for a Z and so copied by EH4, corrected by BDG. Codex A had Ξ for Z, corrected by BDG. The label Λ is hidden within B's gutter.

[238] Operating on Step 20 with *Elements* V.16. We also switch ΘA, PΘ (radii in a circle).

[239] This is not a trivial claim (it is analogous to Step 7 in the previous case, where, however, one needed to compare a chord with its arc; now we make a far more difficult comparison "outside" the circle). Heiberg suggests drawing another tangent from the point Π, on the other side of ΠA from Θ. By symmetry, the two tangents are equal, as are the circumferences associated with them. However, from the axiomatic apparatus of ΣΧ we can show that the two tangents together are greater than the two circumferences together, so that each tangent such as ΘΠ is greater than each circumference such as ΘP.

[240] By construction; Step d. Step 24 follows from Steps 25–26 via *Elements* V.8.

[241] Heiberg's Greek (not his Latin translation) has a very rare misprint: ΠP for NP.

[242] First we invert to AP:NP::(circle ΘHK):(circumference ΘP). Subtracting – and extending the operation into inequalities (*Elements* V.16, 19) – we have: AP:NA>(circle ΘHK):(circumference ΘKP). Notice that circumference ΘKP is understood as the complement to the circumference ΘP.

[243] Proposition 14.

[244] By now the structure of the argument is familiar: ΘA=PA (radii), so that Step 31 translates (via *Elements* V.8) to: AX>AN, but from the construction, diagram and proposition 16 it follows that this cannot be.

COMMENTS

Now that's some sleight of hand. How come we even got to consider the circumference of the circle? What, in all we've accomplished so far in this treatise, or even in this proposition itself, establishes the value of π? And how does the spiral even come in – why does it even matter that the line that measures the circumference of the circle is derived from a spiral in the particular way it is?

One thing, as usual, is clear: Archimedes makes no effort to answer such questions. To be fair, he is committed to a method of proof by contradiction, and this typically does not provide this kind of answer. We do not see *why* the lines are to be equal; merely that their inequality breeds an absurdity. Worse, the particular line of thought adopted here does not at all allow an easy transformation of the negativity of the proof by contradiction into any positive statement. If we look at the extreme case where the assumptions of the contradiction no longer hold and the lines are just equal, and we try to figure out what holds true in such a scenario, we find ourselves staring into a void: for the constructions allowed by propositions 7–8, which form the scaffolds for the proof, no longer apply, so that, in the extreme case where the lines are equal, the very construction required by the proof becomes impossible. Not only proposition 18 but already propositions 7–8 are designed around a proof by contradiction and cannot be adapted to a positive claim.

I need first of all, then, simply to explain Archimedes' reasoning in this proposition and then to try to account for its origins.

We are looking (in the first part, which I will pick as my example) for a contradiction arising from AZ>circumference of the circle. Well, if it is bigger than that circumference, we can find a smaller line AΛ equal to the circumference! (Not really, of course, since actually finding such an AΛ is precisely squaring the circle; but this is immaterial since anyway we are engaged in the counterfactual of a proof by contradiction, so that AΛ cannot be found, in fact, in a deeper sense.) So we have:

$$\text{ZA} > \text{AL (Step b)}$$

An inequality is a good thing to have. Through the technique of proposition 7, we may find, based on this inequality, a line NP satisfying

$$\text{NP} : \Theta\text{P} :: \Theta\text{A} : \text{A}\Lambda \quad \text{(Step c)}$$

or

$$\text{NP} : \Theta\text{A} :: \Theta\text{P} : \text{A}\Lambda$$

But it is not difficult to see that

$$\Theta\text{P} : \text{A}\Lambda < (\text{circumf.}\Theta\text{P}) : (\text{circumf. circle}) \quad \text{(Step 7)}$$

so that we have

$$\text{NP} : \Theta\text{A} < (\text{circumf.}\Theta\text{P}) : (\text{circumf. circle}) \quad \text{(Step 10)}$$

or, through an extension of *Elements* V.18,

$$\text{NA} : \text{AP} < (\text{circumf.}\Theta\text{P}) + (\text{circumf.circle}) : (\text{circumf. circle}) \text{ (Step 11)}$$

But from proposition 15 we also have

$$(\text{circumf.}\Theta\text{P}) + (\text{circumf. circle}) : (\text{circumf. circle}) :: \text{XA} : \text{A}\Theta \text{ (Step 12)}$$

so that we have

$$\text{NA} : \text{AP} < \text{XA} : \text{A}\Theta \text{ (Step 14)}$$

which quickly becomes

$$\text{NA} < \text{XA}$$

Which is absurd (via proposition 13). We see no reason why one should have believed such a proposition in the absence of its proof. It depends, in particular, on proposition 7; and at this point it becomes clear that no one would have sought proposition 7, either, unless one already believed in the truth of proposition 18.

Indeed, we are now in a position, finally, to understand Archimedes' willingness to rely on a neusis, in proposition 7. As pointed out by Knorr, the neusis could have been avoided, had Archimedes sought, in Step c of this proposition 18, to rely not on an equality but on an inequality, which (we now see) is clearly what he required. As I pointed out above, the proof envisaged by Knorr would have been architectonically wrong at that early position of the treatise.

But let us now see in greater detail why a neusis suffices, after all.

The appeal to Step b follows, directly, in the construction upon Step b. Here are two consecutive geometrical stipulations, and I quote them in order:

(b) So, I took a certain line, ΛA, smaller than the line ZA but greater than the circumference of the circle ΘHK . . . (c) Now, it is possible to extend <a line> from A towards the produced <line> AN, so that the <line> between the circumference and the produced <line>, NP, has to ΘP the same ratio which ΘA <has> to AΛ.

The second, we now see, is ultimately based on a neusis: on the realist assumption that a line equal to a straight line can be fitted at a certain position.

But Archimedes does not consider the second on its own; the line AN is considered together with the line ΛA, the two conjured into being so as to allow the comparison upon which Archimedes will build his absurdity. And we immediately notice: the first is actually worse than the second. The specification of the line ΛA asks not that we fit a line equal to a straight line at a certain position; no, it asks we that we fit a line equal to a *curved* line at a certain position. If Step b asks us to assume, in realist fashion, the squaring of the circle, why then can't Step c ask us to assume, in the same realist fashion, the permissibility of a neusis?

To understand proposition 7, we find, we need to understand proposition 18. But the question remains: how to understand proposition 18 itself?

I repeat the difficulty we face. We saw above how Archimedes proved (a part of) proposition 18; but, to prove it, he had to have had some reason to go down that path. This is a very familiar type of puzzle in the historiography of ancient mathematics, where very often it is hard to "read back," from the proof, just why it was formulated to begin with, and what the first principles were from which it could be derived (modern calculus was created to a large extent by practicing mathematicians trying to resolve such puzzles in their reading of Greek geometry: they sought some "first principles" underlying all those various puzzles). And this particular instance is one of the most celebrated of all such puzzles. For indeed the prize is big – a squaring of the circle, and the proof is remarkably frustrating to a reader motivated to find its principles, for it is simultaneously very *simple* and very *opaque*. Many past scholars have tried their hand, including of course Knorr[245] and Dijksterhuis.[246] Heath provides an entire appendix to his *History of Greek Mathematics*, dedicated to this puzzle.[247]

One is naturally led to think – especially if one is modern – of what *nearly* happens. What would happen if we let the line ΘZ slide till it almost coincides with ΘΛ, if we let Θ and H be so near each other that the circumference ΘH becomes nearly indistinguishable from the chord ΘH, if we let the perpendicular to the chord ΘH be nearly indistinguishable from the radius AP? Such is the approach taken by Dijksterhuis. In truth, Dijksterhuis – his usual sober self – does not really claim that his account was by any means Archimedes': he is merely claiming to provide a way for a modern to grasp the truth of Archimedes' claim. Let us review this quickly, translating Dijksterhuis' language of limits into a more geometrical language, and using Archimedes' second diagram (Dijksterhuis also helpfully labels the point where the perpendicular bisects the chord HΘ by the letter E).[248]

As the points become very close, we have:

(1) PE:PΘ::AΘ:AZ (near similarity of triangles), and also

(2) PE:PΘ::XE:(circumference PΘ) (XE becomes nearly the same as PE, PΔ becomes nearly the same as circumference PΘ, both by virtue of ZΘ being a tangent).

But note carefully: XE is the progress the generating line makes as it moves along the circumference PΘ from the position AX to the position AΘ, and so the ratio XE:(circumference PΘ) is simply the ratio of the progress of the generating line to the circumference it traverses during its progress, a ratio which is a constant for whichever stage we take in the growth of the spiral. We might as well take the entire spiral, so that:

245 Knorr 1986: 164–165. 246 Dijksterhuis 1987: 271.

247 Heath 1921: II, 556–561.

248 Dijksterhuis' diagram also applies to the general case where the tangent touches the spiral at an arbitrary point; I revert to Archimedes' treatment of the case of a single rotation.

(3) XE:(circumference PΘ)::AΘ:(circle ΘHK)

Joining (1), (2) and (3) we have now established:

(4) AΘ:AZ::AΘ:(circle ΘHK) or indeed:

(5) AZ=circle ΘHK

This is quite elegant, but it is not clear that anything such as (1) or (2) would even cross Archimedes' mind.

I will not reprise Knorr's and Heath's arguments. They both provide a more elaborate geometrical construction than that of Dijksterhuis, and in their treatments the language of limits is postponed until a stage in the argument in which they make an observation of the character that "if the line is slightly moved it becomes intuitively apparent that it will (impossibly) cross such and such a line." Both reconstructions are a geometrically acceptable, intuitive way of stating a claim equivalent to limits. But there is a fundamental question here as to which puzzle we are even trying to resolve. Is it (a) given the basic arrangement of the diagram of this proposition, how do we find a straight line along AΛ such that it equals the circumference of the circle? Or (b) given a spiral, why would one start looking for a proof that AΛ=circumference? The elaborate construction of Knorr is designed really as an answer to (a), but surely it makes more historical sense to imagine that Archimedes had, already, some inkling of the truth of the result to even bother trying such a proof.[249] Heath does provide an answer to (b), and I am persuaded that, its anachronisms removed, it must be along the right lines. This then is what I now concentrate on (Heath 1921: II, 557): "He must have considered the instantaneous direction of the motion of the point P describing the spiral, using for this purpose the parallelogram of velocities." Kinematic thinking certainly seems to be relevant. Let us think of it as a race: one point – call it *circular point* or simply Θ – begins running around along the circumference of a circle – call it ΘHK. Meanwhile, another point – call it *linear point* or simply A – begins running along a line – call it AΘ. The property of the spiral is that the two points end up meeting at exactly the point Θ: the point Θ completed its rotation even as the point A reached all the way up to the circumference of the circle.

Now let us re-imagine the race. The linear point is still linear and pursues the same path. The circular point, however, is now positioned at a certain point Ψ, along the line perpendicular to AΘ, and such that AΨ is equal to the circumference of the circle ΘHK. They both keep their speeds, and now, once again, they conclude their motions at the same instant: the lengths, as well as the speeds, are, after all, the same. For the circular point to cover the linear length AΨ takes exactly as much time as it took it to complete a rotation along the circumference of the circle: for AΨ is equal to the circumference.

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[249] To be fair, it may be that Knorr would admit as much, and that his reconstruction is designed to answer the very narrow question of why, assuming Archimedes was looking for just this proof, he found it in precisely the way he did. I will return to discuss this question in the comments on proposition 20 below.

We now move on to consider what Heath calls "the parallelogram of velocities" (of which, of course, already the pseudo-Aristotelian Mechanics are aware).[250] Let us imagine that there is but a single moving object that operates in two directions simultaneously: the object moves from Ψ towards A at the speed of the original circular motion; the object also moves from the same point Ψ orthogonally upwards, in the speed of the original linear motion. Well, clearly the two motions would compose to a single motion along the line ΨΘ: we end at the point Θ. So this is a manner of composing two motions – one of a "circular" character, one of a "linear" character – and this manner of composing the two motions ends up at the ending point of the spiral line.

But the spiral line, too, is some kind of composition of the same two motions. And so we begin to wonder how the line ΨΘ and the spiral line interrelate. And at this stage of our thinking it begins to look as if it is not *un*reasonable that ΨΘ should be tangent to the spiral. This is somehow a fair compromise. Had the line ΨΘ passed here inside the spiral instead, it would somehow suggest that its tendency, at this point, would be "more circular than linear." Had it passed here through the spiral on its way out of it (presumably having entered the spiral before), it would somehow suggest that its tendency, at this point, would be "more linear than circular." And if so, Archimedes might well begin to ponder the question whether a tangent to the spiral cuts a line from the perpendicular to the generator of the spiral, which is equal to a circumference of a circle.

However, such words as "tendency, at this point being more circular than linear" etc. are not much different from Heath's words "[considering] the instantaneous direction of the motion of the point Π describing the spiral" – painfully anachronistic words. The difference, in part, is that I do not require Archimedes to have more than intuitions and suspicions. But there is a deeper question. The issue of anachronism is not just a matter of a wrong terminology: it is a matter of a wrong, inappropriate manner of thinking. It is just difficult to imagine Archimedes looking at a line and starting to wonder what the tendency of its tangent might be at a certain point: there was no tradition of proving theorems based on such considerations and no natural way for this problem to present itself to one's mind. Being less anachronistic, verbally, does not really help us.

Let me then try to retrace our steps. In truth, we should start thinking about this at a much more basic level – start thinking about the historical conception

[250] A "proof" of the parallelogram of forces is offered in [ps.-?]Aristotle's *Mechanics* 848b13–23. In a sense, of course, the composition of motions is always at the heart of the operation of *On Spiral Lines*, and, to the extent that we take any reports on Eudoxus seriously, it would have to be essential to Greek geometrical astronomy already by the fourth century (in the much more complicated problem of composing at least two three-dimensional circular motions along spheres. The literature on Eudoxus' astronomical geometry is large, and the evidence is best accessed through Henry Mendell's translation with commentary, on his website: www.calstatela.edu/faculty/hmendel/Ancient%20Mathematics/Philosophical%20Texts/Astronomy/Simplicius%20InDeCael.pdf).

of the spiral. And now it becomes clear that if a Greek mathematician begins to conceive of a line composed of two motions, at fixed speeds, one linear and one circular, he does not do this to prove that, say, the area covered by such a line is such and such or that it has some other properties inherent in it. One does not invent objects for no reason and then study them in abstraction. No: there ought to have been a motivation anterior to the spiral itself. You conceive of such a line for a purpose, and the purpose surely would have been to apply this spiral, with its composite motions, to obtain results concerning the circle (and, ideally, to *square* the circle). And so one must have conceived the composite motion of the spiral to begin with, so as to allow some kind of transformation of the circumference of the circle into a straight line. The most natural way to achieve this is by the following consideration: motions of a fixed speed and a given duration cover the same length, *whether they are linear or circular* (as can be seen from proposition 2, one of the cornerstones of this treatise). And if so, we could square the circle – linearize its circumference – by considering such a circumference as the result of a motion which is equivalent, in speed and duration, to the motion along a straight line. And so the spiral line is conceived for the sake of the triangle of the composition of motions AΘΨ, and one would be intent on looking actively for the position of Ψ along the line AΛ. And that this position is obtained by the tangent would then come as at least one plausible option among many. Indeed, looking at this particular question, one could easily be led to think, locally for this problem (even if such a procedure did not become established as a general mathematical technique), of the property of the tangent as expressing a local composition of motions.

This kind of thinking even has a very obvious parallel elsewhere in Archimedes. I ascribe here to Archimedes the thought that motions of a fixed speed and a given duration cover the same length, *whether they are linear or circular*. Thus one is allowed to move from straight to curved kinematic statements, so that a composition of motions obtained with straight lines can be assumed to be the same as that of curved lines, and a single point would serve the same function with both compositions. In *The Method*, Archimedes explicitly proposed a claim such that if all components of two figures, pair-wise, balance at a point, so would the entire figures, *whether two-dimensional* or *three-dimensional* (or one-dimensional or two-dimensional). Thus one would be allowed to move from two-dimensional to three-dimensional statements of balance, and a single point would serve the same function – balance at a point – for both arrangements (all particular pairs as well as entire figures). In both cases – *On Spirals* as well as *The Method* – the fundamental intuition is physical: why should it matter, as we move things, whether they move along straight or curved lines? Why should it matter, as we balance things, whether we pair them pair-wise or as a whole set?

There is even a potential that Archimedes conceived of such a parallel himself: for the one major group of objects studied throughout his career, which he *avoids* mentioning in *The Method*, are the segments of the spiral line. Could it be because he conceived of the segments of the spiral line as essentially distinct, in their geometrical development, from other figures? Effectively, I suggest that, throughout his major work, Archimedes conceived of his geometrical objects as possessing certain physical affinities. With most other figures, this affinity was with *statics* (geometrical objects possessed, in their physical guise, a center of the weight). With *On Spirals*, this affinity was with *kinematics* (spiral lines, in their physical guise, represented a certain composition of motions). The physical affinity becomes quite pronounced in *On Spirals*: the discussion does have stages which are explicitly kinematic. However, there is no explicit suggestion that the geometrical tangent has a kinematic analogue in the composition of motions at a point, and, in this sense, *On Spirals* does not go as far as *The Method*.

It should be emphasized that Archimedes, in a clear sense, did not square the circle: no construction is offered with which the circle may be squared. It is no less important to note that Archimedes, himself, did *not* emphasize this point. This can be seen, with hindsight, in propositions 16–17. We now note that they make, in fact, a much weaker claim than is mathematically justified. They assert that the angle of the tangent to the line drawn from the start of the spiral is always acute. This, I would say, is no less than misleading. Natural language pragmatics – not to mention the assumptions of a Greek mathematician making as radical a claim as he could – all suggest that if one asserts that a certain angle is acute, nothing more precise can be known about it. The implication, then, is that the angle may change from spiral to spiral (perhaps within the same spiral?) as long as it remains acute.

In fact, the angle of the tangent to the spiral such as ZΘA is a fixed angle, the one of a right-angled triangle such that its two sides AΘ, AZ are in the ratio of a radius to the circumference of a circle (we may say, nearly 81 degrees). Indeed, this is not merely an interesting observation but one that settles the fundamental character of the spiral: we end up showing that all spirals are similar or that really there is only one spiral, not many – that their only difference would be that of scale. Spirals (in the manner defined by Archimedes) are not like ellipses, that can be made more or less elongated – that have a certain parameter according to which they change their form. They are like circles: they always have exactly the same form. And they are also like circles in that they cannot be squared: that is, their defining angle in fact cannot be named. This of course is the reason Archimedes does not dwell on the property of the fixed angle of the spiral. He would not be able to mention it without highlighting the fact that he cannot name this angle or, more generally, solve the problem of finding a tangent to a spiral at a given point (obviously: for this problem is exactly equivalent to the squaring of the circle). Archimedes' silence on this point is magnified by a misleadingly weak claim in propositions 16–17, all serving to present proposition 18 in the best possible light – a tangent that, incredibly, squares the circle (as your mind's eye turns towards the question of the tangent's constructability, there's a flash from a small shiny object).

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And if a line should touch the spiral drawn in the second rotation at its end, and a certain <line> should be drawn from the start of the spiral at right <angles> to the start of the rotation, that <line> shall meet the tangent, and the line between the tangent and the start of the spiral shall be twice the circumference of the second circle.

For let there be the spiral ΑΒΓΘ, drawn in the first rotation, and ΘΕΤ, in the second; and <let> the circle ΘΚΗ be the first <circle> and the <circle> ΤΜΝ, the second <circle>. Let there be a certain line touching the spiral at Θ, <viz.> ΤΖ, and let ΖΑ be drawn at right <angles> to ΤΑ. (1) So, the <line> itself shall meet ΤΖ, (2) on account of its being proved that the angle <contained> by ΑΤ, ΤΖ is acute. It is to be proved that the line ΖΑ is twice the circumference of the circle ΤΜΝ.

(3) For if it is not twice, it is either greater than twice or smaller than twice. (a) Let it first be, if possible, greater than twice. (b) And let a certain line, ΛΑ, be taken, smaller than the line ΖΑ but greater than twice the circumference of the circle ΤΜΝ.[251] (4) So, there is a certain circle, ΤΜΝ, and a given line in the circle smaller than the diameter, <viz.> ΤΝ,[252] and <the ratio> which ΤΑ has to ΑΛ, greater than the <ratio> which the half of ΤΝ has to the perpendicular drawn on it from Α.[253] (c) Now, it is possible to extend ΑΣ from Α towards ΤΝ, produced, so that the <line> between the circumference and the produced <line>, <viz.> ΡΣ, has to ΤΡ the same ratio which ΤΑ <has> to ΑΛ.[254] (d) So, ΑΣ shall cut the circle at Ρ,[255] (e) and the spiral at Χ; (5) and alternately: ΡΣ shall have the same ratio to ΤΑ, which ΤΡ <has> to ΑΛ.[256] (6) And ΤΡ has to ΑΛ a smaller ratio than the circumference ΤΡ to the double of the circumference of the circle ΤΜΝ; (7) for the line ΤΡ is smaller than the circumference ΤΡ[257] (8) while the line ΑΛ <is> greater than the double of the circumference of the circle ΤΜΝ;[258] (9) therefore ΡΣ has a smaller ratio to ΑΡ than the circumference ΤΡ has to the double of the circumference of the circle ΤΜΝ.[259] (10) Now, ΣΑ, in its entirety, has to ΑΡ a smaller ratio than the circumference ΤΡ together with the circumference of the circle ΤΜΝ, counted twice, to the circumference of the circle ΤΜΝ,

[251] Proposition 4. The order of the letters ΤΜΝ is curious.

[252] *Elements* III.7, proposition 13.

[253] *Elements* III.3, V.8, VI.8.

[254] Proposition 7.

[255] The point Ρ is defined only post factum, having been used already in its defined sense in Step c.

[256] *Elements* V.16.

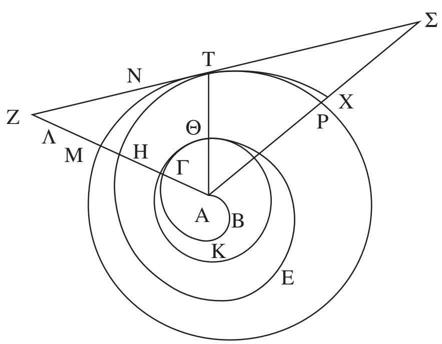
[257] See n. 226 above to proposition 18, Step 8.

[258] Step a: hypothesis. Step 6 follows on the basis of *Elements* V.8.

[259] Between Steps 6,9, line ΤΑ has become ΑΡ (radii in a circle).

counted twice.[260] (11) But the ratio which the mentioned circumferences have <to each other> – XA has that ratio to AT. (12) For this has been proved.[261] (13) Therefore AΣ has to AP a smaller ratio than XA to TA; (14) which indeed is impossible.[262] (15) Therefore the line ZA is not greater than twice the circumference of the circle TMN. (16) And similarly, it shall be proved that neither is it smaller than twice. (17) Now, it is clear that it is twice.

It is to be proved through the same manner: also,[263] if a certain line should touch the spiral, drawn in whichever rotation, at the end of the spiral, and <the line> drawn from the start of the spiral in right angles to the start of the rotation meets the tangent, it is a multiple of the circumference of the circle, counted according to the number of the rotation, in the same number.



Codex C had the line ΣZ pass lower so it "misses" T from beneath. Codex A, which has it pass through A, ends up with the label N attached to the line segment ZT instead of the arc TM, which is the "mistake" I print, as I can well imagine how this is authorial (the reference of N remains clear enough; this mistake is corrected by codices BD, who also position Z at the level of A, followed by Heiberg). H is positioned misleadingly close to the second circle, K – misleadingly close to the spiral (both should lie on the first circle; both corrected by BD; Heiberg, remarkably, misplaced K between the spiral and the *second* circle). Codex A (perhaps also C) had Ξ instead of Z, corrected by BDG. C also might have had Σ instead of E and might have lost P. Λ, Z are lost within B's gutter.

COMMENTS

The explicit proof in this proposition (Steps 1–15) is exactly identical with the first part of the proof in the preceding proposition 18 (Steps 1–18). This gives rise to two questions, one semiotic and the other mathematical. First, how is this identity conveyed? Second – a question with which we are familiar by now – why does Archimedes bother to repeat the very same proof twice?

For the semiotic question: the first thing we note is that Archimedes never says that the two proofs are identical. He does comment in proposition 19, Step 16 that it shall be proved "similarly" that the line is not smaller than twice the circle; but this reads in context not as the statement that this should be proved "similarly" to the corresponding passage in proposition 18 but rather

[260] Extension to inequalities of *Elements* V.18.

[261] Proposition 15 ("corollary").

[262] *Elements* V.8 and the usual implied equality of radii AT=AP.

[263] The manuscripts are missing the word "that." Most likely this is a scribal error, but since the text could marginally make sense without the word, I try translating without it.

that this should be proved "similarly" to the first part of the proof of proposition 19.

It is also noticeable that Archimedes does not truncate the proof of proposition 19 appreciably. Proposition 18 had three steps not present in proposition 19: Step 5 – explaining why the ratio is smaller than the required limit – and Steps 16–17 – explaining why the final result is absurd. Both are very brief backwards-looking γαρ-justifications. Otherwise the flow of the proof is precisely the same, no step skipped. Indeed, it is possible to find moments where Archimedes is more explicit in the later proof. Step 5 in proposition 19, unlike its counterpart Step 6 in proposition 18, explicitly refers to its operation as "alternately," thus signaling its logical ground. Even more striking: in proposition 19 Archimedes spells out (if post factum) the reference of the labels P, X; these are completely unspecified in proposition 18.

I do not think this represents any conscious choice on the part of Archimedes to be more expansive in the latter proposition. Rather, it appears that he made no effort to make the proofs textually identical, instead generating independently the precise wording of the individual claims. Consider the variability of the wording between proposition 18, Step 10 and proposition 19, Step 9:

18, Step 10	19, Step 9
Now, NP, too, shall have to PA a smaller ratio than the circumference ΘP to the circumference of the circle ΘHK.	Therefore PΣ has a smaller ratio to AP than the circumference TP has to the double of the circumference of the circle TMN.

I underline three textual segments in proposition 18: "Now" as against "Therefore" (οὖν / ἄρα); too, absent from proposition 19;[264] "shall have" and not "has." I underline one textual segment in proposition 19: "has" in the second part of the proportion statement (elided in the version of proposition 18).

This elided "has" may hold the key to the textual practice we see here. It is the kind of expression we automatically supply in our head when it is absent, so that making it overt, or failing to do so, is, up to a certain point, not a meaningful choice, perhaps not a conscious choice at all. Perhaps, in a similar fashion, the marker "alternately" may or may not be present, and the statement still could remain the same. To some extent, Archimedes might have felt the same at an even higher level of textual organization: perhaps he felt making explicit the reference of labels, or even asserting a backwards-looking γάρ-justification, are mere variations on the *same* expression. To use a somewhat archaic expression: perhaps such differences were, to Archimedes, mere *surface structure*, the *deep structure* remaining the same.

If so, we end up concluding that the proof of proposition 19 is meant to be exactly identical with that of proposition 18.

[264] The word "too" forces a complete make-over of the word order in English, but this is purely a translation artifact: the position of "a small ratio" is the same in the two versions in the original Greek.

It could be argued that I'm making the wrong comparisons: I looked at exactly those passages where the claims of proposition 19 do not differ, mathematically, from those of proposition 18. But what are those mathematical differences I ignored? Well, the diagram and the construction are different, allowing for a second rotation of the figure. Then, in Step b, line ΛA is assumed to be greater than *twice* the circumference (and not greater than the circumference itself, as in proposition 18). Finally, in Step 11, XA:AT is the same as the ratio of an arc plus two rotations, to two rotations, whereas in proposition 18 it was the same as the ratio of an arc plus a single rotation, to a single rotation. The claim of proposition 18 is based on proposition 15; the claim of proposition 19 is based on the corollary to proposition 15. (But this difference is elided in the texts of propositions 18–19, which merely assert that "this has been proved.") This is all there is: other than those differences, the two proofs are identical.

What becomes clear at this point is that wherever one draws the tangent, the construction of proposition 7 would allow one to generate an absurdity with the assumption that the line intercepted by the tangent is greater than the spiral rotated as far as the point of tangency; for the result of proposition 15 does not determine the position on the rotation of the spiral where the ratios are to be evaluated: there is no singularity associated with the complete rotation. At any rate, the extension to the case of an interim point, not associated with a complete rotation, involves much less mathematical effort than the extension of the claim of the first part of proposition 19 – applying proposition 7 – to the claim made in Step 16, which would require the application of proposition 8.

That Archimedes, then, would move on to provide yet a third identical explicit proof of exactly the same result, now applying to the interim points, is rather shocking.

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If a straight line should touch the spiral drawn in the first rotation, not at the end of the spiral, and a line should be joined from the touching point to the start of the spiral, and a circle should be drawn with the start of the spiral as center, and the joined <line> as radius, and a certain <line> should be drawn from the start of the spiral at right <angles> to the <line> joined from the touching point to the start of the spiral, that <line> shall meet the tangent, and the <line> between both the point at which it falls and the start of the spiral shall be equal to the circumference of the drawn circle – that between the touching point and the section, at which the drawn circle cuts the start of the rotation, the circumference taken towards the preceding <parts> from the point which is in the start of the rotation.[265]

[265] Archimedes takes enormous pains to specify precisely the circumference KMNΔ; see the general comments.

Let there be a spiral on which <is> the <line>[266] ΑΒΓΔ, drawn in the first rotation, and let a certain line, EZ, touch it at Δ, and let AΔ be joined from Δ to the start of the spiral, and let a circle, ΔMN, be drawn with A as center and AΔ as radius, and let that <circle> cut the start of the rotation at K, and let ZA be drawn perpendicular to AΔ. (1) Now, it is clear that that <line> shall fall <on it>.[267] But it is to be proved that the line ZA is also equal to the circumference KMNΔ.

(2) For, if not, it is either greater or smaller. (a) Let it be, if possible, first, greater, (b) and let a certain line be taken, AΛ, smaller than the line ZA and greater than the circumference KMNΔ. (3) So, again, there is a circle, the <circle> KMN, and a line in the circle smaller than the diameter, ΔN,[268] and a ratio, which ΔA has to AΛ, greater than the <ratio> which the half of ΔN has to the perpendicular drawn on it from A.[269] (c) Now, it is possible to extend AE from A towards NΔ, produced, so that EP has to ΔP the same ratio which ΔA <has> to AΛ. (4) For this has been proved to be possible.[270] (5) Now, EP shall also have to AP the same ratio which ΔP has to AΛ.[271] (6) But ΔP has to AΛ a smaller ratio than the circumference ΔP <has> to the circumference KMΔ, (7) because ΔP is smaller than the circumference ΔP,[272] (8) while AΛ is greater than the circumference KMΔ.[273] (9) Now, the line EP has a smaller ratio to PA than the circumference ΔP <has> to the circumference KMΔ; (10) so that AE, too, has to AP a smaller ratio than the circumference KMP to the circumference KMΔ.[274] (11) But the ratio which the <circumference> KMP has to the circumference

[266] The Greek has the feminine form of the article "the," most naturally implying "line." The same formula in proposition 16 used the neuter plural, for the more natural "points"; the feminine here probably reflects the feminine noun "spiral" itself, so that the precise translation might need to be "the spiral on which the <spiral> ΑΒΓΔ." This uneasy expression stems from the uneasy position of the figure of the spiral, studied here extensively perhaps for the first time in the history of Greek mathematics and not yet settled into its formulaic expressions. What appears to be meant is that we envisage the potentially infinite spiral extending all the way from its start outwards; and we pick out the segment associated with a line=spiral such as (in this case) ΑΒΓΔ.

[267] I.e. that ZA shall cut ΔA. Exactly as in the previous two propositions, this follows from the acute angle at AΔZ, which in this case derives from the basic statement of this result in proposition 16.

[268] *Elements* III.7, proposition 13. [269] *Elements* III.3, V.8, VI.8.

[270] Proposition 7.

[271] Implicitly alternating the proportion of Step c (*Elements* V.16) and then using the equality of radii ΔA=AP.

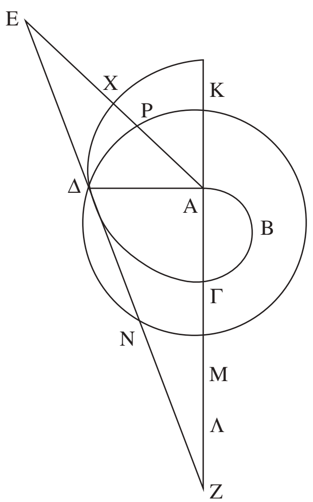
[272] *SC*? Visual intuition?

[273] The hypothesis of Step b. Step 6 following through *Elements* V.8.

[274] An extension to inequalities of *Elements* V.18.

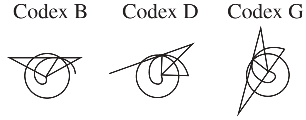
KMΔ – that <ratio> XA has to AΔ;[275] (12) therefore EA has to AP a smaller ratio than AX to ΔA; (13) which indeed is impossible.[276] (14) Therefore ZA is not greater than the circumference KMΔ. (15) Similarly to the above it shall be proved that neither is it smaller. (16) Therefore <it is> equal.

And it shall be proved through the same manner also that if a line should touch the spiral drawn in the second rotation, not at the end of the spiral, and <if> the rest should be constructed the same, the line that meets the tangent, between both the point at which the <line> falls on the tangent and the start of the spiral, is equal to the circumference, in its entirety, of the drawn circle, and yet with the <circumference> between the mentioned points, the circumference taken in the same way; also, that if a certain line should touch the spiral drawn in whichever rotation, not at the end of the spiral, and <if> the rest should be constructed the same, the line between the mentioned points is a multiple of the circumference of the drawn circle, by a number smaller by one than the <number> by which the rotations are counted, and yet equal to the <line> similarly taken between the mentioned points.



The diagram breaks sharply from the horizontal standard; in codex C, the lower point cuts well into the text. It is variously rotated to become more horizontal by BDG (see thumbnails) as well as by Heiberg. Codex C has the point Δ slightly lower than A. The line ΔP is missing (reinstated by BD), a conceivably authorial omission.

Codex B Codex D Codex G



[275] Proposition 14. Once again, Archimedes reverts to the original statement of his results (just as the acute angle required by Step 1 is based on the original statement of proposition 16).

[276] *Elements* V.8 and the usual implied equality of radii.

COMMENTS

Here is the same proof, yet a third time, not even substantially abbreviated. There is one very minor acknowledgment of repetition ("again," in Step 3), and it is also more clearly marked, in Step 15, that the extension of the proof technique to the second case is based not on the first case, but rather on the second case of proposition 18. But there is no strategic statement to the effect that this proof is indeed an exact replica of previous proofs.

There is some variety in the regime of internal cross-references, i.e. the manner in which Archimedes explicitly appeals, or fails to do so, to previous results inside *On Spirals*: in proposition 20, finally, we merely assert (instead of appealing to proposition 13) that the line falls upon the tangent. But proposition 20, all of a sudden, decides to make the appeal to proposition 7 explicit (in Step 4) instead of just suggesting it via its formulaic language – while at the same time proposition 20 drops the explicit reference to proposition 14 (in Step 11). Other than this variability – and of course the "surface structure" variability of choices of mathematical expression – the proof is exactly the same as that of the preceding two propositions (taking just the first case of proposition 18).

What is Archimedes doing? Is it even Archimedes doing it? One attractive option would be to imagine some kind of editorial intervention, expanding a single proof by Archimedes, for a single case, to a series of proofs for various cases. But if so, it is hard to explain the precise arbitrary distribution: why provide the first part only in propositions 19–20? Why have such a degree of variability in the "surface structure"? Why, indeed, have this very clearly stated generalization at the end of proposition 20? Finally, no section seems to stand on its own: it is difficult to see how proposition 18, out of the entire series, could have been the only proposition proved: would Archimedes not have gestured towards a generalization (the way our text does at the end of proposition 20)? Such judgements are always subjective, but the text of propositions 18–20 has the variety, and confidence, of an authorial voice.

If authorial, we need to understand Archimedes' motivation. There ought to be something which he felt that an explicit recapitulation did, and which a mere statement of extendibility would not. Archimedes felt that he needed to spell out how to repeat the proofs for the cases of 19–20; he could not just *state* that they were repeatable (in the manner in which he states this at the end of proposition 20, for further extensions).

Let us then consider the options available to Archimedes. As we saw already for propositions 16–17, he could have made one sweeping, general claim which does not ask for repeatability, and simply applies to all cases directly; or he could have pursued one single case and then claimed its repeatability.

How could Archimedes provide one sweeping, general claim? This, on the face of it, does not appear to be so difficult. All we need to do is to pick any of the diagrams of propositions 18–20 and in the text refer to the spiral in general terms and the point at which the tangent touches the spiral – as a "random" one. Say, with the diagram of proposition 20, the text would ask that a spiral is drawn (with no reference made to its being drawn during its first rotation), and that a tangent touches it at an arbitrary point Δ. The general definition of goal would ask that we prove a claim roughly the same as the generalization at the end of proposition 20 (of which 18 and 19 are, in a sense, special cases). The reductio assumption – and therefore the definition of the point Λ – will be once again open-ended ("let us assume that it is not the multiple-and-then-some that the point Δ represents; so it is either greater or smaller, etc."). Then, in the equivalent of proposition 20, Step 11, the ratio of the generating lines and the circumference will have to be expressed in a general language (which has been achieved already, after all, at the ends of propositions 14–15).

I can see two reasons Archimedes would not like this. First, as suggested above for propositions 16–17, he might have felt uncomfortable with the generalization of a spiral figure. This is important, if we consider that the diagram of one (and the diagram accompanying a proof for spirals must represent a particular spiral) is to be taken to cover them all: for it would always involve a stretch of the imagination to have a multiply coiled spiral representing a single one, or a singly coiled spiral representing a multiple one.[277] He might have felt that even though the proof did apply under such a general perspective, the worry would still lurk that some unsuspected coil along the way could still cut the proof's progress. Clarifying the proof by having, essentially, multiple cases for the multiple diagrams, certainly removed this worry.

Second, and probably key to this, there is an obvious price to pay for a more general statement. Had Archimedes used such language, he would have had to give up the language of proposition 18 – he would have lost the opportunity of dropping a bomb where, out of the growth of the deductive structure of the treatise, suddenly a claim was made that *a certain line is equal to the circumference of the circle*. He could still have left this as an implicit consequence – but this would have been to give up the opportunity to highlight the rectification of the circle. Or he could have asserted it as an explicit corollary – but since the proof, in its generality, would make such special corollaries mathematically otiose, such a special statement would appear much more marked and would make the statement of the rectification less elegant – no longer a bomb dropped but instead a laboriously contrived statement.

Why not have just proposition 18, followed by a longish statement, such as the one at the end of proposition 20, asserting the repeatability of the proof? There are surely drawbacks to that: the longish statement would end up very long, and very opaque. Archimedes has a clunky structure all right, with his repetition of the same proof thrice. But the alternative has its own clunkiness. The main concern appears to be the epistemic one, mentioned above: how

[277] It is interesting that while the text of the proofs differs only trivially, the diagrams are as distinct as they can be. Their fundamental orientation is given by the position of the tangent line which is (in the first case, repeated by all three propositions) variously at the bottom (18), top (19) and left (20) of the spiral. It would be very difficult to judge, based on these three diagrams, that they represent three special cases of the very same proof.

would it be obvious that the proof is repeatable, when the visual layout is so different? Indeed, would there not be a constant, specific worry that the particular case of a single, *full* rotation has a specialness that does not translate to the other cases? Dijksterhuis, sensing this, offers a proof once and then asks us to extend it, but the single case he takes up (1987: 268–270) is that of proposition 20: he clearly feels that the case of a spiral drawn with less than a rotation is more obviously "representative," or at least more difficult (it is easy enough to intuit that if the proof holds for less than one rotation, it will hold for exactly one; but the opposite is slightly less clear). The need to showcase in just the right way the special case of the rectification of the spiral is in tension with the need to pick a representative case, easy to generalize. Faced with this choice, Archimedes prefers to showcase the rectification of the spiral. Indeed, this goes back all the way to the letter to Conon: for there, the claim of propositions 18–20 was stated merely for the special case of the rectification of the circle (indeed, the very word "spiral," in that context, seems to be used to refer to that drawn during the first rotation): see Archimedes' introductory letter, pp. 20–21 above. Deep inside the treatise, Archimedes is still bound by his context of communication.

What Archimedes seems to resist is a mid-way position between the explicit repetition and the implicit suggestion of repeatability. Perhaps, to our sensibility, the most elegant way out would have been to provide a single case (maybe proposition 18, after all, to accommodate Archimedes' need to showcase that particular result), followed by a more discursive meta-deductive statement: that the proof will obtain even if the tangent touches at any other point on the spiral drawn during any rotation, since (and here we part from the very spirit of Archimedes' discourse) one will always be able to find a line according to propositions 7 or 8, and the results of propositions 14–15 will always apply to derive exactly the same impossibility.

What we find, then, is that Archimedes is reduced to a certain inelegance because he definitely has to avoid the kind of meta-deductive statement suggested above. And indeed, even in more expansively meta-theoretical works such as *The Method*, within the text of the proofs themselves Archimedes never resorts to such observations on how results can be obtained. Once again we find that Greek mathematicians prefer example to instruction. Their works are to some extent explanatory in the sense that, pursuing their approach step-by-step, one becomes familiar with the mathematical possibilities and so can learn how to proceed. Following propositions 18–20, it is indeed much easier for me to envisage how to cast the proof for the cases Archimedes omits to prove and instead merely suggests at the end of proposition 20 (or the second case omitted in both of propositions 19 and 20). But these works are not explanatory in the sense that Archimedes ever explains much *in his own words*. And so, sometimes, so as not to be too opaque, one has to pile up examples.

Could there have also been an advantage, a specific gain from the repeated iteration? I have suggested as much in my comments on proposition 17, as I pointed out that, from proposition 10 onwards, the text of *On Spirals* takes up the principle of "dual – and more." Results always repeat each other – and then add a bit more. This, I suggested, could somehow be metaphorically related to the structure of the spiral: it is a discursive spiraling structure answering the spiraling structure of its object. This suggestion can now be given more concrete and less metaphorical meaning.

Archimedes needs to have the repetition of propositions 18–20, I suggest, primarily because of the intricate structure of the spiral, a single object which appears to take many forms in its rotation. There is thus a very simple sense in which the "dual – and more" structure of propositions 18–20 directly stems from the "dual – and more" structure of the spiral. Propositions 18–20 follow the coils (first outwards, 18 expanding to 19; then backwards, to less than a coil, in 20; and then, in the unproved statement, expanding outwards yet again).

What now appears to be the case is that the "dual – and more" structure from proposition 10 onwards was there largely in anticipation of the structure of propositions 18–20: it provides a context against which the structure of propositions 18–20 appears less unexpected and therefore less otiose.

And so we may begin to understand Archimedes' overall architecture: faced with a complex, apparently (or deceptively) variegated object; preferring example to instruction; foregrounding the rectification of the circle – and so forced almost to the iterated, coiling structure of propositions 18–20; an iterated, coiling structure which is then allowed to characterize the entire sequence of propositions from the moment in which the treatise begins to take off in earnest, in proposition 10.

Yet this was just one turn of the spiral. Let us pursue Archimedes to the next coil.

/21/

Taking the area contained by both the spiral drawn in the first rotation and the first line in the start of the rotation, it is possible to circumscribe a plane figure around it and inscribe another, composed of similar sectors, so that the circumscribed is greater than the inscribed by a <magnitude> smaller than any given area.

Let there be a spiral, on which <is> the <line> ΑΒΓΔ, drawn in the first rotation, and let the point Θ be the start of the spiral, ΘΑ <the> start of the rotation, the <circle> ΖΗΙΑ the first circle, and the diameters ΑΗ, ΖΙ its diameters, at right <angles> to each other. (1) So, the right angle ever again being bisected, and the sector containing the right angle, the remainder of the sector[278] shall be smaller than the given;[279](a) and let the sector have come to be, <as> the <sector> ΑΘΚ, smaller than the given area. (b) So, let the four right angles be divided into the angles

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[278] The expression "the remainder of the sector" envisages the process of bisection as one in which we always divide a sector into two, tossing out one half and keeping the other. "The remainder of the sector," then, is simply the last sector obtained in the process of continued bisection.

[279] *Elements* X.1. This is distinct from the lemma ("Archimedes' Axiom") evoked at the end of the introductory letter.

equal to the <angle contained> by ΑΘ, ΘΚ, (c) and let the lines making the angles be drawn as far as the spiral. (d) So, let the point at which ΘΚ cuts the spiral be Λ, (e) and let a circle be drawn with Θ as center and ΘΛ as diameter; (2) its circumference shall fall, towards the preceding circumference, inside the spiral, and towards the following circumference, outside.[280] (f) So, let the circumference [ΟΜ] be drawn, as far as it <extends> to fall on ΘΑ, [at Ο], and <as far as it extends to fall> on the <line> falling on the spiral beyond the line ΘΚ.[281] (g) So, again, let the point at which ΘΜ cuts the spiral be Ν, (h) and let a circle be drawn with Θ as center and ΘΝ as diameter, as far as the circumference of the circle <extends> to fall on ΘΚ and on the <line> falling on the spiral beyond ΘΜ, (h) and, similarly, let circles be drawn through all the other <points> at which the <lines> making the equal angles cut the spiral, with Θ <as> center,[282] as far as each circumference <extends> to fall on the preceding line and on the following; (3) so, there shall be a certain <figure> composed of similar sectors, circumscribed around the taken area, and another inscribed.

And it shall be proved that the circumscribed figure is greater than the inscribed by a <magnitude> smaller than the given area. (4) For the sector ΘΛΟ is equal to the <sector> ΘΜΛ, (5) and the <sector> ΘΝΠ to the <sector> ΘΝΡ, (6) and the <sector> ΘΧΣ to the <sector> ΘΧΤ, (7) and also: each of the other sectors in the inscribed figure is equal to the sector having a common side,[283] among the sectors in the circumscribed figure. (8) Now, it is clear that all the sectors shall be equal to all the sectors;[284] (9) therefore the inscribed figure is equal to

[280] In terms of the diagram: Ο is inside the spiral and Μ – outside.

[281] "The line falling on the spiral beyond the line ΘΚ" is the line continuing from ΘΣΜ to the spiral. Archimedes has to use a periphrastic expression because he did not commit a diagrammatic label to the point at which this line meets the spiral. Step f has a strange redundancy, asserting the identity of the drawn circumference more directly as ΟΜ, and less directly by referring to such periphrastic expressions; for this reason, Heiberg thought the reference to ΟΜ might be a late explanatory scholion. This is possible, but it has to be noted that such scholia are not very apparent in the text of *SL* as a whole.

[282] Since Archimedes does not have a generalized way of referring to the radii of the various circles, he uses a rare construction: circles are drawn through a point, with a center (defined by a center and a point on the circumference, instead of by a center and a radius).

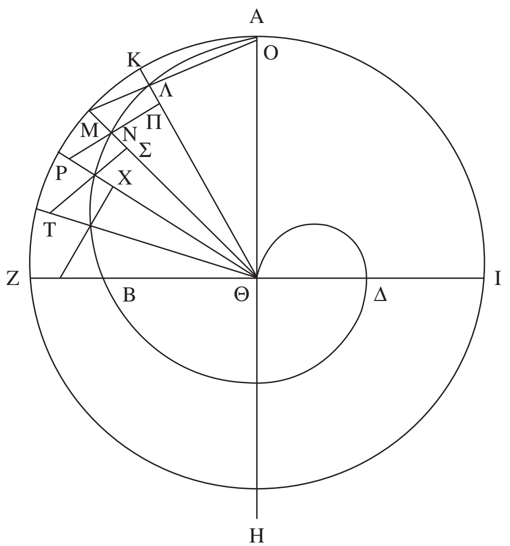
[283] Having a common side: in the sense in which ΘΛΟ, ΘΜΛ have the common side ΘΛ. Having a common side is not only the sign allowing us to identify the pairs which Archimedes wishes to equate, but also provides the grounds for the claim of equality: all sectors are similar (by construction), so that, with a common side, they must also be equal.

[284] This means, I think, that it is clear that the method we have identified of picking pairs of sectors involves no double-counting, so that the two sets of sectors have an equal number of members. It is clear thanks to the procedure implied in the order of Steps 4–6, counter-clockwise, so that there is no movement back and forth and each sector is counted exactly once.

the figure in the area circumscribed around the area without the sector ΘAK; (10) for this alone is not taken among the <sectors> in the circumscribed figure.[285] (11) Now, it is clear that the circumscribed figure is greater than the inscribed, by the sector AKΘ, (12) which is smaller than the given.[286]

/COROLLARY/

And from this it is obvious that it is possible to draw a figure around the said area,[287] as was said, so that the circumscribed figure is greater than the area by a <magnitude> smaller than any given area, and again: to inscribe, so that the area, similarly, is greater than the inscribed figure by a <magnitude> smaller than any given area.[288]



Codex A had the smaller arcs drawn as arcs (faithfully copied by the descendants of A, though those of codex 4 are nearly indistinguishable from straight lines). Codices AC do not have Γ, which BD reinstate (an omission which may be authorial). H has X for A, A for Λ. A is completely hidden in codex C. Codex E copies here the next diagram, and omits entirely the present one (it leaves a blank space where the next diagram should go).

[285] We need to recognize that the last sector area, right near the start of the spiral and on the other side of the start of the rotation from ΘKΛ, contains a sector in the circumscribed figure, but no sector (or, if we wish, a zero sector) in the inscribed figure. Thus the inscribed figure as a whole has one sector fewer than the circumscribed figure as a whole. Archimedes does not clarify this point (which, by definition, calls for zooming in on a minuscule part of the figure and so does not lend itself to diagrammatic treatment).

[286] Step 12, elegantly, reverts to the very first construction of Step a.

[287] I.e. the area contained by the spiral line and the start of the rotation.

[288] Since the magnitude (circumscribed-inscribed) is smaller than the arbitrarily given magnitude, and circumscribed>area>inscribed (this is taken to be visually intuitive), it follows that (circumscribed-area) and (area-inscribed) are each smaller than (circumscribed-inscribed), hence smaller than the arbitrarily given magnitude.

COMMENTS

The argument of this proposition – based on equal, adjacent sectors – is aesthetically pleasing, all the sectors effortlessly sliding and clicking, as we follow the logic of the proof, into position. The visual basis of the argument makes it also directly compelling, even if the verification of the precise claim remains taxing: it is easy to see how in principle the difference between the two figures, circumscribing and circumscribed, can be recomposed into a single sector. To ground this visual argument, we also begin a series of very compelling diagrams, with the clock arrangement of a circle divided into sectors serving as a scaffold for interesting smaller lines – the symmetry of the circle balanced against the coiling asymmetry of the spiral. One wonders, incidentally, how well the fine detail of the individual sectors was represented in ancient papyrus; our medieval, parchment diagrams are often splendid. I chose the figure of this proposition, from the Archimedes palimpsest, for the cover of this volume. In codex A, it appears that the various circular arcs were drawn as arcs; codex C turned them into straight lines, simplifying somewhat the resolution. I am not sure which is to be preferred as authorial.

The astute reader will understand that we begin the campaign of measuring the area intercepted by the spiral. The proposition makes no use of the preceding results: it is a fresh start, belonging in a sense to the introductory passage of propositions 1–11. Indeed, it is functionally like propositions 4 and 7–8, that allowed Archimedes to find a straight line squeezed, in its magnitude, in between two given lines (straight or curved) and then to transfer this line into a geometrical configuration involving the spiral line and a circle. The combination of propositions 4, 7–8 provided Archimedes with the "opening" he needed so as to apply the method of exhaustion in the measurement of the line intercepted by the tangent, just as the sequence of propositions 21–23 provides Archimedes with the "opening" he will soon exploit so as to apply the method of exhaustion in the measurement of the spiral area. Why did Archimedes position propositions 4, 7–8 in the introductory passage, but propositions 21–23 here in the final stage of results? This may well be because the application of propositions 21–23 is so obvious: positioning them too early in the book would have been to give the game away, to announce in advance just what the proof strategy is for finding the area of the spiral.

The text made a discontinuous transition. Its overall texture, however, did not change: the path of deductive sequence breaks while the discourse keeps its identity. The following set of three propositions will present the same "dual – and more" structure we are now familiar with. They are redundant in exactly the same sense in which propositions 18–20 are redundantly repeated.

The property I refer to as the "example rather than instruction" is visible *inside* the proposition, as well. The basic strategy of comparing the two figures is based on:

1. a pair-wise equation of adjacent sectors. This Archimedes does not say directly: rather, he asserts the equation for three pairs (Steps 4–6) and only then, in Step 7, is the procedure generalized.

2. an equation of the number of sectors in the inscribed figure, and the number of sectors in the circumscribed figure minus the first, greatest sector. To see this, we need to verify that the "tail" of the inscribed figure, but not of the circumscribed figure, vanishes at the last sector, so that the total number of circumscribed sectors is greater by one than the total number of inscribed sectors. This Archimedes does not do: not only does he take the equality of the number of sectors as a given (without arguing for the vanishing tail), he does not even mention that we need this equality of numbers.

In the first case, we see Archimedes avoiding instruction, piling up examples instead. As the book of Deuteronomy says (19.15): "A single witness shall not prevail . . . only of the evidence of two witnesses, or of three witnesses, shall a charge be sustained." Archimedes of course could have used one example, or even none at all, merely asserting the equality of adjacent pairs, as he does in Step 7. In this sense he piles up examples. And he avoids instruction: he merely hints, by the words chosen in Step 7, how we know that all such adjacent pairs are equal, never asserting the grounds for the equality. Note that we see here the discursive pattern of propositions 18–20, repeated precisely in a lower level of analysis: instead of clarifying a procedure, Archimedes lets it sink in through triple repetition.

In the second case, we see Archimedes avoiding instruction altogether, leaving the argument opaque. There is of course nothing invalid about the deductive structure: the construction, indeed, is elegant and precise. But precision is to be supplied by the reader and is not made part of the surface texture of the proof.

A comparable example is the last sentence at the end of the proof – obviously its goal: that it is possible to circumscribe and inscribe so that the difference from the spiral area is smaller than any given magnitude. Archimedes has a precise argument, but it is left in part to be supplied by the reader – and the lack of explicit, second-order explanation makes it appear at first confusing. The natural reading of the words "from this it is obvious that it is possible . . . " is that, following the same construction strategy, one could also find a circumscribed figure (or an inscribed one) fulfilling the condition that its difference from the spiral area is smaller than a given magnitude. But this is impossible: the construction strategy cannot be extended to a different case, where the circumscribed figure is compared directly with the spiral area. It relies on there being two figures, circumscribed and inscribed. What Archimedes actually means is that the construction we have been given is *already* one that satisfies the condition that the circumscribed (or inscribed) figures differ by a smaller magnitude. One wonders, indeed, if it would not have been simpler to state this goal at the enunciation of the problem – that is, setting the task to begin with as the finding of circumscribed and inscribed figures differing from the spiral area by a smaller magnitude – and then assert the last sentence not as a kind of "corollary" but as a step within the proof itself. For, after all, what Archimedes means when he says "from this it is obvious that it is possible . . . " is that one can write exactly such a proof.

So why not write it, instead of presenting it as some kind of *extension* of a proof?

Throughout, Archimedes is happy to have his results derive as extensions of something else: not argued directly, but understood on the basis of examples. And, naturally enough, yet another example of the same argument now follows.

/22/

Taking the area contained by the spiral drawn in the second rotation and by the line which is the second among the <lines> at the start of the rotation, it is possible to circumscribe around it a plane figure composed of similar sectors and to inscribe another, so that the circumscribed is greater than the inscribed by a <magnitude> smaller than any given area.

Let there be a spiral, on which <is> the <line> ΑΒΓΔΕ, drawn in the second rotation, and let the point Θ be <the> start of the spiral, ΑΘ <the> start of the rotation, ΕΑ the second line among the <lines> at the start of the rotation, and let the ΑΖΗ circle be second, and ΑΗ, ΖΙ its diameters at right <angles> to each other. (1) Now, again, with the right angle being bisected as well as the sector containing the right angle, the remainder shall be smaller than the given.[289] (a) And let the sector have come to be <as> the <sector> ΘΚΑ, smaller than the given area. (2) So, dividing the right angle into the angles equal to the <angle> contained by the <lines> ΚΘΑ and the rest constructed according to the same <constructions> as before, the circumscribed figure shall be greater than the inscribed figure by a <magnitude> smaller than the sector, ΘΚΑ; (3) for it shall be greater by the difference, by which the sector ΘΚΑ exceeds the <sector> ΘΕΡ.[290]

/COROLLARY/

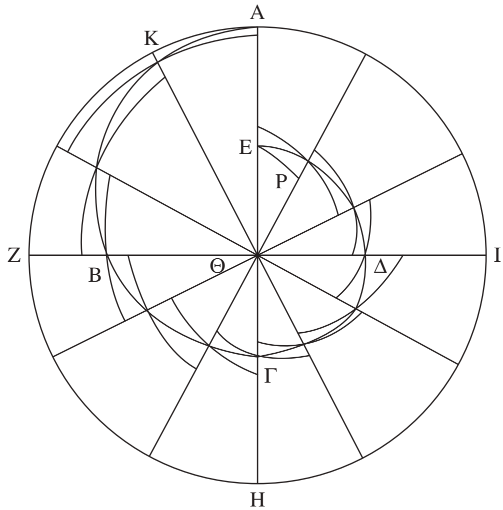
Now, it is clear that it is also possible that the circumscribed figure be greater than the taken area by a <magnitude> smaller than any given area, and, again, that the taken area be greater than the inscribed figure by a <magnitude> smaller than any given area.

Through the same manner it is obvious: that it is possible, taking the area contained both by the spiral, drawn in whichever rotation, and by the line in

[289] *Elements* X.1.

[290] In the case of the double rotation, the "tail" of the inscribed figure does not vanish into nothing, but remains as a minimal figure – that of ΘΕΡ. Thus the difference between the sum of circumscribed sectors and the sum of inscribed figures is not the greatest sector, but rather the difference between the greatest sector and the smallest sector.

the start of the rotation, counted by the same number, to circumscribe a plane figure, as has been said, so that the circumscribed figure is greater than the taken area by a <magnitude> smaller than any given area, and again to inscribe, so that the taken area is greater than the inscribed figure by a <magnitude> smaller than any given area.



Division into twelve sectors is original (against the letter of the text, which stipulates 2^n sectors: Heiberg has eight sectors instead, as he will do in all following continued bisections). Codex C misses the internal circular arc passing through the hours 10 to 12. Both codices A and C miss the internal circular arc passing through the hours 11 to 1 (that is, passing through E: as a consequence, P appears in codex A as if it were on the spiral, not on a circular arc; this omission may be authorial but, if so, it is a fatal slip that has to be corrected). Heiberg adds the first rotation of the spiral. Codex A omitted Θ (not reinstated by any copy); codex C omitted P altogether, and misplaced E to the end of the arc passing through E and cutting AΘ.

COMMENTS

By now, we predict an Archimedean repetition, and I will not comment on it. This brief proposition, however, presents a number of smaller questions.

First, there is a strange expression used twice – in the enunciation and in the "corollary" at the end – where we are asked to circumscribe a "plane" figure. Why specify this? Whoever thought we were dealing with solids, anyway? An intriguing possibility is that here and there we see, perhaps, some trace, in Archimedes' thought, of the approach taken in the alternative proof reported by Pappus: for Archimedes must have considered the problem of the spiral, among other things, in the context of the cone (for this, see the comments on 24 below).

Second, we see Archimedes once again extending a problem to beyond the first rotation. This gives rise to a basic unresolved question. Is the spiral a coiled object such that, beyond its first rotation, it begins to turn upon itself? Would the area "taken by the spiral" then involve some kind of double-coverage, the area of the spiral taken by the first rotation counted twice towards the area of the spiral taken by the second rotation? No: Archimedes means the area contained by the line AE and the coil of the spiral in its second rotation alone – the coil from E to A. This is signaled by the diagram, which simply does not draw the inner, first rotation of the spiral. One therefore concludes that the expression "the spiral drawn during the second rotation" means just that – a single coil, and not the entire spiral consisting of both coils (such, indeed, is the implication of the language of the setting-out of proposition 19, which distinguishes between the first rotation, ΑΒΓΘ, and the second rotation, ΘΕΤ). The phrase "spiral drawn during the second rotation" is actually quite clear: it is a spiral segment, not a spiral line, extending all the way from the start of the spiral. And yet both Heiberg and Dijksterhuis (as well as the authors depending on them) never draw the spiral segment, always drawing a complete spiral whenever a spiral is discussed. This is quite an important distinction: perhaps, if we envisage the second rotation of the spiral as an anchorless linear segment, we see it as quite radically distinct from the centrally anchored, "semi-closed" first rotation, which in turn may explain Archimedes' need to repeat his proofs for the cases of the first and second rotations, allowing the second rotation then to generalize directly to the third and higher rotations (as, after all, *visually*, second and higher rotations are the same). Note that we will have an even more "unanchored" figure in the following proposition.

Finally, note an elegant move in Step 3, "greater by the difference, by which . . . exceeds . . . ." This is very powerfully reminiscent of the language of propositions 10–11 and must be here either as an unconscious reflex of Archimedes' anticipation of the application of the propositions or, more interestingly and (I think) more likely, as a conscious, slightly misleading, clue: for while we are about to consider the application of propositions 10–11 to the case of the sectors, we will do so by considering the difference not of very separate sectors, as Archimedes does here, but by considering the difference of *adjacent* sectors.

/23/

Taking the area contained both by the spiral, which is smaller than the <spiral> drawn in a single rotation, not having the beginning of the spiral as an end, as well as by the lines drawn from the ends of the spiral, it is possible to circumscribe around the area a plane figure composed of similar sectors and to inscribe another, so that the circumscribed figure is greater than the inscribed by a <magnitude> smaller than any given area.

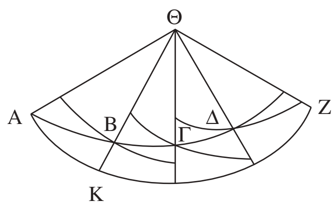
Let there be a spiral, on which <is> the <line> ΑΒΓΔΕ, and its ends Α, Ε, and let Θ be <the> start of the spiral, and let ΑΘ, ΘΕ be joined. (a) So, let a circle be drawn with Θ as center and ΘΑ as radius, (b) and let it meet ΘΕ at Ζ. (1) So, with the angle at Θ as well as the sector, ΘΑΖ, ever again being bisected, the remainder shall be smaller than the given area.[291] (c) Let the sector ΘΑΚ be smaller than the given <area>. (d) So, similarly to the previous <propositions> let circles be drawn through the points, at which the spiral cuts the lines making equal angles at Θ, (e) so that the circumference of each falls on both

[291] *Elements* X.1.

the preceding and the following <line>. (2) So, there shall be a certain plane figure circumscribed around the area contained by both the spiral ΑΒΓΔΕ as well as the lines ΑΘ, ΘΕ, composed <=the plane figure> of similar sectors, and another inscribed, and the circumscribed exceeds the inscribed by a <magnitude> smaller than the given area. (3) For the sector ΘΑΚ is smaller <=smaller than the given area>.[292]

/COROLLARY/

From this it is obvious that it is possible to circumscribe a plane figure around the mentioned area, as has been said, so that the circumscribed figure is greater than the area by a <magnitude> smaller than any given area.[293]



Codex C omits E (as do codices EG). Codex D swaps E, Z. Heiberg has a very different figure where the spiral is drawn from its start.

COMMENTS

This is a surprising move: we have expected, based on the progress of propositions 18–20, to have the proof provided for the case of a spiral fragment extended from the start outwards, smaller than a single rotation (and then, presumably, extended to all non-integer rotations). It turns out that propositions 21–23 follow a somewhat different route from that of propositions 18–20. In the sequence 21–23, we do not have a general result for non-integer areas, but merely one type of those – the unanchored spiral fragment contained within the first rotation. It is not self-evident, from the procedure followed in this proposition, that other non-integer areas may be seen to fall under the logic of this case (even though Dijksterhuis effectively assumed that, by picking this one case to illustrate Archimedes' approach in propositions 21–23). Indeed, from the practice of propositions 18–20, we are led to expect that, had Archimedes wished to assert that the result is valid for any non-integer area, he would have said so.

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[292] Now the reader has to supply the entire argument: the circumscribed is greater than the inscribed by the difference ΘΚΑ-ΘΕΔ (that this is true is left as a – visual? – exercise) which in turn is of course smaller than ΘΚΑ.

[293] Heiberg supplies, following Rivault, the complementary claim for the inscribed figure; but I find it more likely that the omission is authorial and is a mark of the radical ellipsis Archimedes employs at this stage of the sequence of propositions 21–23.

On the other hand, we recognize with surprise that propositions 18–20, after all, did not cover *everything*. We have considered the tangent associated with any point on the spiral, effectively considering it as the end point of an outwardly extended spiral line arising all the way from the start of the spiral. It was in the nature of the object studied there that the question never arose whether or not the spiral actually did extend all the way back to the start, and nothing so far in the treatise ever prepared us for the notion of a spiral fragment. It was indeed only in proposition 22 that it became clear that the second rotation of the spiral is, in some sense, a "fragment" of a spiral (and should not be understood as the extended combination of first and second rotations).

The treatise begins with an introduction mentioning in explicit, precise terms the object of the spiral; followed, many propositions later, by an explicit set of definitions (following proposition 11) setting out the spiral more explicitly; but it remains for the further turns and coils for the treatise to unpack, in somewhat clearer terms, just what is this object we are studying.

Now, finally, we will measure it.

/24/

The area contained by both the spiral drawn in the first rotation, as well as the first line among the <lines> at the start of the rotation, is a third part of the first circle.

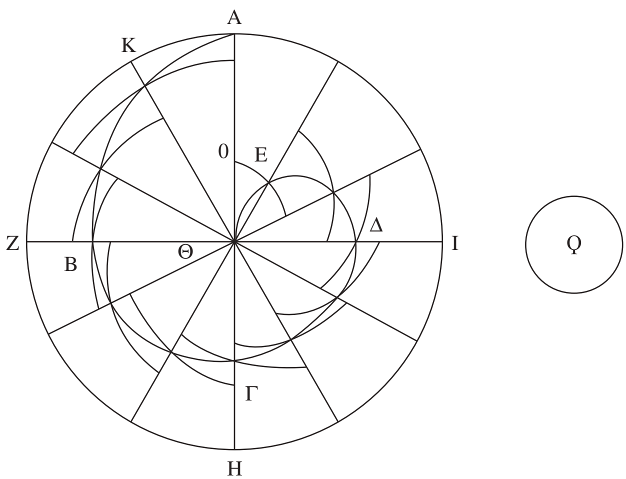
Let there be a spiral drawn during the first rotation, on which <is> the <line> ΑΒΓΔΕΘ, and let the point Θ be <the> start of the spiral, the line ΘΑ first among the <lines> at the start of the rotation, and the circle ΑΚΖΗΙ, first <circle>, of which let the circle, in which is Ϙ, be a third part. It is to be proved that the mentioned area is equal to the Ϙ circle.

(1) For if not, it is either greater or smaller. (a) Let it first be, if possible, smaller. (2) So, it is possible to circumscribe around the area contained by both the spiral ΑΒΓΔΕΘ and the line ΑΘ a plane figure composed of similar sectors, so that the circumscribed figure is greater than the area by a <magnitude> smaller than the difference by which the circle Ϙ exceeds the mentioned area.[294] (b) So, let it be circumscribed, and let the greatest of the sectors, of which the mentioned figure is composed, be the <sector> ΘΑΚ, and the smallest <of the sectors>, ΘΕΟ. (3) Now, it is clear that the circumscribed figure is smaller than the circle Ϙ.[295] (c) So, let the lines making right angles at Θ be produced as far as they <extend> to fall on the circumference of the circle. (4) So, there are certain lines – those falling on the spiral

[294] Proposition 21.

[295] By combinations of hypotheses in Steps a and 2. Indeed, the entire point of the construction of proposition 21 is to be able to "squeeze" a figure which is (in this part of the proof) greater than the spiral area, but smaller than the circle which is one-third of the first circle.

from Θ – exceeding each other by an equal <difference>,[296] of which ΘA is <the> greatest, while ΘE <is the> smallest, and the smallest is equal to the difference. And there are also certain other lines, those falling on the circumference of the circle from Θ, equal to those <=lines falling on the spiral> in multitude, while each is equal in magnitude to the greatest <=among the lines falling on the spiral>, and similar sectors have been set up on all <the lines>, both on the <lines> exceeding each other by an equal <difference> as well as on the lines equal both to each other as well as to the greatest; (5) therefore the sectors on the <lines> equal to the greatest are smaller than triple the sectors on the lines exceeding each other by an equal <difference>; (6) for this has been proved.[297] (7) But the sectors on the <lines> equal both to each other as well as to the greatest are equal to the circle AZHI, (8) while the sectors on the <lines> exceeding each other by an equal <difference> are equal to the circumscribed figure; (9) therefore the circle AZHI is smaller than triple the circumscribed figure. (10) And <it is> three times the Ϙ circle; (11) therefore the circle Ϙ is smaller than the circumscribed figure. (12) But it is not <smaller>, but greater.[298] (13) Therefore the area contained by both: the spiral ΑΒΓΔΕΘ as well as ΑΘ is not smaller than the area Ϙ.



DG swap the position of the two circles; B has the smaller circle somewhat higher (B, interestingly, only draws a subset of the small arcs required for the proof: hours 12/2, 1/3, 10/12, 9/11). Codices AC did not have Θ, inserted by D (the omission may be authorial). C may have lost A, and seems to have lost K, B. The figure of C is positioned together with the next one, even though each part of the proposition is counted separately and even though a scribal note at this position states "the diagram follows." (That is, the scribe considered that this, and not the next page, was the appropriate position for the first diagram of 24.) See the next figure.

[296] Proposition 12, taken for granted at this point.

[297] Proposition 10, just in case you ask. One may imagine the proposition lying in the dark for the last fourteen propositions, waiting for this moment – when the spiral finally gets home – and then the lights turn on, confetti falls down: it's the treatise's birthday.

[298] Assumption of Step a.

Nor, furthermore, greater. (d) For let it be, if possible, greater. (14) So, if possible, it is possible to inscribe a figure inside the area contained by the spiral ΑΒΓΔΕΘ and by the line ΑΘ, so that the mentioned area is greater than the inscribed figure by a <difference> smaller than <the difference> by which the mentioned area exceeds the circle Ϙ.[299] (e) So, let it be inscribed, and let the <sector> ΘΡΞ be the greatest among the sectors, of which the inscribed figure is composed, and the <sector> ΘΕ the smallest.[300] (15) Now, it is clear that the inscribed figure is greater than the circle Ϙ. (f) So, let the <lines> making the equal angles at Θ be produced so far as to fall on the circumference. (16) Now, again, there are certain lines exceeding each other by an equal <difference> – those falling on the spiral from Θ[301] – of which ΘΑ is greatest, while ΘΕ <is> smallest, and the smallest is equal to the difference, and there are also other lines, those falling on the circumference of the circle from Θ, equal to those <=lines falling on the spiral> in multitude, while each is equal in magnitude to the greatest <=among the lines falling on the spiral>, and similar sectors have been drawn on all <the lines>, both on the <lines> equal to both each other as well as to the greatest, as well as on the <lines> exceeding each other by an equal <difference>. (17) Therefore the sectors on the <lines> equal to the greatest are greater than triple the sectors on the <lines> exceeding each other by an equal <difference> without the <sector> on the greatest <line>. (18) For this has been proved.[302] (19) And the sectors on the <lines> equal to the greatest are equal to the circle ΑΖΗΙ, (20) while the <sectors> on the <lines> exceeding each other by an equal <difference>, apart from the <sector> on the greatest, are equal to the inscribed figure; (21) therefore the circle ΑΖΗΙ is greater than triple the inscribed figure. (22) And it is triple the circle Ϙ; therefore the circle Ϙ is greater than the inscribed figure. But it is not, but smaller; (23) therefore nor is the area, contained by both the spiral ΑΒΓΔΕΘ as well as by the line ΑΘ, greater than the circle Ϙ. (24) Therefore it is equal [to the <area> contained by the spiral and the line ΑΘ].[303]

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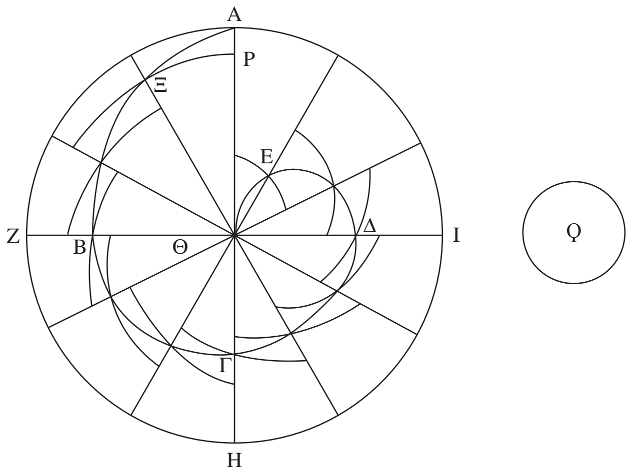
[299] Proposition 21.

[300] Heiberg emends the text to read "the sector OΘE," following Moerbeke. In fact, not only is the letter O missing from the text: it was also missing from the diagrams of codices AC, reinstated by BDG (obviously, on the model of the diagram for the first case). Clearly, Archimedes moves to a more shorthand description of the sector.

[301] Proposition 12. [302] Proposition 10.

[303] Heiberg brackets the final words. Indeed, the "it" in Step 24 must pick up the subject of Step 23 and refer to the area which is then said to be equal to the circle. The final words, however, assume that the "it" refers to the circle. Some hasty scholiast was nonplussed by the brief ending of this major proposition.



COMMENTS

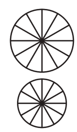
The proof consists of two deductive chains. Each consists of no more than twelve Steps (2–13, 14–24), which involve hardly any geometrical argument at all: one merely sets up the construction whereby the circumscribed/inscribed figure is squeezed between the spiral area and circle Ϙ, assuming them to be unequal (Steps 2–3, 14–15), whereupon all that needs to be done is to recall proposition 10 and show in detail how it applies in this case – what the various abstract terms specified in proposition 10 mean in the context of proposition 24 (Steps 4–9, 16–21). Following this, all that's left is to note that this contradicts the preliminary construction.

The double reductio structure makes the proposition, well, double, while the construction of the figure composed of sectors, and especially the application of proposition 10, involves complex details, so that the proposition as a whole feels like it is of a complex structure. It is not at all: it is nothing more than a construction followed by the direct application of a result. Its conceptual complication resides, however, in the cleverness of the argument of proposition 21 – and especially the opaqueness of the argument of proposition 10 which makes its application here truly surprising and, indeed, magical: for the result follows directly once the application has been understood, no further deductive work required – even though the original proposition 10 appeared totally unrelated to the matter of spirals!

Indeed, how could one even come up with this deductive route? With this result? Not an easy question to answer in general, and perhaps not too much weight should be accorded to any particular speculation in this regard. But

Codex C postponed, as noted above, the diagram for the first case, and the diagrams end up amalgamated as in the thumbnail, the first case above the second. Perhaps as a consequence, the small circles are omitted. BG have the small circle somewhat higher. H4 have the arc PΞ fall on the circumference of the circle; perhaps also in A. B has only the arcs in the hours 1–3, 7–9, 8–10, 9–11, 10–12. E misses the hours 8–10; G misses the hours 12–2. C missed the letter A and may have missed B. Codices AC did not have Θ at the start of the spiral (perhaps an authorial omission) which was inserted by BD, who also insert an O at the intersection of the arc passing through E and the line AΘ. (E4, however, position the letter E there, perhaps also in codex A; perhaps this is the preferred reading.)

Codex C:



there is a legitimate historical question, analogous to that of the other main result of *On Spirals*, in Proposition 18. In this case the speculation is not entirely idle. A passage in Pappus (translated as Appendix 1 to this volume) is relevant, as Knorr has shown, to this discussion. So it is time to bring in the evidence for the other ancient treatment of the volume of the spiral. This will also be an opportunity to do justice to Knorr 1978a, the most important study dedicated to Archimedes' *On Spirals*.

Pappus' *Collection* IV.21–25 is a survey of the spiral. Pappus makes the briefest of historical introductions; provides a construction of the spiral and a set of definitions; finds the spiral area; and finally mentions a few extensions. Everything he says raises serious problems. The brief historical introduction consists of the claim that the result was Archimedes' response to a study proposed by Conon – i.e. the exact opposite of the scenario outlined by Archimedes in his introduction. The proof differs from that of Proposition 24. The extensions are not those offered in *On Spirals* as it now stands.

Now, there's one possible deflating response: that Pappus' *Collection* is very much Pappus' original work, and so we need not perhaps be so struck by his offering a different proof from that of 24 (and then adding to it some different extensions); even the historical statement need not be so problematic: Archimedes is clearly very respectful of Conon, and it is easy for a reader to come away from *On Spirals*, as we know it, with a sense that Conon, above all, was responsible for the studies put forward by Archimedes (and it is never certain that Pappus read *On Spirals* as we know it, rather than in some mediated – or superior? – form). Such appears to have been the position taken by past scholars before Knorr, who dismissed the passage as evidence for Archimedes himself, apparently taking it to be no more than Pappus' own variation on Archimedes' original thought, as we find it in *On Spirals*.

This deflating response, however, goes against the simple reading of Pappus, who introduces this survey with the emphasis that – while the study was indeed proposed by Conon – "Archimedes proved it using an amazing manner of approach"[304] (Hultsch 1876: I, 234.2–3). We can now emphasize ourselves – following Sefrin-Weis 2010 – that the entire goal of Pappus' *Collection* IV is the survey of ever more complicated geometrical methods. The reason Pappus chooses to discuss the spiral, at this point, is that its study

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304 επιβολή; Sefrin-Weis prefers "line of attack." I prefer to note the similarity between epi-bole an eph-odos (ἐπιβολή, ἔφοδος), a striking-into and a way-into, dynamic and static ways of viewing the same thing: an Archimedean approach – ἔφοδος, of course, being the original title of what came to be known in the English language as *The Method*.

In the following discussion a certain familiarity with Archimedes' *Method* is assumed. Briefly: Archimedes sent to Eratosthenes a letter-treatise, titled ἔφοδος (standardly translated "Method"), where many previously established results (as well as two new ones) were shown to be true in a non-rigorous fashion (based on the summation of infinitely many indivisibles, as well as on mechanical considerations; only in *Meth.* proposition 14 are indivisibles used alone). Archimedes does not claim that such proofs are rigorous, claiming, however, the usefulness of knowing the truth of a claim through such procedures, as a starting-point for the finding of a valid proof.

by Archimedes involves a remarkable approach, or methodological ingenuity. It is thus almost certain that the basic proof-idea on display in Pappus' proof was one Pappus, at least, took to be by Archimedes. And, as Knorr has observed, this proof-idea is very different from that of the extant *On Spirals* (see the figure on p. 183). The circle is associated with a rectangle KΠΛN, so that the ratio of the sector ABΓ to the sector ZBH is shown to be the same as that of the square on KN to the square on MN – or, equally, the circle with KN as radius to circle with MN as radius, or, equally, the cylinder with the KN-radius circle as base to the cylinder with the MN-radius circle as base, provided the two cylinders have the same height. So, as long as the cuts to the circle's circumference – as well as to the side KΠ – are in the same proportion, we may make the ratio of sectors to sectors the same as the ratio of cylinders to cylinders – so that the ratio of the whole circle to the whole figure in the spiral is the same as the ratio of the whole cylinder (with the KN-circle radius as base, and NΛ as height) to the whole figure in the cone (with the KN-circle radius as base, and the same NΛ as height). Whereupon Pappus comes to a stop: "but the cylinder is three times the cone; therefore the circle, too, is three times the said figure <=the spiral area>."

This is amazing in two ways: first, in the brilliant idea of mapping a spiral/circle configuration into a cone/cylinder configuration; second, in the direct transition from a result for a figure enclosed within a cone to the cone itself. This, for once, appears like a leap based on the intuition of a limit – even though Archimedes is, of course, in full possession of the tools of the double reductio!

Now, we should not rush immediately to assert that Archimedes, himself, made the transition as direct as that. First, Pappus is clearly compressing Archimedes' proof, and it is conceivable that he merely abbreviated (as obvious, and yet cumbersome) the details of a double reductio (while possible, I find this less likely: as it stands, the trick is worthy of the label "amazing" and of its relatively late position in Pappus' ascending order of complex proofs; without the intuitive argument based on a limit, less so). Second, even if Archimedes makes a leap based on the intuition of a limit, he need not necessarily mean it as a rigorous proof. It could be offered as just that – an intuitive, suggestive argument that makes a certain claim appear plausible. In other words, the argument in Pappus – in many ways suggestive of the use of indivisibles throughout *The Method* (and especially in *Meth.* 14: this comparison was made already by Knorr 1996) – could have been put forward by Archimedes in the same spirit in which the arguments of *The Method* have been proposed.

Such was not Knorr's view: he took the argument reported by Pappus to be a treatise offered by Archimedes on a par with the extant *On Spirals*. Indeed, he took it to be Archimedes' *first published* version of *On Spirals*. I must say I am quite confident Knorr was wrong about *that*. The crux of the matter comes in Knorr 1978a: 67–68, where Knorr mentions in passing that "it appears from our discussion of Pappus' theorems that this list [what we call *The Big Letter* to Conon] announced not new discoveries, but new proofs – at least as far as *SL*, 24 was concerned." Knorr is correct about this implication of his interpretation – which should be seen as a reductio ad absurdum. For the introduction to *On Spirals* makes clear that *The Big Letter* was intended as a major challenge, inviting responses that carried meaning because they carried danger: they could in principle have exposed their proponents as well as Archimedes himself. And so it contained a set of claims whose proofs are awaited by Dositheus (and surely others represented by him); as well as false theorems. All of this would immediately evaporate if the promise were merely that of an alternative proof to a known, already proven result. Why the anticipation? Whence the danger? How would Archimedes now prove that this, and not that, was his intended alternative proof? The challenge of finding an alternative proof is weak to the point of being meaningless; whereas that of finding a proof to an unproved result is obviously powerful and rich in agonistic potential. Which is pretty much what Archimedes says in his introduction in very clear Greek: "those theorems, about which you keep . . . asking that I write down the proofs" – which plainly means, the results, whose proofs are *not yet* written down.

Knorr's article begins with a comparison to *The Method* and returns to it repeatedly, insisting on the idea that Archimedes' ideas emerged in a more heuristic fashion than that displayed in most of his preserved writings. And yet Knorr never for a minute considered the possibility that the context of the publication of the alternative *On Spiral Lines* proof was indeed that of *The Method*: a *retrospective* claim that a certain approach leads to faith in a particular result *whose rigorous proof was already published*. I do not see how Knorr could rule this out.

Why did Knorr wish to rule that out? Because he was committed to his own version of the dynamics of Archimedes and his audience – where an impetuous Archimedes, ever set on his heuristic, original approaches, is stifled by the Alexandria Thought Police that demand he conform to established procedure. The first, heuristic version of *On Spiral Lines* was a failure; Archimedes went on to publish another, corrected version. Once again, I find this account wildly at variance with the self-assured figure of the author speaking haughtily about the failure of his entire audience to come up with any response. Archimedes was no recanting Galileo – in part, because Alexandria had no inquisition with which to threaten him. Greek individuals, and Greek society as a whole, had no place for recantation. So that, had anyone disapproved of Archimedes, he surely would have thundered down with a vigorous response – and, as surely, would have won the day. Knorr's image is noble, and worthy of Knorr's own nobility. But it is also entirely of Knorr's own making.

And yet Knorr was right about the crucial thing: Pappus' evidence most probably represents another, lost work on spiral lines by Archimedes (almost certainly, then, published *later* than the treatise we are now reading). And so it seems reasonable to look in it for clues to Archimedes' thinking about the Spiral. Indeed, it is conceivable that Pappus' evidence derives from a source directly comparable to *The Method*, where Archimedes explicitly states that he has discovered the result of *On Spirals* 24 through just this route. A caveat is required: even if Archimedes explicitly said that he had discovered the result in that way, this does not prove he actually did. But it is worth contemplating the implied path of discovery.

So let us go back to the fundamental assumption: that the spiral is invented in order to achieve a quadrature of the circle. If so, two questions become most natural: the relation between the spiral line and the circle's circumference (thinking along those lines, and considering the figure dynamically, gives rise, as we saw above, to the suspicion that proposition 18 is true); and the relation between the spiral area and that of the circle enclosing it. And so we contemplate a circle and a spiral, and it is perfectly natural to consider an arbitrary sector. While it is still impossible to compare the circle's sector to that of the spiral area it encloses, it is a very simple idea (and the one most entrenched in Archimedes' practice, certainly inherited from Eudoxus himself) to compare the circle's sector with a smaller sector directly circumscribing or inscribed in the spiral. That the circle sector is to the sector associated with the spiral as the squares on the intercepted radii are is now obvious.

At some point, one's mind might wonder what would happen when more sectors are brought into play; and it is clear that, as more sectors are brought in, the ratio is always that of the square on the radius of the circle to the square on the radius intercepted by the spiral. The first term is a constant; the second slides, squared, along a continuous progression.

My main observation now is that the mind contemplating this fact is extremely well trained in the problem of the ratio of the cylinder to the cone. Twice in his introductions (to *On the Sphere and the Cylinder* I and to *The Method*) Archimedes singled out this result – *Elements* XII.10 – as Eudoxus' crowning achievement. Arguably, Archimedes' major geometrical project was the extension of Eudoxus' result on the ratio of the cylinder to the cone, to other curvilinear figures. And if so, it would not be at all unlikely that at some stage, as he pondered the pattern of variation in the ratios of circle sectors to sectors associated with the spiral enclosed by the circle, that he would also notice that this is reminiscent of the ratios of cylinder cuts to cuts associated with the cylinder enclosed by the cone.

I add the following in passing. *Elements* XII.10 considers the cone as the limit, so to speak, of a pyramid with a many-sided polygonal base. This is indeed the most direct way of measuring the cone and was most probably Eudoxus'. However, there is some evidence that early Greek thinkers could also consider the cone as the limit, so to speak, of a series of narrowing cylinders. This comes from Plutarch's polemic with Stoicism, in the course of which he criticizes Chrysippus' resolution of a dilemma suggested by Democritus (*Comm. Not*. 1079e, revising Cherniss' Loeb translation):

Look at the way in which he [Chrysippus] attacked Democritus who had raised – in a concrete[305] and vivid fashion – this difficulty: if a cone is cut by a plane parallel to the

[305] φυσικῶς, i.e. the opposite of λογικῶς, "a purely verbal/abstract squibble." Plutarch's overarching theme is that the Stoa claims to be close to intuitive principles but is in fact involved with ad hoc unnatural principles, so in this context the emphasis is on Democritus' more down-to-earth approach (compared with Chrysippus' own retort).

base, how should we conceive of the arising surfaces of the segments – equal or unequal? For if they are equal the cone is then uneven, possessing notches and rough points; but if they are unequal, the segments are equal and the cone would appear to have the property of a cylinder, composed of equal, and not of unequal circles.[306]

Now, in the introduction to *The Method*, Archimedes himself provides our – unique – evidence that Democritus made a contribution to the study of the ratio of a cone to a cylinder: the latter is praised for having first made the claim that the ratio is a third, even if without stating a proof (which Eudoxus was the first to publish; thus, says Archimedes, we see that the mere making of a claim can provide a real contribution to mathematical study). Now, it is not *certain* that Democritus made the observation concerning the ratios of the cone and the cylinder in the same context in which he offered the dilemma quoted by Plutarch. But this hypothesis is likely enough.[307] And if so, we find that the Democritean observations concerning the cone – surely prominent in Archimedes' mathematical thinking – would involve the contemplation of the cone as a series of stacked cylinders – which is, needless to add, the way the conoid is envisaged in *On Conoids and Spheroids*, already worked out in some sense, at the time *The Big Letter* was sent out . . .

In short, the image of the cone as related to a series of stacked cylinders would be prominent in Archimedes' mind, so that there is a good likelihood that, having considered the ratios of sectors of the circle to their related spiral sector, Archimedes would come to the observation that the two relations are identical: sectors of circles to related sectors defined by spiral; segments of cylinder to related segments defined by cone. At which point one is already in possession of the proof-idea reported by Pappus and is no longer in doubt concerning the measurement of the spiral area.

I note that Pappus' sketch does not provide any particular way of cutting the circle (hence, the cylinder): any two cuts would suffice to suggest that the same result could be obtained for any number of cuts and so to suggest that the circle is to the spiral as the cylinder to the cone. Sefrin-Weis' notes as follows (2010: 122 n.1): "The ratio for the division is not specified. Most likely, it is $1{:}2^n$," and I suppose this is the standard reading of this proof. This indeed makes sense, if the goal is some kind of a formal proof, where one could use a

306 Chrysippus' response seems to comes close to postulating a tertium between equality and inequality: everything here, from Democritus' original paradox through Chrysippus' response, tantalizingly suggests a sophisticated reflection upon the problem of the infinitesimal – all mediated through a single, oblique passage in Plutarch. See Sedley 2008: 323–325 for a recent discussion of the problem in Democritus, with references in n. 49.

307 If pressed to offer my own guess, I would imagine Democritus making a series of observations concerning the shape a cone would take under strict mathematical atomism: its edges would become uneven, and its base would become not a circle, but a large polygon. If so, it becomes natural to say that the cone is in fact indistinguishable from a pyramid (in turn indistinguishable from a stacked series of polygonal prisms) – which could possibly be all Eudoxus needed to motivate his proof, as recounted by Archimedes.

specific division algorithm – say, continuous bisection – to show that the composition of slices may approximate the figures to a difference smaller than any given magnitude. And it is in fact possible to turn Pappus' approach into a valid proof. All one would need is such a division algorithm, such a proof of approximation, and then a double reductio showing that the ratio of the circle to the spiral area is the same as the ratio of the cylinder to the cone (for if not, it is either smaller or greater; and either way one gets an absurdity once the circumscribed/inscribed figure is "squeezed" in). Such was Archimedean standard practice, and Archimedes would have immediately realized that a proof such as that reported by Pappus could easily be made rigorous in such a way.

This opens a number of questions. First of all, once again, could Pappus' report actually be a compressed account of such a formal proof? As I pointed out above, this cannot be entirely ruled out, but it is at least unlikely.[308]

Second, why did Archimedes not present the proof in such a rigorous fashion? This, first of all, is yet another argument – if one were required – against Knorr's thesis, according to which the proof reported by Pappus was published prior to the extant *On Spirals*. For had Archimedes possessed the proof-idea reported by Pappus, it would have been trivial for him to transform it into the Eudoxean, rigorous form. However, if the proof was actually published in a *Method*-like context, of Archimedes claiming to have found the result in such a manner, then it makes perfect sense for him to present this as a mere sketch. Indeed, what this is most reminiscent of is not the detailed theorems of *The Method*, but rather the meta-theoretical passage following the second theorem: having discovered that the sphere is four times the cone on its great circle then, following on the idea that a circle is equal to a triangle whose base is the circumference, its height the radius – which in turn suggests that a sphere is equal to a cone whose base is its, the sphere's, circumference, and its height the sphere's radius – so that one now suspects that the surface of the sphere is four times its great circle . . . So the circle being like a triangle (which

[308] The main observation is that, the proof being made rigorous, it would no longer be "amazing." Knorr suggests that even without the use of an argument based on limits, the very analogy of areas and solids would also have been deeply problematic from a formal point of view – so much so that this, indeed, would have been the reason why the proof of the extant *On Spiral Lines* was offered in the first place (Knorr 1978b: 56: "in a plane investigation . . . the use of solids is viewed as inappropriate"). This claim hangs by the thinnest of threads. All we have to rely upon are Pappus' own classifications of types of problem which do mention the terms "solid" and "plane" but in a completely different sense. Even aside from my own hypothesis (Netz 1998a, 2004a) that such classifications represent the interests of late ancient mathematicians and so need not necessarily reflect an ancient practice, Knorr's interpretation lands him in considerable difficulties: he needs to assume that Pappus does not understand his own classification, and, above all, he needs to assume a mathematical context where a proportion involving planes on the one side, solids on the other, is illegitimate, while a neusis is not! (Both at Knorr 1978a: 65.) Such proportions are common and unproblematic in Greek mathematics and are of course taken for granted in *Meth.* proposition 15, the valid proof of *The Method*!

is proved) *should be* like a sphere being like a cone – not yet a proof but already grounds for suspecting that a certain result is true which in turn is enough for looking for a proof.[309]

Finally, why would Archimedes not present the rigorous transformation of the proof in Pappus in the extant *On Spirals*? Assuming that this was, indeed, his original line of thought (and it certainly could have been that), why has he not just stopped there? Why remove the auxiliary solid construction and bring in, instead, the much more cumbersome proposition 10?

Now, the main thing to emphasize is that, at this point, it should not be hard for Archimedes to stumble upon proposition 10. Although Archimedes' original line of thought – just as the sketch reported by Pappus – need not have involved the division algorithm by continuous bisection, it would still be very natural for Archimedes to consider just such an arrangement, and at this point it would be natural to ask if there are other cases where the very same relation holds (maybe the spiral could be connected to yet another configuration?). What are such cases? Well, one would then characterize them as cases where areas or volumes behave as squares on lines arranged in an arithmetic progression. Wait a minute: so any series of squares on an arithmetic progression has the property that the sum is bounded by the third of the same number of squares on the greatest one – and so one is already in possession of proposition 10. No need to look for further geometrical configuration – the relation for lines and squares, as such, is good enough. And so we see that *finding* proposition 10 is easy, once one notices the possible decomposition of cones and cylinders into equal segments. For such a decomposition is a concrete case of proposition 10 which proves, in its very structure, that proposition 10 must be true. Beneath its hideous abstract appearance, proposition 10 has a heart of cone.

Still, why would Archimedes not wish to present his proof directly in terms of the cone and the cylinder? At this point we are left with nothing other than speculation. But let us consider the alternatives side by side. On the one hand, the "Pappus-derived" rigorous line of proof: showing the algorithm for circumscribing and inscribing both spiral and cone via sectors and cylinders – and showing that this algorithm may yield differences smaller than any given magnitudes; showing the proportion of cylinder and circle, composite-cylinders figure and composite-sectors figure; and then deriving the claim that the circle is to the spiral area as cylinder to cone. Difficult, cumbersome and, all along – from the first moment one breathes the words "cone" and "cylinder" – as obvious as daylight. On the other hand, consider the rapid magic of proposition 24 as one discovers, with a gasp of delight, just how proposition 10 applies to the matter of the spiral. Speculation, no doubt: but I would suggest that the difference between the "Pappus-derived" rigorous proof and

309 If this is correct, one consequence is that the passage in Pappus loses some of its edge: it is no longer an example of an Archimedean proof based on the intuition of a limit, but rather an Archimedean admission that he could rely on the intuition of a limit as a suggestive stepping-stone on the way to what he himself would consider a valid proof.

the one in our extant *On Spirals* has most do to with the different conceptual distances between tool and result. The tool of the cone and cylinder is quite close to the result of spiral and circle (which is why it could have served, in the first place, as a natural process of discovery leading to just this result!). The tool of proposition 10 is, conceptually, very distinct from that of proposition 24 – and the intervention of proposition 18, in the middle, serves to widen that gap and make the application least obvious. More precisely, then: proposition 10 is a deliberate way of hiding the cone – its abstract *mask*.

Idle speculation? Perhaps. So let us concentrate on the claims that do not depend on such interpretations of Archimedes' rhetorical practices and are instead simple historical observations. The proof reported by Pappus is either by Archimedes or it isn't; I think it is much likelier to be by Archimedes (in this I follow Knorr against all previous scholars, who tended to see in it no more than a variation invented by Pappus himself: while the question is debatable, it is not in my view in serious doubt). It was published either before or later than the extant *On Spirals*; I think it is much likelier to have been published later (in this I differ from Knorr; once again, I believe my position is not in serious doubt and that Knorr's view is, in this regard, manifestly wrong). And so I conclude with the strong conviction that, at some point after his publication of the extant *On Spirals*, Archimedes went on to publish the proof reported by Pappus.

This, I repeat, is historical fact, debatable, but as solid as such a fact can be. And so we end up with a picture of the growth of Archimedes which is the exact opposite of that provided by Knorr. Instead of the impetuous youth tamed by Alexandrian pedantry, we find an author offering first a bold, improbable result, and returning later to double the stakes and make it even bolder. Archimedes was never tamed.

/25/

The area contained by both the spiral which is drawn in the second rotation, as well as the second line among the <lines> at the start of the rotation, has that ratio to the second circle which 7 has to 12; which is the same <ratio> as <the ratio> which <the areas> taken together, both the <rectangle> contained by the radius of the 2nd circle and by the radius of the 1st circle, as well as the third part of the square on the difference by which the radius of the 2nd circle exceeds the radius of the first circle, have to the square on the radius of the 2nd circle.[310]

[310] The radii of the circles associated with the spiral's rotation form an arithmetical progression whose difference is equal to the first term. In this case let us take as numerical examples the values *first circle's radius=3, second circle's radius=6*. The ratio turns out to be $(6*3)+(3^2/3):6^2$, or obviously 7:12. A more geometrical approach would be to take the square on AE as basic – call it *the small square. The rectangle* contained by AΘ, ΘE is clearly twice the small square (since AE=EΘ); the square on AΘ – call it *the big square* – is clearly four times the small square. So we are looking at the ratio of twice the small

Let there be a spiral, on which <is> the <line> ΑΒΓΔΕ, drawn in the second rotation, and let the point Θ be <the> start of the spiral, the line ΘΕ, first <among the lines> at the start of the rotation and the <line> ΑΕ, second <among the lines> at the start of the rotation, and let the circle ΑΖΗΙ be the second, and the diameters ΑΗ, ΙΖ, at right <angles> to each other. It is to be proved that the area contained by both the spiral ΑΒΓΔΕ as well as the line ΑΕ has to the circle ΑΖΗΙ the ratio which 7 has to 12.

(a) Let there be a certain circle, Ϙ, <and let> the radius of the circle Ϙ be equal in square to both the <rectangle> contained by ΑΘ, ΘΕ, as well as the third part of the square on ΑΕ. (1) So, the circle Ϙ shall have to the <circle> ΑΖΗΙ as seven to twelve,[311] (2) since its radius, too, has to the radius of the circle ΑΖΗΙ, that ratio in square.[312] Now, the circle Ϙ shall be proved to be equal to area contained[313] by both the spiral ΑΒΓΔΕ as well as the line ΑΕ.[314]

(3) For if not, it is either greater or smaller. (b) So, let it first be, if possible, greater. (4) So, it is possible to circumscribe, around the area,[315] a plane figure composed of similar sectors, so that the circumscribed figure is greater than the figure by a <difference> smaller than the <magnitude> by which the circle Ϙ exceeds the area.[316] (c) Let it be circumscribed, and let the greatest <of the sectors> of which the circumscribed figure is composed be the sector ΘΑΚ, and <the> smallest, the <sector> ΘΟΔ. (5) Now, it is clear that the circumscribed figure is smaller than the circle Ϙ.[317]

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square plus a third of the small square, to four times the small square, or 2:4 or 7:12. This then appears as a trivial reformulation, and the reader may well wonder, at this point, why Archimedes chose just this and not the many others he could have come up with. He could, after all, have equally said, "which is the same as the ratio of the Gates of Thebes to the Labors of Heracles." More in the comments.

[311] Greek mathematical language has two main expressions for proportion, one with the verb "to be" and the other with the verb "to have": "A is to B as C to D," "A has to B the ratio which C <has> to D." The infelicity in my translation – an expression beginning with "have," ending with "is" – is in the original.

[312] *Elements* XII.2 – though obviously in such a context just taken for granted. The remarkable thing is that Archimedes points *this* out – that circles are to each other as the squares on their radii – but does not indicate the calculation whereby 7:12 follows from the geometrical construction.

[313] Archimedes is momentarily infected by the language of rectangles and uses for this area the participle form used for contained rectangles (περιεχόμενον, instead of περιλαφθέν).

[314] The circle Ϙ is introduced, in this proposition, later than the definition of goal, and so it has to be mentioned again within a secondary definition of goal.

[315] "The area," in this proposition, means "the area contained between the spiral and the line."

[316] Proposition 22 Corollary. [317] Follows directly from Step 4.

(d)[318] Let the lines making equal angles at Θ be produced so far as to <extend> to fall on the circumference of the second circle. (6) So, there are certain lines exceeding each other by an equal <difference>[319] – those falling on the spiral from Θ – of which ΘA is <the> greatest, while ΘE <is the> smallest, and other lines, those falling from Θ on the circumference of the circle AZHI, smaller than these <=lines falling on the spiral> in multitude by one, while each is equal in magnitude both to each other as well as to the greatest <=among the lines falling on the spiral>, and similar sectors have been drawn on the lines equal to the greatest <line>, and on the <lines> exceeding each other by an equal <difference>, but it has not been drawn on the smallest <line>. (7) Therefore the sectors on the <lines> equal to the greatest have to the sectors on the <lines> exceeding each other by an equal <difference>, apart from the <sector> on the smallest <line>, a smaller ratio than the square on the greatest <line>, ΘA, to the <areas> taken together: both the <rectangle> contained by AΘ, ΘE as well as the third part of the square on EA; (8) for this has been proved.[320] (9) But the sectors on the <lines> equal both to each other as well as to the greatest are equal to the circle AZHI, while the sectors on the lines exceeding one another by an equal <difference> apart from the <sector> on the smallest <line> are equal to the circumscribed figure;[321] (10) therefore the circle has to the circumscribed figure a smaller ratio than the square on AΘ to the <areas> taken together: both the <rectangle> contained by AΘ, ΘE as well as the third part of the square on AE.[322] (11) But that ratio which the square on ΘA has to the <rectangle> contained by ΘA, ΘE and the third part of the square on AE, the circle AZHI has to the circle Ϙ.[323] (12) Now then, the circle AZHI has a smaller ratio to the circumscribed figure than

[318] The original Greek has an asyndeton. Remarkably, the same happens in Step g in the second part of the proposition (though not in Steps c, f of the preceding proposition) – which is why Heiberg offered no emendation. One must conclude that this is intended as the beginning of a new paragraph: Steps 1–5 set up the first arm of the reductio; the second arm of the reductio begins afresh with Step d.

[319] Proposition 12, taken for granted at this point.

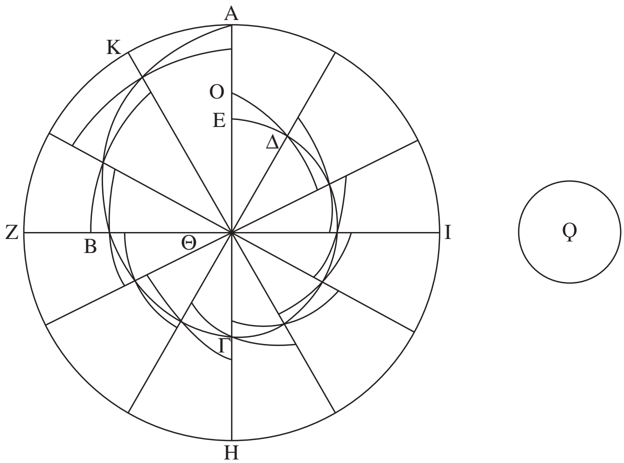
[320] Proposition 11 Cor. This is where proposition 25 differs from the previous proposition 24: relying on proposition 11 instead of 10.

[321] Original case arrangement was "to the sectors … is equal the circle," which I inverted for reasons of English style (I preferred not to invert the word order, as the logical flow depends on the re-identification of the sectors as the circle and not vice versa).

[322] Step 10 uses Step 9 to substitute one term in Step 7 ("circle" instead of "sectors on equal lines"), and then relies on *Elements* V.8 to replace another term with another greater than it ("circumscribed figure" instead of "sectors on unequal lines"; that the circumscribed figure is greater than the sectors on the unequal lines is taken for granted visually).

[323] ΘA is the radius of the circle AZHI, and then one needs to apply Step a, as well as *Elements* XII.2.

to the circle Ϙ.[324] (13) Thus the circle Ϙ is smaller than the circumscribed figure.[325] (14) But it is not <smaller>, but greater.[326] (15) Therefore the circle Ϙ is not greater than the area contained by both the spiral ΑΒΓΔΕ as well as the line ΑΖ.



Heiberg has an 8-division and draws a complete spiral (he also adds a mysterious label Λ on the intersection of the spiral and the line ΘΙ). BG have the smaller circle higher (B also has it to the left). Θ was omitted by A, reinstated by BD, and possibly had Ξ instead of Z, corrected by BDGH. G omits A. C omits the smaller circle as well as the labels ΟΕΒΓ; apparently also the label K.

Nor, furthermore, smaller. (e) For let it be, if possible, smaller. (16) Now, again, it is possible to inscribe inside the area contained by both the spiral and the line ΑΕ a plane figure composed of similar sectors, so that the area contained by both the spiral ΑΒΓΔΕ and the line ΑΕ is greater than the inscribed figure by a <difference> smaller than the <magnitude> by which the same area exceeds the circle Ϙ.[327] (f) Now, let it be inscribed, and, among the sectors of which the inscribed figure is composed, let the sector ΘΚΡ be <the> greatest, and the <sector> ΘΕΟ <the> smallest; (17) now, it is clear that the inscribed figure is greater than the circle Ϙ.[328]

(g) Let the <lines> making equal angles at Θ be produced so far as <to extend> to fall on the circumference of the circle. (18) Now, again, there are certain lines exceeding each other by an equal <difference>[329] – those falling on the spiral from Θ – of which ΘΑ is <the> greatest, while ΘΕ <is the> smallest, and other lines, those

[324] Step 10 is (circle ΑΖΗΙ):(circumscribed figure)<(square ΘΑ):(combination of areas).

Step 11 is (square ΘΑ):(combination of areas)=(circle ΑΖΗΙ):(circle Ϙ).

So Step 12 easily derives (circle ΑΖΗΙ):(circumscribed figure)<(circle ΑΖΗΙ):(circle Ϙ).

[325] *Elements* V.10. [326] Step 5. [327] Proposition 22 Cor.

[328] Follows directly from Step 16.

[329] Proposition 12, taken for granted at this point.

falling from Θ on the circumference of the circle, smaller than these <=the lines falling on the spiral> in multitude by one, while each is equal in magnitude both to each other as well as to the greatest <=among the lines falling on the spiral>, and similar sectors are drawn on the <lines> exceeding each other and on the <lines> equal to the <greatest>. (19) Therefore the sectors on the <lines> equal to the greatest have to the sectors on the <lines> exceeding each other by an equal <difference>, apart from the <sector> on the greatest <line>, a greater ratio than the square on ΘA to the <areas> taken together: both the <rectangle> contained by AΘ, ΘE as well as the third part of the square on EA.[330] (20) But the sectors on the <lines> exceeding each other by an equal <difference>, apart from the <sector> on the greatest <line>, are equal to the figure inscribed in the area, (21) while the other <sectors = those on the lines equal to the greatest> are <equal> to the circle.[331] (22) Now then, the circle AZHI has a greater ratio to the inscribed figure than the square on ΘA to the <rectangle> contained by ΘA, ΘE and the third part of the square on AE,[332] (23) that is, the circle AZHI to the circle Ϙ.[333] (24) Therefore the circle Ϙ is greater than inscribed figure;[334] (25) which indeed is impossible; (26) for it was smaller.[335] (27) Therefore the circle Ϙ is not smaller, either, than <the> figure contained by both the spiral ABΓΔE, as well as the line AE. Thus <it is> equal.

/COROLLARY/

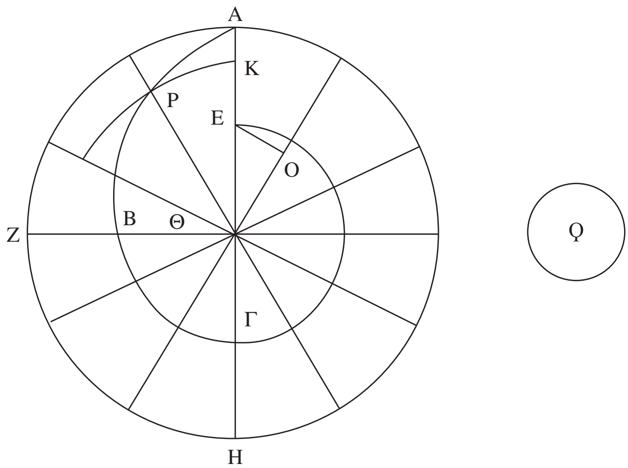
Through the same manner it shall also be proved that the area contained by both the spiral drawn in whichever rotation, as well as the line counted by the same number as the rotations, has to the circle counted by the same number as the rotations a ratio which both, taken together: the <rectangle> contained by the radius of the circle <counted> by the same number, and by the radius of the <circle> counted by the <number> of rotations smaller by one, as well as the one-third of the square on the difference, by which the radius of the greater circle of those mentioned exceeds the radius of the smaller circle of those mentioned, have to the square on the radius of the greater circle of those mentioned.

[330] The Step referring to the result of proposition 11 (such as Step 8 above) is now omitted: the result is now routinized. A lovely detail is added by the reliance of the two parts of this proof on two separate clauses of the same corollary to proposition 11.

[331] In both of Steps 20–21 I rearrange the cases, as in Step 9 above.

[332] This restates Step 19 based on the re-identifications of Steps 20–21, in the manner of Step 12 above.

[333] This is the claim of Step 11 above. [334] *Elements* V.10. [335] Step 17.



Heiberg: an 8-division a complete spiral (I now recant the complete spiral drawn in Netz et al. 2011: 179: C seems to have the same topology here as A). Heiberg also inserts a letter Δ where the continuation of the line ZBΘ cuts the spiral again, and a second Z (!) where it cuts the circle again. EO was drawn as a straight line by codex A, turned into an arc by codices BD (codex C is impossible to read here). Codex C may have missed the letter B. Codex A had Ξ for Z, corrected by BDG. Codex 4 positions the letter Γ at the hour 5, instead of 6.

COMMENTS

The proposition, once again, is no more than a construction together with a direct application of two ideas: the identification of a difference between two series (proposition 22); the summation of such a series (proposition 11). It brings no new difficulties of mathematical understanding. However, together with its corollary, it brings up in a new way the question of generality, and the manner in which a particular claim is meant to support a general conclusion.

I will explain: Greek mathematics always operates via particular specimens (in some sense) displayed as diagrammatic objects. The proof is general because the operations upon the particular specimens are seen to be repeatable for any case in which the particular specimen would correspond to the general formula for which the general claim is made. Generalization depends on the clarity of the relation between general formula and particular specimen. This is all spectacularly eroded in this proposition.

First, Archimedes commits a sleight of hand that undermines our sense of what the general formula is even supposed to be. The enunciation makes not one claim, but two: that the spiral area studied would have to the circle the ratio of 7:12; and that this ratio is the same as the ratio of certain radius constructions. So what does the proposition even *claim*? That Spiral: Circle::7:12, or that Spiral:Circle::(a certain radius construction):(another radius construction)? The definition of goal asserts:

It is to be proved that the area contained by both: the spiral ΑΒΓΔΕ as well as the line ΑΕ, has to the circle ΑΖΗΙ the ratio which 7 has to 12.

This seems to establish that the claim is Spiral:Circle::7:12 and that 7:12:: (a certain radius construction):(another radius construction) is a mere “extra” thrown into the enunciation without any motivation. Indeed, the equivalence is never accounted for in the course of the proof, and the reader has to verify it for himself (as I provide in n. 310 above) already in the course of the enunciation. Thus the enunciation serves – a remarkable exception! – already as the site of a demonstrative claim.[336]

The next stage of the argument complicates the picture further. An additional circle Ϙ is constructed in Step a, so that its radius stands to that of the second circle in the ratio described in the ratio of the constructions of radii in the enunciation. It is then asserted in Steps 1–2 (once again, no argument being made for the validity of the equivalence other than a general reference to the relationship between squares and circles from *Elements* XII.2) that the circle Ϙ would also stand in the ratio 7:12 to the second circle. The definition of goal is then reformulated to refer to the specific circle Ϙ. The confusion resides in the failure to articulate which property of the circle Ϙ would be relevant to the proof. Is the proof valid as long as it relies upon the circle Ϙ being at the ratio 7:12 to the second circle? Or is the proof valid as long as it relies upon that circle's radius standing in a particular ratio? The enunciation suggests the former (it claims directly the result 7:12, bringing in the ratio of radii as an afterthought). The construction suggests the latter (the circle is primarily constructed in terms of the ratio of radii, which is equated as an afterthought with the ratio 7:12).

The question is moot within the terms of the second spiral alone. So far, the proposition merely toyed with the practice of obfuscating one's goals and thus obscuring the grounds for generality. There is confusion aplenty, but no logical difficulty. This changes as we move into the corollary – which is obviously, in some sense, the entire point of the proposition. For now we are told that "it shall be proved" "in the same manner" that the same result holds for further rotations. And at this point we need to understand just what "the same result" is – just what was the result obtained in this proof?

Clearly the generalization has nothing to do with 7:12, and no numerical terms are mentioned explicitly at all. So one is to generalize the construction of radii which, however (and now the problem explodes), was never set out in generalizable terms.

The enunciation (as well as Step a, in the particular terms of the diagram) set up the construction in the following terms:

<the ratio> which <the areas> taken together, both: the <rectangle> contained by the radius of the 2nd circle and by the radius of the 1st circle, as well as the third part of the square on the difference by which the radius of the 2nd circle exceeds the radius of the first circle, have to the square on the radius of the 2nd circle.

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[336] Even this (elementary) claim is not entirely transparent, however, as the relation is verified through transferring the square on the difference between AE, EΘ to the square on EΘ. Why does the proposition not refer directly to the square on EΘ, then? Because it relies in general on this square for the difference, which only in this particular happy case allows the easy reformulation of the ratio. The difference between the two adjacent lines (the value AΘ-EΘ) just so happens, in this case, to be the same as the smallest line (EΘ): the same will not be true for further rotations, and it will no longer be so easy to resolve the ratio into numerical terms.

As we wish to generalize this from "second rotation" to "nth rotation," we assume that any mention of "second" should be translated into "nth." This should give us:

<the ratio> which <the areas> taken together, both: the <rectangle> contained by the radius of the nth circle and by the radius of the 1st circle, as well as the third part of the square on the difference by which the radius of the nth circle exceeds the radius of the first circle, have to the square on the radius of the nth circle.

The corollary, however, stipulates a different ratio (using the same "nth" terminology):

<the ratio> which <the areas> taken together, both: the <rectangle> contained by the radius of the nth circle and by the radius of the **n-1th circle**, as well as the third part of the square on the difference by which the radius of the nth circle exceeds the radius of the first circle, have to the square on the radius of the nth circle.

In other words, we never had a meta-enunciation anticipating the generalization to any rotation: at this logical level, the proof works as a direct transition from a particular case to its general statement in a formula, and clearly a particular case can be seen to fall under various, different formulae, so that it cannot justify, in and of itself, any particular transition to generalization.

In this case a genuine ambiguity is at stake: the formula under which EΘ is supposed to fall. Does it function as the radius of the first circle, or does it function as the radius of one-before-second circle? Archimedes, if anything, tends to lead us in the wrong direction here, since in his enunciation he refers to "1st circle," not to "the circle counted by the number smaller by one than the 2nd" – as well he should, however, seeing that the enunciation does not serve as a specimen of the meta-generalization into the case of any rotation, but rather as the general formula for the specific case of the second circle!

The only way, indeed, to derive the meta-formula would be to look at the application of the construction of Step a – the way in which the construction of the circle justifies the steps of the proof.

Here it is clear, indeed, that being at the ratio 7:12 is a consequence of, and not an underlying reason for, the validity of the argument. The numerical ratio never plays a role in the proof. However, the construction of radii comes in twice, in both parts of the exhaustion argument, the derivation from Step 6 to 7 and the derivation from Step 18 to 19. Those derivations claim that since the series of sectors in the circumscribed or inscribed figure are on sides in an arithmetical progression, it follows that the ratio of "all the sectors on the line equal to the greatest" (=the outer circle, AZHI) to the "sectors on the sides exceeding each other by an equal difference" (circumscribed or inscribed figure) is greater or smaller than the ratio defined by the construction of radii – all according to the terms of proposition 11.

And so we need to find what ΘE – the smallest of the lines – means in terms of proposition 11. Does it mean what is here "the radius of the circle n–1"? Or does it mean what is here "the radius of the smallest circle"?

This in fact does not emerge directly from proposition 11 itself, whose terms, of course, do not refer to any particular circles. Proposition 11 is stated in the general terms of progressions. So this makes our work harder. But there is worse than this, since, in fact, proposition 11 is not merely highly general: it is also obfuscating *in exactly the same manner of proposition 25*. It stated, as we recall, a more general claim about any arithmetical progression which was then spelled out in terms that seemed to refer to a particular arithmetical progression whose first term was equal to the difference of the progression.

It is this very first term which translates, in proposition 25, into the radius ΘE. And so, as we seek guidance, from proposition 11, as to the identity of the crucial radius, we find an ambiguous answer – indeed, the casual reader is likely to assume (for reasons which were set out in my comments on proposition 11) that the proof there was indeed meant to hold for the case where the smallest term is equal to the difference, and so we need to assume here that ΘE is equal to the "difference."

Both proposition 11 and proposition 25 (referring to the labels of the original figures on pp. 88, 157) suggest two different ways of being interpreted as a generalization, both based on two different interpretations one may assign to the same term.

In proposition 11, the two interpretations arise from the two separate functions of the line ΞN: it can serve just as the smallest term in a (generalized) arithmetical progression ΞN, ΛM, IK, etc.; or it can serve as the difference between the terms in this progression (equal to ϘM, ϡK, etc.).

In proposition 25, the two interpretations arise from the two separate functions of the line ΘE. It can serve either as the radius of the first circle, or as the radius of the circle-short-by-one of the second circle.

We face the fantastic spectacle of the equivocation of the generalization in the corollary of proposition 25, seeking resolution in the equivocation of the proof of proposition 11 – and following upon the equivocation of the terms of the enunciation of proposition 25.

The constraint on our interpretation is that proposition 11 is to apply in proposition 25 – which settles both questions simultaneously. For proposition 11 can make no sense in proposition 25 if it is meant to apply for the smallest term which is also equal to the difference (this would have been the side of the smallest sector – which, in this case, is no longer part of the circumscribed or inscribed figures, for, as we move beyond the first circle, our sectors begin "in the middle" of the series, and the arithmetical progression no longer sets out from the smallest member equal to the difference). Thus ΘE stands for the term in which an arithmetical progression begins whose final term is ΘA. Clearly, in general this would be the radius of the "n–1" circle. This is how we can discover what ultimately is the *manner* of this proof through which we can go on proving "in the same manner."

I do not see that past readers of Archimedes were much perturbed by all of this. Dijksterhuis notes that (1987: 277) "in order to elucidate Archimedes' result we give the discussion – which is indirectly synthetical in his treatise – in the analytical form," and on the next page he merely adds, "To the . . . sectors of circles applies the proposition [spir. 11] provided it is freed of the restrictive condition that the common difference . . . should be equal to the least term."

This would suggest that the difficulty of the proposition is an inevitable result of its synthetic form, and that the application of proposition 11 is a mere detail, perhaps an authorial oversight.

But I find it hard to believe that the presentation of proposition 11 is an authorial oversight. Archimedes was certainly not so naive as to miss the difficulty – and, once noticed, it could have been easily fixed by the mere addition of a single label to the diagram. Whatever is going on in proposition 11, it is intentional.

And the same must be true for proposition 25 as well. In fact, the difficulty has little to do with synthesis vs. analysis (Dijksterhuis' presentation is not so much "analytical," in fact, as it is discursive). It arises primarily from the reference to the circle, in the enunciation, as "first," rather than as "smaller by one than the second" or, more deeply, from the very treatment of the particular proof of the case of the second rotation as if it provided in and of itself the formula for the generalization into the case of any rotation. What is at stake, logically, is not the Greek practice of presenting results in a synthetic manner, but rather the Greek practice of letting generalization arise from following the manner in which a general formula is manipulated in a particular case – which makes it very hard to construct generalizations of generalizations. The only safe way to avoid this difficulty, in fact, would have been to take the statement of the corollary as the enunciation of the proof, and then take the second rotation as an arbitrarily selected rotation to which the proof applies as well as to any other.

Now, it is reasonable to suggest that at least one reason Archimedes did not make the claim of the corollary in the enunciation of proposition 25 was that he did not want to get rid of the ratio 7:12. For, after all, having stated the general claim for any rotation, it would have been a redundant detail to add that the value turns out to be a particular numerical ratio in the case of the second rotation. This accounts, in a stroke, for both of our problems: why Archimedes sets out a dual enunciation; and why it is so difficult to wring out of the proposition its correct generalization. The root cause is Archimedes' desire to highlight the detail of one particular configuration among the many to which the proof applies: this, then, would be exactly analogous to Archimedes' choice to highlight the particular result of proposition 18 – with its own concomitant failure of generality, based on the desire to highlight a simple numerical ratio.

In short, the goal around which this proposition is structured is the elegance of the ratio of 7 to 12. It is indeed a striking ratio, bringing together two highly resonant numbers. I was only half-facetious when suggesting, in a footnote, that Archimedes could equally have called it, as far as the reader was concerned, "the ratio of the Gates of Thebes to the Labors of Heracles." The number "appears" meaningful. Indeed, one is most closely reminded of *The Sand-Reckoner*, where the discussion ends up with the value – the number that exceeds the number of sand (Heiberg 1913: II, 258.4–5):

A' myriads of the seventh numbers

A long calculation involving the earth, the sun, the entire cosmos and a complicated system of numbers ends up with a minimally elegant expression around the number “7.” Something not dissimilar happens here, as the complexities of the diagram, the construction of radii – as well as the barely legible proposition 11! – now combine in this neat pattern with its ratio of 7 to 12.

If there is one principle underlying our difficulties with both propositions 11 and 25 it is that the elegance of the outcome seems to come at a premium. And if a consequence of such elegance of the outcome is a certain mystification concerning the process, then this is by no means a hindrance but is, rather, embraced by Archimedes as a feature of his style. I was struck, in this case, by the coincidence of the mystification at the heart of proposition 25 with that at the heart of proposition 11. It could well be just a coincidence; but we should not rule out the possibility that this is an intentional pattern, Archimedes seeking out, explicitly, the elegant structure of a double mystification.

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The area contained both by the spiral, which is smaller than the <spiral> drawn in one rotation, not having the start of the spiral as its end, as well as by the lines drawn from its ends to the start of the spiral, has to the sector – having the radius equal to the greater of the lines drawn from the ends to the start of the spiral, and the circumference which is between the mentioned lines on the same side as the spiral – that ratio which <the areas> taken together, both the <rectangle> contained by the lines drawn from the ends to the start of the spiral, as well as the third part of the square on the difference, by which the greater of the mentioned lines exceeds the smaller, have to the square on the greater of the lines joined from the ends to the start of the spiral.

Let there be a spiral, on which <is> the <line> ABΓΔE, smaller than the <spiral> drawn in one rotation, and let its ends be A, E, and let <the> start of the spiral be the point Θ, and let a circle be drawn with center Θ and radius ΘA, and let ΘE fall on its <=the circle’s> circumference at Z. It is to be proved that the area contained by both the spiral ABΓΔE and the lines AΘ, ΘE has to the sector AΘZ that ratio which <the areas> taken together, both the <rectangle> contained by AΘ, ΘE, as well as the third part of the <square> on EZ, have to the square on ΘA.

(a) So, let there be a circle, in which <are> QX, having the radius equal in square to both the <rectangle> contained by AΘ, ΘE as well as the third part of the <square> on EZ, (b) and <let there be> at its center an angle equal to the <angle> at Θ.[337] (1) So, the sector QX has

[337] A meaningless direction in and of itself, meant to be explicated by common sense as well as by the diagram: the angle in the auxiliary circle is equal to the angle AΘE containing the spiral area under consideration.

to the sector ΘAZ the same ratio which the <rectangle> contained by AΘ, ΘE and the third part of the square on EZ have to the square on ΘA; (2) for the radii have to each other the same ratio in square.[338] So, the sector ϘX shall be proved to be equal to the area contained by both: the spiral ABΓΔE, as well as the lines AΘ, ΘE.

(3) For if not, it is either greater or smaller. (c) Let it first be, if possible, greater. (4) Now, it is possible to circumscribe a plane figure around the mentioned area <=sector of the spiral> composed of similar sectors, so that the circumscribed figure is greater than the said area by a <magnitude> smaller than as much as the sector ϘX exceeds the said area.[339] (d) So, let it be circumscribed, and let <the> greatest of the sectors of which the circumscribed figure is composed be the sector ΘAK, and <the> smallest, the <sector> ΘOΔ. (5) Now, it is clear that the circumscribed figure is smaller than the sector ϘX.[340] (e) So let the lines making equal angles at Θ be drawn through so far as <to extend> to fall on the circumference of the sector ΘAZ. (6) So, there are certain straight <lines>[341] exceeding each other by an equal <difference>[342] – those falling on the spiral from Θ – of which ΘA is <the> greatest, while ΘE <is the> smallest, and there are also other lines, those falling from Θ on the circumference of the sector AΘZ, smaller than these <=lines falling on the spiral> in multitude by one, while each is equal in magnitude both to each other as well as to the greatest <=among the lines falling on the spiral>, and similar sectors have been drawn both: on all the lines equal both to each other and to the greatest <line>; as well as on the <lines> exceeding each other by an equal <difference>; but on ΘE <a sector> is not drawn. (7) Now, the sectors on the <lines> equal both: to each other as well as to the greatest <line> have to the sectors on the <lines> exceeding each other by an equal <difference>, apart from the <sector> on the smallest <line>, a smaller ratio than the square on ΘA, to the <areas> taken together: both the <rectangle> contained by AΘ, ΘE as well as the third part of the square on EZ.[343] (8) But the sectors on the <lines> equal both to each other and to the greatest <line> are equal to the

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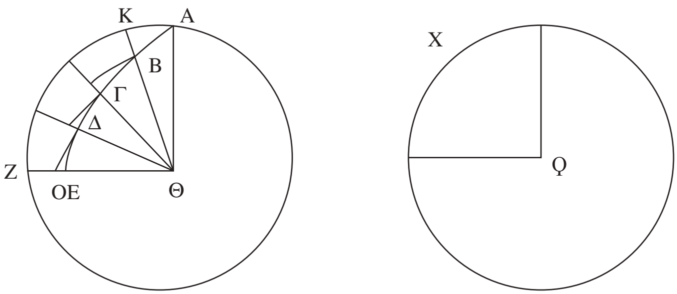
[338] Step 2 restates Step a and recalls *Elements* XII.2. The extra requirement that sectors are to each other as their angles is not explicitly recalled, perhaps because it is not indeed in Euclid: this, once again, belongs to the field of extensions of proportion theory to sectors of circles, which this book often requires and always takes for granted.

[339] Proposition 23 Cor. [340] A restatement of Step 4.

[341] I usually do not distinguish εὐθεῖαι (“straight”) from γραμμαί (“lines”) when both are used in the meaning “straight lines,” either noun or adjective implied; I translate here εὐθεῖαι by “straight <lines>” to signal the surprising transition (this phrase so far always used γραμμαί, starting at proposition 10 itself).

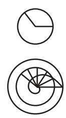
[342] Proposition 12, taken for granted at this point. [343] Proposition 11 Cor.

sector ΘAZ, (9) while the <sectors> on the <lines> exceeding each other are equal to the circumscribed <figure>.[344] (10) Now then, the sector ΘAZ has to the circumscribed figure a smaller ratio than the square on ΘA to the <areas> taken together, both the <rectangle contained> by ΘA, ΘE as well as the third part of the <square> on ZE. (11) But the ratio which the <square> on ΘA has to the mentioned <areas> is that ratio which the sector ΘAZ has to the sector ϘX;[345] (12) so that the sector ϘX is smaller than the circumscribed figure.[346] (13) But it is not <smaller>, but greater.[347] (14) Therefore the sector ϘX shall not be greater than the area contained by both the spiral ABΓΔE and the lines AΘ, ΘE.



Beginning with this figure, those for codex C have not been drawn in (although space has been prepared for them). Heiberg's diagram is radically different, as in the thumbnail.

Heiberg



Nor, furthermore, smaller. (f) For let it be smaller, and let the rest be constructed the same. (15) So, again, it is possible to inscribe inside the area a plane figure composed of similar sectors, so that the mentioned area is greater than the inscribed by a <magnitude> smaller than as much as the same area exceeds the sector ϘX.[348] (g) Now, let it be inscribed, and let <the> greatest of the sectors of which the inscribed figure is composed be the sector ΘBΓ, and <the> smallest, the <sector> OΘE. (16) Now, it is clear that the inscribed figure is greater than the sector ϘX.[349] (17) Now, again, there are certain lines

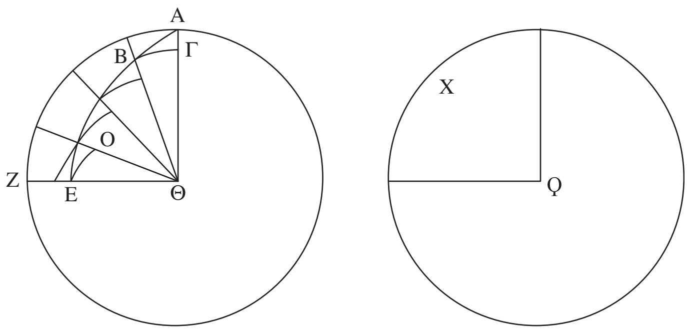
[344] In both of Steps 8–9, the original case arrangement was "to the sectors . . . is equal . . . ."

[345] Recalling Step 1. Step 10 is: (sector ΘAZ):(circumscribed figure)<(square):(areas). Step 11 is (square):(areas)::(sectorΘAZ):(sector ϘX). The two taken together imply: (sector ΘAZ):(circumscribed figure)<(sectorΘAZ):(sector ϘX). The following Step 12 relies on this implicit result.

[346] This is based on the implicit result obtained in the preceding footnote, and then *Elements* V.10.

[347] Step 5. [348] Proposition 23 Cor. [349] Restating Step 15.

exceeding each other by an equal <difference>[350] – those falling on the spiral from Θ – of which ΘA is <the> greatest, while ΘE <is the> smallest, and there are also other lines, those falling from Θ on the circumference of the sector ΘAZ, apart from ΘA, smaller than the lines exceeding each other by an equal <difference> in multitude by one, while each is equal in magnitude both to each other as well as to the greatest <=among the lines falling on the spiral>, and similar sectors are drawn on each <line>, but <a sector> is not drawn on the greatest of the <lines> exceeding each other by an equal <difference>. (18) Now, the sectors on the <lines> equal both to each other and to the greatest <line> have to the sectors on the <lines> exceeding each other by an equal <difference>, apart from the <sector> on the greatest <line>, a greater ratio than the square on ΘA to the <rectangle contained> by AΘ, ΘE and the third part of the square on EZ; (19) so that the sector ΘAZ, too, has to the inscribed figure a greater ratio than to the sector ϘX.[351] (20) Thus the sector ϘX is greater than the inscribed figure.[352] (21) But it is not <greater>, but smaller.[353] (22) Therefore the sector ϘX is not smaller, either, than the area contained by both the spiral ABΓΔE as well as the lines AΘ, ΘE. (23) Therefore <it is> equal.



Codices BDG do not extend the arc from the "nine o'clock" position to the "half past ten" position all the way, leaving only its segment from the "quarter to ten" position to the "half past ten" position. Heiberg has a very different diagram, as in the thumbnail.

Heiberg

[350] Proposition 12, taken for granted at this point.

[351] The transition from Step 11 to 12 above was elliptic; now the entire passage homologous with Steps 8 to 11 above is elided. The line of thought is by now indeed familiar. If Step 18 is A:B::X:Δ, then we have the following manipulations: (A) the sectors on the equal lines are the same as the sector of the circle; (B) the sectors on the unequal lines are the same as the inscribed figure; (C:D) the complex ratio of areas is the same as the ratio of the sector of the circle to the area of the spiral. We now substitute all the terms of Step 18 to derive Step 19.

[352] *Elements* V. 10. [353] Step 16.

COMMENTS

First, a note on the use of labels in the diagrams. The use of the terms “QX” is at first glance surprising: we would have thought that the reason there are two labels and not one, in this case, is because we label separately the sector required for the proof as well as its complementary. Instead, the sector under study is (implicitly) understood to be QX as a whole. Thus the two labels are there to bind a single sector. From the diagram, it appears that Q is understood to stand at the center, while X stands at an arbitrary point on the sector’s circumference: the function of the letter X, then, is to pick out the correct sector by picking out the correct circumference via an arbitrary point established upon it. This is a trivial detail, but it suggests an interpretation for the use of auxiliary circles in the preceding propositions as well: was the label Q perhaps always understood to refer to the center of the auxiliary circle, and only by extension to the circle as a whole? If so, the expression “the circle in which Q is” referred to a concrete geometrical relation (the circle in which the center point Q is) and not to a more abstract, purely “quantitative” object such as “the circle designated by Q.” So even the detached, “abstract” objects of the diagram are considered in terms of a geometrical configuration.

Now, as for the contents of this proposition: as is by now familiar, it is not at all clear why proposition 26 requires more explicit statement than the preceding corollary. After all, it would be technically true to state that “Through the same manner it shall also be proved that [the partial spiral area is to the circular sector surrounding it in the same ratio defined by radii].” The proof flows from exactly the same observation as those of propositions 24–25: that equiangular lines extended from the center to the spiral define an arithmetic progression, which entails the application of proposition 11 (or, in the special case where the series begins at a line equal to the difference, i.e. right from the start of the spiral, it entails the application of proposition 10), and nothing in the application of proposition 11 is affected if we choose a progression that goes through an entire “circuit” of the circle, or through just a segment of that circuit (Dijksterhuis 1987: 280, on proposition 26: “The argument of proposition 25 can be used unchanged”).

Just how obvious is that to the reader? For most readers – including of course me – this is not obvious at all. The proof is sufficiently difficult for the reader to suspend judgement and just give Archimedes the benefit of the doubt that the bookkeeping was done properly. And, once reading is based on such suspension of judgement, the reader can no longer tell whether the same bookkeeping would hold in some other related configuration: for, you see, the reader no longer has access to the bookkeeping.

This, then, is a good moment to review the bookkeeping of propositions 10–11, 24–26. Bear with me.

Let us first transform the terms of propositions 10–11 into the geometric configurations of a series of circular sectors. Obviously, the series of *squares on lines equal to the greatest* becomes, simply, the circle (or any sector thereof).

The “squares on the lines exceeding each other” have a meaning when we begin at the very start of the spiral different from that when we start elsewhere.

At the very start of the spiral, the *squares on the lines exceeding each other* becomes the figure composed of sectors circumscribing the spiral area (quite obviously); while *squares on the lines exceeding each other without the greatest* becomes the figure composed of sectors inscribed in the spiral area. This is because the circumscribed and the inscribed figure differ by a sector equal to the greatest sector.

When we begin at a point other than the start of the spiral, the difference between the circumscribed and inscribed becomes more complicated: the circumscribed is greater, in that it has the greatest sector, but it is smaller, in that it does not have the smallest sector. Thus the circumscribed figure is the "squares on the lines exceeding each other, greatest included, smallest excluded," while the inscribed figure is the "squares on the lines exceeding each other, greatest excluded, smallest included." Either way, we exclude one term in the sequence so that the number of terms in the sequence of "squares on the lines exceeding each other" is, in this case, smaller by one than the total number of terms in the entire series from smallest to greatest. To rebalance the bookkeeping, then, the number of terms in the sequence of "squares on the lines equal to the greatest" is made, in this case, to be smaller by one than the number of terms in the entire series from smallest to greatest.

At this point, it is easy to translate proposition 10 corollary into the terms of geometrical configuration: that the circle is less than three times the circumscribing figure, but more than three times the inscribed figure.

Proposition 11 translates in a somewhat more complicated way. Broadly, we see a similar result: the circle is less than [a certain ratio] to the circumscribing figure, but more than the same ratio to the inscribed figure. To establish the correct bookkeeping, though, we make the number of terms equal to each other, smaller by one than the number of terms in the series from smallest to greatest. This is achieved by having the series from smallest to greatest contain the sectors in question "from both ends" – in the case of proposition 25, the series has nine terms from ΘA to ΘE, as against the eight lines in the series of terms equal to the greatest ΘA; in the case of proposition 26, the series has five lines from ΘA to ΘE, as against the four from ΘA to the extension of ΘΔ which are the terms equal to the greatest ΘA.

We can now indeed verify that the "squares on the lines exceeding each other, greatest included, smallest excluded" correspond, indeed, to the circumscribed figure; "squares on the lines exceeding each other, greatest excluded, smallest included" correspond, indeed, to the inscribed figure. If there is any reason to distinguish between propositions 25 and 26, it is that the relation between the two series becomes apparent somewhat more obviously in proposition 25: the series of terms from smallest to greatest counts "the same line" twice over – as ΘA and as its smaller segment ΘE. The series taken "from both ends" coils, in this case, upon itself. The same series in proposition 26 does not yet coil upon itself, and so we simply compare the four sectors to the five lines surrounding them from both sides. Perhaps Archimedes thought this difference is what merited a separate treatment (indeed, the same difference does not recur in the corollary to proposition 25, which provides exactly the same series: it is difficult to generalize from proposition 25 to its corollary for the different reasons explained in the preceding comments – that Archimedes never explained what general formula derives the particular ratio of proposition 25 so that formally it resists generalization).

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Of the areas contained by both the spirals and the lines in the rotation, the 3rd is twice the 2nd, the 4th <is> three times, the 5th <is> four times, and always: the following <area is>, according to the numbers in sequence, a multiple of the second area; while the 1st area is a sixth part of the second.

Let there be the spiral set forth, drawn in the first rotation as well as the second and however many following <rotations>, and let the point Θ be <the> start of the spiral, and the line ΘE, the start of the rotation, and among the areas: let the <area> K be the 1st, the <area> Λ – the 2nd, the <area> M – the 3rd, the <area> N – the 4th, the <area> Ξ – the 5th. It is to be proved that the area K is a sixth part of the following <area>, while the <area> M is twice the <area> Λ, the <area> N is 3-times the <area> Λ, and always: the following <area> of those in sequence is a multiple of the <area> Λ, according to the numbers in sequence.

First now, that the <area> K is a sixth part of the <area> Λ shall be proved like this. (1) Since it has been proved that the area KΛ[354] has that ratio to the second circle which 7 has to 12,[355] (2) while the second circle <has> to the first circle as 12 to 3 ((3) for it is clear),[356] (4) and the first circle has to the area K, as 3 to 1,[357] (5) therefore the area K is a 1/6th <part> of the <area> Λ.[358] (6) And again, it has been proved that the area KΛM, too, has that ratio to the third circle which <the areas> taken together, both the <rectangle contained> by ΓΘB and the third part of the square on ΓB, to the square on ΓΘ,[359] (7) while the third circle has to the second circle <the ratio> which the square on ΓΘ has to the <square> on ΘB,[360] (8) and the second circle has to the area KΛ <the ratio> which the square on BΘ has to the <areas> taken together, both the <rectangle contained> by BΘ, ΘA as well as the

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354 Understood to mean the areas K and Λ taken together. 355 Proposition 25.

356 *Elements* XII.2. The explicit statement that this is clear refers to the observation that the first and second lines together are double the first line.

357 Proposition 24.

358 K and Λ together are as 7 to 12; K alone is as 1 to 12; so Λ alone is as 6 to 12 and K is to Λ as 1 to 6.

359 Proposition 25 Cor.

360 *Elements* XII.2. Implied result of Steps 6–7: (area KΛM):(second circle)::(rect.(ΓΘB) + $\frac{1}{3}$sq.(ΓB)):(sq.(ΘB)) (we basically just have to "cut out the middleman," the third circle on one side of the sequence of ratios, the square on ΓΘ on the other side: this follows *Elements* V.22).

third part of the square on AB.[361] (9) Therefore the <area> KΛM, too, has to the <area> KΛ a ratio which the <rectangle contained> by ΓΘ, ΘB and the third part of the <square> on ΓB has to the <rectangle contained> by BΘ, ΘA and the third part of the <square> on AB.[362] (10) But these have to each other a ratio which 19 <has> to 7.[363] (11) Thus the area KΛM, too, has to the area ΛK that ratio which 19 <has> to 7. (12) Now, the <area> M itself has to the <area> KΛ a ratio which 12 <has> to 7,[364] (13) while the <area> KΛ has to the <area> Λ a ratio which 7 has to 6.[365] (14) Now, it is clear that the <area> M is twice the <area> Λ.[366]

And it shall be proved that the following have the ratio of the numbers in sequence. (15) For the <area> KΛMNΞ has to the circle, whose radius is ΘE, that ratio which <the areas> taken together, both the <rectangle contained> by EΘ, ΘΔ as well as the third part of the square on ΔE, have to the square on ΘE,[367] (16) while the circle, whose radius is ΘE, has to the circle, whose radius is ΘΔ, that ratio which the square on ΘE has to the square on ΘΔ,[368] (17) and the circle, whose radius is ΘΔ, has to the area KΛMN that ratio which the square on ΘΔ has to the <areas> taken together, both the <rectangle contained> by ΘΔ, ΘΓ as well as the third part of the square on ΔΓ;[369] (18) therefore the <area> KΛMNΞ, too, has to the <area> KΛMN a ratio which the <rectangle contained> by ΘE, ΘΔ and the third part of the <square> on ΔE have to the <rectangle contained> by ΔΘ, ΘΓ and the third part of the <square> on ΔΓ.[370] (19) Dividedly,[371] the area Ξ, too, has to the <area> KΛMN a ratio which the difference between the two: the <rectangle contained> by EΘ, ΘΔ together with the third part of the <square> on EΔ; and the <rectangle contained> by ΔΘ, ΘΓ

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[361] Proposition 25. Combining this with the previous implied result we gain the result of Step 9, through a similar application of *Elements* V.22 (which we may also imagine applied just once, on the combination of Steps 6–8 taken together).

[362] Step 9 was lost (a homoioteleuton, if we adopt, following Heiberg, Commandino's emendation) by the time the scholia were inserted. See scholia (Appendix 2).

[363] If we scale each line segment such as ΘA to be a unit, we derive ΓΘ*ΘB=6 unit squares, ΘB*ΘA=2 unit squares, and the third of the squares (equal to each other) each being, obviously, a third of a unit square, so that the ratio becomes 6-and-a-third to 2-and-a-third or indeed 19 to 7. Archimedes specifies none of this and does not even elaborate, anywhere, on the equality of all those "thirds of a square."

[364] A restatement of Step 11: *Elements* V.17.

[365] Step 6 restated, this time through *Elements* V.18.

[366] Steps 12–13 together with *Elements* V.22 yield M:Λ::12:6.

[367] Proposition 25 Cor. [368] *Elements* XII.2. [369] Proposition 25 Cor.

[370] Once again *Elements* V.22 is applied to Steps 15–17 (as it was before with the derivation of Step 9 from Steps 6–8: this time I did not provide the interim conclusions, which are essentially redundant).

[371] Asyndeton in the original; perhaps just a scribal omission.

together with the third part of the <square> on ΓΔ, has to <the areas>, both the <rectangle contained> by ΔΘ, ΘΓ as well as the third part of the <square> on ΔΓ.[372] (20) But the <areas> taken together exceed the <areas> taken together[373] <by the magnitude> by which the <rectangle contained> by ΕΘΔ, too, <exceeds> the <rectangle contained> by ΔΘΓ,[374] (21) and it exceeds <=the rectangle mentioned above> by the <rectangle contained> by ΔΘ, ΓΕ;[375] (22) therefore the <area> Ξ has to the <area> ΚΛΜΝ a ratio which the <rectangle> contained by ΘΔ, ΓΕ <has> to the <rectangle contained> by ΔΘ, ΘΓ and the third part of the square on ΓΔ.[376] (23) Through the same <arguments> the <area> Ν, too, shall be proved to have to the area ΚΛΜ that ratio which the <rectangle> contained by ΘΓ, ΒΔ has to the <areas> taken together, both the <rectangle contained> by ΓΘΒ and the third part of the square on ΓΒ; (24) therefore the <area> Ν has to the area ΚΛΜΝ that ratio which the <rectangle contained> by ΘΓ, ΒΔ <has> to the <rectangle contained> by ΘΓ, ΒΔ and the <rectangle contained> by ΘΓ, ΘΒ and the third part of the <square> on ΓΒ[377] [(25) and inversely];[378] (26) but these <=the terms in the final consequent of the proportion above> are equal to both the <rectangle contained> by ΔΘ, ΘΓ and the third part of the square on ΓΔ.[379] (27) Now, since the area Ξ has to the <area> ΚΛΜΝ that ratio which the <rectangle contained> by ΘΔ, ΓΕ has to the

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[372] *Elements* V.17. The "dividedly" operation takes A+B:B::C+D:D and turns it into A:B::C:D. In this case A+B is the area ΚΛΜΝΞ, B is the area ΚΛΜΝ; C+D is (rect.(ΕΘ, ΘΔ) + $\frac{1}{3}$sq.(ΕΔ)), while D is (rect.(ΔΘ, ΘΓ) + $\frac{1}{3}$sq.(ΔΓ). C, in this case, turns out to be a contrived subtracted object, the difference between C+D and D.

[373] "Exceed," in the original Greek, is the verb from the noun form "difference." We refer to the "C" as described in the previous footnote or to "the difference between the two: the <rectangle contained> by ΕΘ, ΘΔ together with the third part of the <square> on ΕΔ; and the <rectangle contained> by ΔΘ, ΘΓ together with the third part of the <square> on ΓΔ" from the previous Step, which is here simplified.

[374] It is now finally conceded – if implicitly – that all the "thirds of the square" will be thirds of equal squares, so that both can be commonly removed.

[375] The claim is that rect.(ΘΔ, ΘΕ)-rect.(ΔΘ, ΘΓ)=rect.(ΔΘ, ΓΕ) which follows directly from *Elements* II.1, as ΓΕ is the difference between ΘΕ and ΘΓ (we do need to reshuffle the terms mentioned in Step 20, though, to see that).

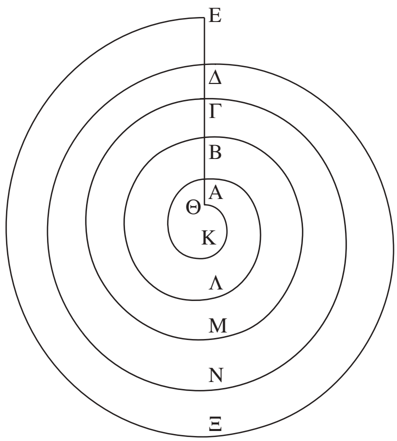
[376] Restating Step 19 via Steps 20–21. [377] *Elements* V.18.

[378] This draws the conclusion that the ratio in Step 24, A:B::C:D is also valid as B:A:: D:C (*Elements* V.7 Cor.). Heiberg brackets it, although this result would indeed be picked up by Step 27 below. I assume he did so because he considered that it interfered with the understanding of the immediately following pronoun "these," but it appears to me the pronoun remains perfectly transparent even with those two words inserted. Heiberg is right, however, that even though the hand of the scholiast has not been heavy on this treatise as a whole, the extant scholia to this proposition (see Appendix 2) are suggestive of this kind of intervention.

[379] Once again *Elements* II.1: rect.(ΘΓ, ΒΔ)+rect.(ΘΓ, ΘΒ)=rect.(ΘΓ, ΔΘ), as ΒΔ + ΘΒ=ΔΘ.

<areas> taken together, both the <rectangle contained> by ΔΘΓ and the third part of the rectangle on ΓΔ,[380] while the <area> KΛMN has to the <area> N <the ratio> which the <areas> taken together, both the <rectangle contained> by ΔΘΓ and the third part of the square on ΓΔ, <have> to the <rectangle contained> by ΘΓ, ΔB,[381] (28) therefore the <area> Ξ has to the <area> N the same ratio which the <rectangle contained> by ΘΔ, ΓE <has> to the <rectangle contained> by ΘΓ, BΔ.[382] (29) But the <rectangle contained> by ΘΔ, ΓE has to the <rectangle contained> by ΘΓ, ΔB the same ratio which ΘΔ has to ΘΓ (30) since ΓE, BΔ are equal.[383] (31) Now, it is clear that the <area> Ξ, too, has to the <area> N that ratio which ΘΔ has to ΘΓ.[384]

(32) Similarly the <area> N, too, shall be proved to have to the <area> M that ratio which ΘΓ <has> to ΘB, (33) and the <area> M to the <area> Λ <the ratio> which BΘ <has> to AΘ; (34) but the lines [EΘ],[385] ΔΘ, ΓΘ, BΘ, AΘ have the ratio of the numbers in sequence.



In the manner of propositions 10–11, the diagram contains numbers related to a scholion which is an obvious later commentary; also, redundant straight lines, obviously related to the inserted numerals. (See Appendix 2.)

[380] Recalling Step 22. [381] Restating Step 24 via Steps 25–26.

[382] In a magic trick, *Elements* V.22 is applied, and the unwieldy combinations of areas disappear, leaving us with an area to an area as a rectangle to a rectangle.

[383] The equality of the line segments is once again acknowledged; in this case we consider two instances of a double line segment. And then apply *Elements* VI.1.

[384] Steps 28–29.

[385] Heiberg brackets "EΘ" because it is not included among the terms referred to in Steps 31–33; however, the purpose of Steps 32–33 is not to add in two more terms to the single term of Step 31 but rather to imply, through two examples, how the result of Step 31 is universally generalized, and the reference to the "extra" term EΘ points directly to that generalization. What Steps 31–34 taken together claim, then, is that each area is to its internally contiguous area as the number, smaller by one in its position in the sequence of numbers, to a number smaller by one than that: the third area is twice the second, the fourth is three times the third, etc. See the comments for the practice of generalization in this proposition.

COMMENTS

The problem of generalization becomes internal to this proposition which collapses, into a single claim, the many-layered structure of previous clusters of propositions. Archimedes makes two separate claims: that the first area is one-sixth the second; and that all further n areas are to the n+1 area in the ratio n–1:n (the second is to the third in the ratio 1:2, etc.). It is not surprising that Archimedes chooses to treat the first, particular result, separately: it arises from the general treatment only as a special, limiting case, and even then it results not from the line of argument as a whole but rather from an interim conclusion reached in Step 22.[386] For this reason Archimedes does not derive the limiting case from an application of Steps 15–22, but rather calculates it directly in Steps 1–5. What is surprising is that Archimedes proves the result twice for the simplest non-limiting case, namely the one where the ratio of the areas is 1:2. This result, proved in Steps 6–14, could in principle have been seen directly as a particular case from Steps 15–34. Thus Archimedes offers three separate proofs for a theorem that bifurcates naturally into two cases, first for the first (indeed, limiting) case, then again for the second (entirely typical) case, and only then generalizing the result. This is the usual structure of two-and-more we have seen so often.

The redundancy does serve the exposition: the three proofs are independent of each other, but they gradually build up the requisite conceptual apparatus. The first proof, in Steps 1–5, brings in the elementary idea of combining the results from propositions 24–25 – for the ratio of a spiral area and a circle – together with the result of *Elements* XII.2 that circles are to each other as the squares on their diameters. The second proof, in Steps 6–14, takes this up and brings in the further idea of taking three ratios in a row and nesting them within each other: an area to a circle, a circle to a circle, a circle to an area – so that one finally finds the ratio of an area to an area. The third proof, in Steps 15–34, begins (in Steps 15–18) with an exact repetition of the previous proof-idea, to which several new refinements are added. A relatively minor detail is that the ratio of combined shells is transformed into the ratio of individual shells (Steps 19–22). A deeper idea introduced in Step 23 is to nest together two sets of results: not only the ratio of shell n+1 to shells n and lower, but also shell n to shells n–1 and lower. This allows us immediately to compare how two shells, n+1 and n, stand in a ratio to the same set of shells n and lower, from which one finds the ratio of the shells n+1 and n (Steps 24–34). Both the second and third proofs take up the previous proof-ideas and elaborate on them

[386] If we generalize the argument of Steps 15 onwards to hold across any arbitrary area, then the claim of Step 22 is that "shell" area n is to the set of all areas up to and including area n–1 as rect.(n–1, 2):rect.(n–1, n–2) + $\frac{1}{3}$. In the case of the innermost comparison – the second shell Λ to the innermost K – (n–2) is zero, while n–1 is 1: we therefore derive directly 2:$\frac{1}{3}$, or the ratio of 1 to 6. The continuation of the argument from Step 23 onwards crucially depends on the non-zero value of n-2 and so does not apply to this limiting case.

based on the idea of nesting together *several* ratios which may then be simplified.

The structure is engaging and perhaps also helps in the understanding of the proof: certainly one follows the claim of Steps 15–18 more easily, following the simple setting-out of the same argument in Steps 6–9. What is clear is that the structure also enacts the "spiralling-out" it sets out to study. Once again, one wonders if the structure of "two-and-more" may not be a deliberate aesthetic choice, fitting to the object chosen for study.

The price paid for this redundant fragmentation of the proof is a certain loss in the clarity of the generalization. Obviously, the proposition offers a redundant proof of a particular case. It also fails – once again, in a manner reminiscent of previous sequences – to make its generality explicit.

The spiral is not said to be produced "arbitrarily," to a certain or to a chance rotation. Instead, Archimedes draws a big enough spiral, namely one of five rotations (once again, the same five used to escape the danger of *small numbers*). However, this is not large enough: the first special case compares the first and second rotations; the second compares the second and the third; and while the general proof begins with the totally unrelated (and therefore apparently "floating in space" and so "arbitrary") case of the fifth and the fourth rotations; it also brings in the extra nested case of the fourth and the third rotations – hitting right up against the special second case. The reader may well wonder, until the end of the proof, if the purpose is not to build the chain of proportions down to the third shell (discussed in Steps 6–14) and so construct some kind of inductive ladder leading up from it to the fourth and fifth shells, so that the claim of Steps 15–34 might end up, still, to be about a particular result which only then could be generalized. It is only right at the end of the proof that it becomes possible to argue that nothing hinges on the particular location of the shells chosen for the third part of the proof and that Steps 15–34 are already, directly, the general proof (which would then make the second case, of Steps 6–14, redundant). But since it was not apparent to the reader, while reading the proof, that Steps 15–34 serve such a general function, there was no prompt to the reader to exercise his or her caution in verifying that the arguments made are indeed generally repeatable. The overall structure of the proof tends to obscure the arguments required for sustaining its general validity.

/28/

If two points should be taken on the spiral drawn in whichever rotation – not the end points[387] – and lines should be joined from the points taken to the start of the spiral, and circles should be drawn, with the start of the spiral as center, and the <lines joined> from the points to the start of the spiral as radii, the area contained by both the greater of the circumferences between the lines, and the spiral between the same

[387] That is, the two points taken are not on the start of the rotation.

lines, as well as the produced line,[388] has that ratio to the area taken by both the smaller circumference and the same spiral,[389] as well as the <line> joining their ends,[390] which the radius of the smaller circle together with two third parts of the difference by which the radius of the greater circle exceeds the radius of the smaller circle <has> to the radius of the smaller circle with one third part of the same difference.

Let there be a spiral, on which the <line> ΑΒΓΔ,[391] drawn in a single rotation, and let two points be taken upon it, Α, Γ, so that the point Θ is <the> start of the spiral, and let <lines> be joined from Α, Γ to Θ, and with Θ as center, and ΘΑ, ΘΓ as radii, let circles be drawn. It is to be proved that the area Ξ has to the <area> Π the same ratio which both ΑΘ and two third parts of ΗΑ, taken together, have to both ΑΘ and one third part of ΗΑ, taken together.

(1) For the area ΝΠ has been proved to have to the sector ΗΓΘ that ratio which both the <rectangle contained> by ΗΘ, ΑΘ and the third part of the square on ΗΘ have to the square on ΗΘ;[392] (2) therefore the <area> Ξ itself has to the <area> ΝΠ that ratio which the <rectangle contained> by ΘΑΗ, together with two third parts of the square on ΗΑ, has to the <areas> taken together, both the <rectangle contained> by ΑΘΗ and the third part of the <square> on ΗΑ.[393] (3) And since the area ΝΠ has to the sector ΝΠΞ that ratio which both, taken together,

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[388] "Produced" is used here in the same sense as "joined," referring to the relevant "joined" line among the two lines joined earlier.

[389] Meaning in this case the very same *segment* of a spiral line.

[390] This now refers to the line joining the ends of the smaller circumference and the spiral segment; which happens to be a segment of the other "joined" line of the original construction.

[391] The diagram has the letters ΑΓΕ (corrected by B, perhaps Coner's hand, and by Heiberg). The simplest way to account for the discrepancy would be to assume that the original text had the line ΑΒΓΔΕ, Ε dropped from the text and Β, Δ dropped from the diagram. Since we are reduced now to a single Byzantine source, since the reading as it stands remains mathematically possible – and since this is a recurrent feature of some of the diagrams – I do not attempt an emendation.

[392] Proposition 26.

[393] Step 1 is: ΝΠ:sector::rect.(ΗΘΑ) + $\frac{1}{3}$sq.(ΗΑ):sq.(ΗΑ), or (*Elements* V.7 Cor.)

sector:ΝΠ:: sq.(ΗΑ):(rect.(ΗΘΑ) + $\frac{1}{3}$sq.(ΗΘ)). But the difference between the sector and ΝΠ is the Ξ; hence Ξ:ΝΠ should be in the same ratio as sq.(ΗΘ)-(rect.(ΗΘΑ) + $\frac{1}{3}$sq.(ΗΑ)) is to (rect.(ΗΘΑ) + $\frac{1}{3}$sq.(ΗΑ)) (*Elements* V.17). We are looking at sq.(ΗΘ)-(rect.(ΗΘΑ) + $\frac{1}{3}$sq.(ΗΑ)).

By the kind of reasoning that goes into *Elements* II.4, sq.(ΗΘ) may be decomposed most naturally into sq.(ΗΑ)+rect.(ΗΘΑ)+rect.(ΘΑΗ).

It follows immediately that sq.(ΗΘ)-(rect.(ΗΘΑ) + $\frac{1}{3}$sq.(ΗΑ)) is

$$\text{sq.}(\mathrm{H}\mathrm{A}) + \text{rect.}(\mathrm{H}\Theta\mathrm{A}) + \text{rect.}(\Theta\mathrm{A}\mathrm{H}) - (\text{rect.}(\mathrm{H}\Theta\mathrm{A}) + \tfrac{1}{3}\text{sq.}(\mathrm{H}\mathrm{A}),\ \text{or}$$

$\frac{2}{3}$sq.(ΗΑ)+rect.(ΘΑΗ), hence Step 2. There is quite a lot packed in here, though all of a very elementary nature.

the <rectangle contained> by ΘA, ΘH and the third part of the <square> on HA have to the square on ΘH,[394] (4) while the sector NΠΞ has to the sector N that ratio which the <square> on ΘH <has> to the square on ΘA,[395] (5) the area NΠ, too, shall have to the <area> N the same ratio which both, taken together, the <rectangle contained> by ΘA, ΘH and the third part of the <square> on HA <have> to the <square> on ΘA;[396] (6) therefore the <area> NΠ has to the <area> Π a ratio which both, taken together, the <rectangle contained> by HΘA and the third part of the <square> on HA <have> to both, taken together, the <rectangle contained> by HA, ΘA and the third part of the square on HA.[397] (7) Now, since the area Ξ has to the <area> NΠ that ratio which both, taken together, the <rectangle contained> by ΘAH and two third parts of the square on HA have to the <areas> taken together, both the <rectangle contained> by HΘA and the third part of the <square> on HA,[398] (8) while the area NΠ has to the <area> Π that ratio which the <areas>, taken together, both the <rectangle contained> by HΘA as well as the third part of the square on HA, <have> to both, taken together, the <rectangle contained> by HAΘ and the third part of the square on HA,[399] (9) <therefore> the <area> Ξ, too, shall have to the <area> Π that ratio which both, taken together, the <rectangle contained> by ΘAH and two third parts of the <square> on HA have to both, taken together, the <rectangle contained> by ΘAH and the third part of the <square> on HA.[400] (10) But the <areas> taken together, both the <rectangle contained> by ΘAH and two third parts of the <square> on HA, have to both, taken together, the <rectangle contained> by ΘAH and the third part of the square on HA that ratio, which both, taken together, ΘA and two third parts of HA have to both, taken together, ΘA and the third part of HA;[401] (11) now, it is clear that the area Ξ, too, has to the area Π that ratio which both, taken together, ΘA and two third parts of HA <have> to both, taken together, ΘA and the third part of HA.

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[394] This only slightly reformulates the claim of Step 1.

[395] An extension of *Elements* XII.2. [396] *Elements* V.22.

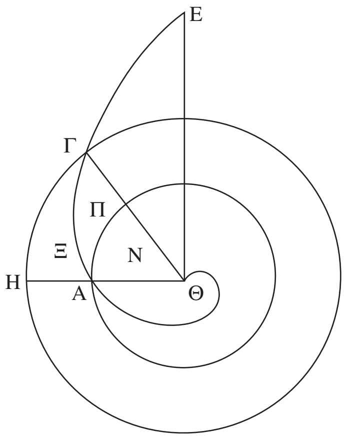
[397] Step 5 is $N\Pi : N :: (\text{rect.}(\Theta A,\Theta H)+\frac{1}{3}\text{sq.}(HA)) : \text{sq.}(\Theta A)$. Π, obviously, is the difference between NΠ and N, so from *Elements* V.19 Cor. we know that:

$$N\Pi : \Pi :: (\text{rect.}(\Theta A, \Theta H) + \tfrac{1}{3}\text{sq.}(HA)) : (\text{rect.}(\Theta A, \Theta H) + \tfrac{1}{3}\text{sq.}(HA)) - \text{sq.}(\Theta A).$$

However, as noted already through the reasoning of n. 393 above, rect.(ΘA, ΘH)-sq.(ΘA) is rect.(ΘA, AH). Hence the result of Step 6.

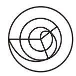
[398] Step 2 recalled. [399] Step 6 recalled. [400] *Elements* V.22.

[401] *Elements* VI.1: all the terms in the first proportion are parallelograms (rectangles or squares) one of whose sides is under the height HA.

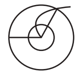


See n. 391 for the revisions by B and Heiberg, whose diagrams are completely different (see thumbnails: B, followed by Heiberg, also has Δ instead of E, and B inserted between A and Γ). The diagrams contain scholiastic numerals: see Appendix 2. Codex D has the line ΘH tilted upwards.

Codex B



Heiberg



COMMENTS

The two final propositions involve areas marked directly by letter labels (instead of being marked through the lines of their perimeter, in turn marked through points on vertices or along a periphery). Such "quantitative" labeling of areas is common – and is indeed employed in this treatise as well – for isolated areas (such as the "quantitative" area Ϙ in propositions 24–5) or lines (as in proposition 10). The rule is that when such objects become immersed in a larger setting, labels are attached to marked points (as, in a qualified way, in proposition 26; and to lines in proposition 11). Here, however, the use of direct labeling of areas is maintained even within a rich configuration. This is somewhat less marked in proposition 27, where the two labeling systems run alongside each other – a series of areas corresponding to a series of line segments. It is quite striking here in proposition 28, where the two systems cross each other so that sector HΘΓ of Step 1 becomes sector NΠΞ from Step 3 onwards (enharmonic notation, if you will). Whether intended as a marked variation or not, the labeling is in fact mathematically motivated and suggests the special kind of mathematical work being accomplished. Consider, first, proposition 27. There, the areas are marked directly by letters, and this allows us to combine them transparently: KΛMNΞ minus Ξ is KΛMN (whereas, had the spiral shells been labeled via their perimeters, one would have to struggle to identify which is the sum of which). While standing in a particular geometrical configuration, the shells are characterized mathematically via their quantitative, not spatial properties. The same remains true here in proposition 28 – a large part of the work is purely quantitative – but some geometrical configuration is brought back into play, with the resulting duality of labeling. What we see, then, is the tendency of *On Spirals* to reduce its geometrical terms into quantitative terms, ensuing from the reduction of geometrical areas to the underlying sums of squares provided in propositions 10–11. The treatise concludes on an ambiguous note, suggesting that the more abstract interlude of propositions 10–11 may not have been an extraneous aside after all: abstract relations have been used to discover geometrical relations, but, in the process, geometrical objects have become somewhat more abstract.

We note the dependency of proposition 28 for its overall "spirit" on the preceding proposition. And, indeed, while proposition 28 does not depend deductively on 27, it carries forward its main proof-idea of the nesting of ratios, some spiral-to-circle, some circle-to-circle – together with the elementary process of "peeling" away terms in proportions via the proportion theory operation "dividedly." Steps 1–2 involve such a "peeling" exercise; Steps 3–5 nest together ratios of spiral-to-circle and circle-to-circle, "peeled" to obtain Step 6; Steps 2, 6 are then recalled at a markedly close distance as Steps 7, 8, nested together to obtain Step 9 – which requires a very simple manipulation so as to obtain the required result (this final manipulation and is analogous to the final steps of the preceding proposition). Indeed the proposition does little else but repeat this basic proof-idea. On the one hand, the – considerable – computation involved in verifying the "peeling" and "nesting" operations is all delegated to the back, as it were, or served as a challenge to the reader: my notes 393 and 397 explicate in detail complicated claims that Archimedes glosses over. On the other hand, the working of this proof-idea is presented in an extremely explicit way, which involves that strange repetition of claims in close proximity. This is done solely so that the nesting relation between Steps 2, 6 would be made more evident. This is the usual trend away from an explicit setting-out of proofs, leaving just the broad outlines of a proof. This – even though this result (as well as 24, 18 and 27) was originally offered in the Letter to Conon and so was marked, even in the introduction to this treatise, as being among the key goals!

The treatise thus ends on the note of a sketch – and it ends abruptly. Just as we have come to expect the standard move of "two-and-more," the two final propositions enact a bit *less* than two. They are "a proposition – and a hasty extension." This is in part because the previous proposition compresses two separate results into one (about the ratio of the two innermost shells; as well as about all ratios); and in part because this proposition missed out on an obvious opportunity for a corollary. Proposition 28 asserts that its result holds for "whichever" rotation. In fact the proof obtains only for the first rotation. This is seen in the figure but extends to the underlying logic, since Step 1 flows from proposition 26, which is explicitly proved only for the case of the first rotation. Archimedes missed out on a corollary back in proposition 26, asserting that the result would hold even for further rotations; and then, as a consequence, he misses out on the same corollary here. But this is worse than just an observation omitted. Since the corollary to proposition 26 is absent, the general claim of proposition 28 is apparently false[402] – unless, that is, we are invited to re-insert the missing corollary to proposition 26 as part of our reading of Step 1 of proposition 28. But how are we to know, as we read proposition 28, that such an extension is required? Would it not appear instead possible to read Step 1 for what it literally claims (and what it appears to demand in the diagram), as a claim about the first rotation alone, expecting the generalization to the case of further rotations to emerge later in the proof?

Recall proposition 11. Read in its place, there was nothing suggesting it referred to anything other than case where the smallest term is equal to the difference; it is only through its application to proposition 25 that one may have gathered that it is to be taken – against its literal reading, as well as its diagram – as referring to the general case of a progression whose smallest term need not be equal to its difference. So here: proposition 26, read in context, makes merely a claim for the first rotation, and it is only as one reads proposition 28 that it becomes clear that one needs to consider, in one's mind, the extension to potential further rotations. The theme of a precarious generalization – of a proof-method just barely being adequate to its result, the very gap no more than half-evident – is sounded again. Here, however, at this abrupt ending of an abruptly elided proposition, it acquires the added significance of a resolution avoided.

At the end of Nabokov's *The Gift*, the lover-protagonists accompany the girl's family to the train station and finally have the apartment left all alone just for them. We follow them home and realize, through tiny clues left by the author, that the door is in fact locked, and they have forgotten to take the keys with them. The consummation hangs precariously beyond the limits of the book. So, too, with *On Spirals*.

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[402] For this reason, Heiberg even wished to consider the emendation of the enunciation of proposition 28 from "whichever" to "first"!

APPENDIX 1 PAPPUS' *COLLECTION* IV.21–25[1]

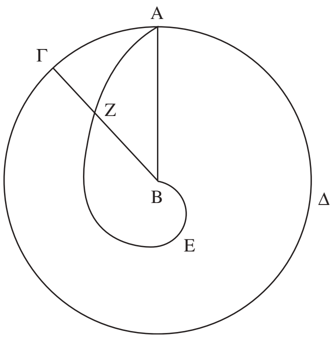
The study of the spiral drawn in the plane was proposed by the geometer Conon of Samos, but it was Archimedes who proved it using a certain amazing approach.[2]

The line's manner of origin is such: let there be a circle whose center is B, its radius AB. Let the line AB be moved in such a way so that B remains in place, while A is carried uniformly along the circumference of the circle, and simultaneously with it <=with AB> let a certain point, setting out from B, be

[1] Book IV as a whole is translated with commentary by Sefrin-Weis (2010). As she points out, the book appears to be a study in methods, growing in complexity from the elementary plane geometry using strict Euclidean techniques, to reach ultimately problems that involve curves that cannot be produced unless mechanically. Archimedes – as usual – is the key author, and the major transition is from his study of the Arbelos, IV.14–20 (very striking results based on elementary techniques) to this study of the Spiral, IV.21–25. The book ends at IV.53–54 with Pappus' explicit construction of the neuses required by the extant *On Spiral Lines*, so that this Archimedean field is marked as focal to the book as a whole. I translate the passage IV.21–25 for the possibility that it may include a reflection of another lost work by Archimedes touching on the spiral: see the comments on proposition 24.

[2] This seems flatly to contradict the introduction to the extant *On Spiral Lines*, where Archimedes appears to credit himself as initiating the study, Conon being no more than a – *highly worthy* – passive recipient. But in fact Archimedes takes credit in the extant *On Spiral Lines* for this alone: the first to claim to have a proof of *SL* 24, 18, 27, 28. It would, still, be strange to imagine Conon proposing a study of the spiral without making some concrete claims about it, and then it would be even stranger if such claims did not coincide with the four mentioned by Archimedes (Knorr suggested the obvious lacuna: the use of the spiral for angle trisection – but why would anyone capable of doing that fail to notice the results related to the squaring of the circle, a central problem for sure, to which the spiral is so obviously related?). Perhaps the likeliest account is that Archimedes claimed that it was Conon who suggested to him the study *in private*. Needless to say, other accounts are possible as well: Pappus or his intermediaries may have failed to understand the original source.

carried uniformly along it <=along AB> towards A, and let both the point B pass through BA, and the point A <pass through> the circumference of the circle in an equal time; so, the point moved along BA shall draw during the rotation a line such as is BEZA, and the point B shall be its start, while the line BA shall be the start of the rotation, and the line itself is called spiral. And its primary property is this: indeed, whichever line be drawn through towards it, such as BZ, and is produced, it is: as the whole circumference of the circle to the circumference AΔΓ, so the line AB to BZ. And this is most easy to grasp from the manner of origin: for <in the time> in which the point A passes through the whole circumference of the circle, in that <time> B, too, <passes through> BA; while <in the time> in which A <passes through> the circumference AΔΓ, in that <time> B, too, <passes through> BZ. (And the motions are of equal speeds each with itself, so that it is also proportional.)[3] And this too is obvious: that whichever lines may be drawn through from B towards the line, containing equal angles, they exceed each other by an equal <difference>.



There is extant only one independent manuscript for Pappus, Vat. Gr. 218, and the diagrams reproduced here are derived from it.

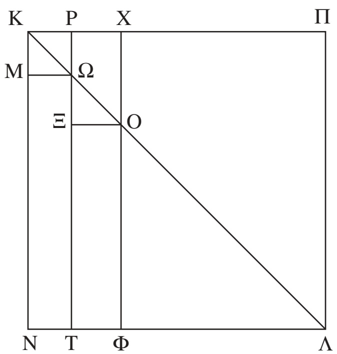
And the figure contained by both: the spiral, as well as the line in the start of the rotation, is proved to be a third part of the circle containing it. (a) For let there be whichever circle, (b) and the line mentioned above <i.e. a spiral>, (c) and let a right-angled parallelogram be set out, <namely> KNΛΠ, (d) and let the circumference AΓ be taken, a certain part of the circumference of the circle, (e) and the line KP, the same part of KΠ, ((f) and let both ΓB as well as KΛ be joined), (g) and PT parallel to KN, (h) and ΩM <parallel> to KΠ, (i) and the circumference ZH around B as center. (1) Now, since it is: as the line AB to AH, that is BΓ to ΓZ[4] (2) the whole circumference of the circle to ΓA ((3) for this is the primary property of the spiral).[5] (4) But as the circumference of the circle to ΓA, ΠK to KP, (5) while as ΠK to KP, ΛK to KΩ,[6] (5) that is PT to PΩ,[7] (7)

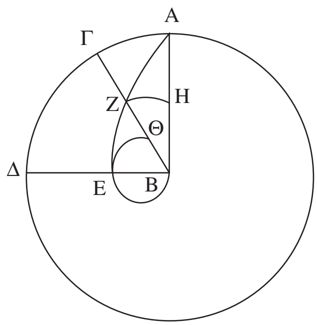
[3] The idea is that of *SL* 2: in motions, each of which is of uniform speed, the lengths traversed during equal times are proportional.

[4] Equality of radii in circles. [5] Here transformed via *Elements* V.17.

[6] *Elements* VI.4. [7] *Ibid.*

therefore also as ΒΓ to ΒΖ, ΤΡ to ΡΩ, (8) therefore convertedely, as well, as the square, too, on ΒΓ, to the <square> on ΒΖ, so the <square> on ΡΤ to the <square> on ΤΩ.[8] (9) But as the <square> on ΒΓ to the <square> on ΒΖ, so the sector ΑΒΓ to the sector ΖΒΗ,[9] (10) while as the <square> on ΡΤ to the <square> on ΤΩ, so the cylinder on the parallelogram ΚΤ around the axis ΝΤ to the cylinder on the parallelogram ΜΤ around the same axis,[10] (11) therefore, also, as the sector ΓΒΑ to the sector ΖΒΗ, so the cylinder on the parallelogram ΚΤ around the axis ΝΤ to the cylinder on the parallelogram ΜΤ around the same axis. (12) And similarly, if we set ΓΔ equal to ΓΑ, and ΡΧ equal to ΚΡ, and construct the same things, it shall be: as the sector ΔΒΓ to the <sector> ΕΘΒ, so the cylinder on the parallelogram ΡΦ around the axis ΤΦ to the cylinder on the parallelogram ΞΦ around the same axis. (13) Proceeding[11] along the same manner we will prove that as the whole circle to all the figures inscribed within the spiral, so the cylinder around the axis ΝΛ to all the figures from <the> cylinders inscribed within the cone on the triangle ΚΝΛ around the axis ΛΝ; (14) from which it is obvious that as the circle to the figure between spiral and the line ΑΒ, so the cylinder to the cone; (15) but the cylinder is three times the cone; (16) therefore the circle, too, is three times the said figure.



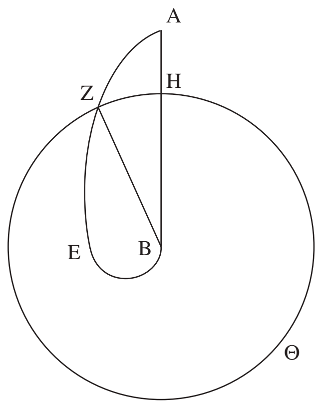


[8] *Elements* V.19, VI.22. [9] An extension of *Elements* XII.2.

[10] *Elements* XII.2, 11. The expression "cylinder on parallelogram" is confusing; the Greek particle απο translated here as "on" really has the function of "the cylinder whose construction *has its starting-point with* the parallelogram" (such, indeed, is the meaning of "on" in the expression "the square on the line"). The sense here is probably the cylinder whose base is the circle whose radius is one side of the parallelogram (in this case, a rectangle), and whose height is the other side of the parallelogram. Perhaps "around the axis ΝΤ" means not so much the measure of the height of the axis, as the direction of the rotation: it is meant to spell out which of the sides of the rectangle is taken as the base for rotation and which for the height.

[11] "proceeding" – in the original Greek, the participle ἐφοδεύσαντες, derived from the noun ἔφοδος, "approach" – the original title of Archimedes' *Method*.

In the same manner we will prove also that if a certain line is drawn through inside the spiral, such as BZ, and a circle is drawn through Z around the center B, the figure contained by both: the spiral ZEB as well as the line ZB is a third part of the figure contained by both: the circumference ZHΘ as well as the lines ZBΘ. As for that, the proof is somewhat the same; in what follows we will prove a theorem, holding for the same line, which is worthy of study.[12]



The label E is omitted by the manuscript.

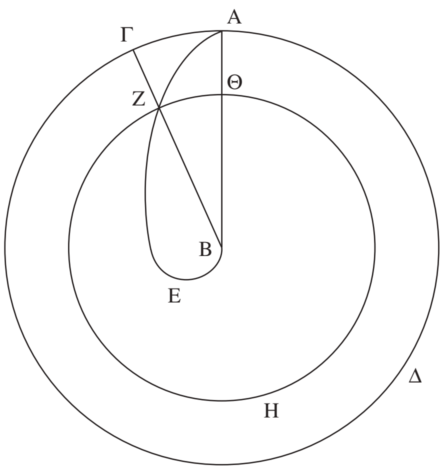
For let there be both: the circle mentioned above (in the generation <of the spiral>) as well as the spiral itself, AZEB. I say that whichever line is drawn through, such as BZ, it is: as the figure contained by the whole spiral and by the line AB to the <figure> contained by the spiral ZEB and the line BZ, so the cube on AB to the cube on ZB.

(a) For let a circle, <namely> the <circle> ZHΘ be drawn through Z around B as center. (1) Now, since it is: as the figure contained by the line AZEB and by the line AB is to the figure contained by the line ZEB and by the line ZB, so the circle AΓΔ to the figure contained by the circumference ZHΘ and by the lines ZBΘ, (2) for each was proved a third part of each,[13] (3) while the circle AΓΔ has to the area taken by the lines ZBΘ and by the circumference ZHΘ the ratio composed of both: the <ratio> which the circle AΓΔ has to the circle ZHΘ, as well as the <ratio> which the circle ZHΘ has to the area taken by the lines ZBΘ and by the circumference ZHΘ, (4) but as the circle AΓΔ to the circle

[12] The "theorem" is in Greek θεώρημα, the same as the "study" originally said to be proposed by Conon. The juxtaposition could be intentional, in which case it reads most naturally as an opposition: this was proposed by Conon, that is *also* worthy of investigation, despite its going beyond the original scope of study. Perhaps such was the transition within Archimedes' original treatise; I think it could also mark a transition to observations taken from sources other than Pappus' Archimedean source, perhaps even a transition to Pappus' own original contribution. In other words, even if it is reasonable to assume that the original proof-idea connecting spirals to cones is from a lost treatise by Archimedes, the same no longer holds for what follows.

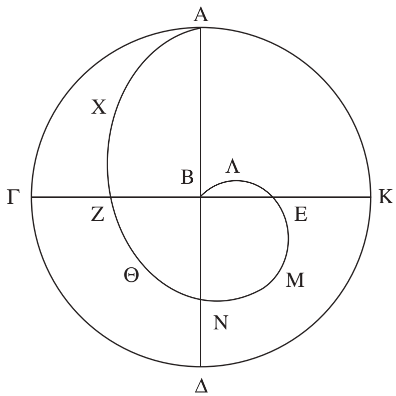
[13] In the preceding proof with its extension to segments of rotations.

ZHΘ, so the <square> on AB to the <square> on BZ, (5) while as the circle ZHΘ to the said area, its whole circumference to the <circumference> ZHΘ, (6) that is the circumference of the circle AΓΔ to the <circumference> ΓΔA, (7) that is, because of the property of the line, (8) the line AB to the <line> BZ, (9) therefore also: the figure between the spiral and the line AB has to the <figure> between the spiral and BZ a ratio which is the one composed of both: the <ratio> of the <square> on AB to the <square> on ZB, as well as the <ratio> of AB to BZ. (10) And this ratio is the same as the <ratio> of the cube on ZB to the cube on BZ.



So, from this it is obvious that if – the spiral and the circle around it being assumed – AB is produced to Δ and ΓZ, EK are drawn at right angles to it, of as much as the area between the straight <line> BΛE and the line BE is one, of that much the area between the line NME and the straight <lines> NBE is seven, the area between the line ZΘN and the straight <lines> ZBN 19, and the area between the line AΞZ and the straight <lines> ABZ 37 – for these are clear from both: the preceding theorem – as well as <the fact> that of as much as AB is 4, ZB is three, BN two, and BE one – for this too is clear both from the property of the line as well as from the circumferences AΓ, ΓΔ, ΔK, KA being equal.[14]

[14] This strikes one as an effort to emulate the striking numerical results of *SL* 27; the results, however, are fairly unexciting, and they are obtained in a rather pedestrian way. This adds a little to the probability that this passage is from later, post-Archimedean efforts, perhaps by Pappus himself.



Θ, X omitted by the manuscript.

APPENDIX 2
SCHOLIA TO *ON SPIRALS*[1]

/PROPOSITION 3/

For this has been proved in the 1st theorem of *On the Sphere and the Cylinder*.

[1] There is a continuum between a commentary written as an independent piece of scholarship, to the notes jotted down by a teacher (or a student). Bear in mind that even the most sophisticated of the commentaries from late antiquity are products of the teaching environment. In the case of Archimedes, we may see Eutocius moving from the schoolmasterly set of notes for *SC* I to the more ambitious scholarship of *SC* II. The scholiastic material present in the manuscript tradition to *On Spiral Lines* clearly falls at the very schoolmasterly end and represents no more than an attempt to engage students, whose background is in Euclid's *Elements* as well as school calculation, with this text. The scholia consist of: (i) a handful of tiny marginalia (propositions 3, 8, 16), (ii) numerals inserted into the figures of 10, 27, 28, (iii) a more extended treatment of proposition 10 based on the inserted numerals and (iv) a more extended treatment of proposition 27, largely independent of the inserted numerals. It is quite possible that the teacher responsible for this material had more to say orally (or perhaps we have lost some text) concerning at least propositions 27, 28 with their strange numerals. Most of the treatment of proposition 10 was taken by codex A out of the marginalia and constituted a separate, tiny "commentary" treatise with the title "scholion to the 10th theorem," positioned right at the end of the book. (That is, an appendix in the manner of the one you are now perusing.) The rest of the material is either inserted into the diagram or into the margin, and even the scholion to proposition 10 could have been originally placed in the margins (to which it is located by codex E). Codex C does not carry the diagrams for 27, 28 at all, but in proposition 10 – where its diagram is barely legible – it seems to carry a (badly executed?) copy of the same inserted numerals. Since the texts of codices A and C are closely interwoven, it seems likely that their common source is proximate enough, and the simplest assumption is that codex C had some such thing as the same scholia in its source and then made a conscious choice not to copy the scholiastic material, misunderstanding the inserted numerals as authorial to the diagram (it is almost certain that codex C did not copy Eutocius, either). The overall lesson we may draw from all of this is that Archimedes' *On Spiral Lines* was subsumed, at least once – likely in late ancient or in early Byzantine times – into a fairly elementary educational context.

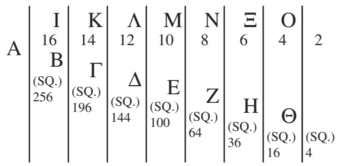
/PROPOSITION 8, STEP 2/

(1) For ΚΓ has to ΓΛ a greater ratio than to ΞΓ, (2) and through this ΞΓ is greater than ΓΛ.[2]

/PROPOSITION 8, STEP 6/

(1) For the <rectangle contained> by ΙΚ, ΝΙ is equal to the <rectangle contained> by ΞΙ, ΙΛ; (2) for two lines cut each other in a circle;[3] (3) while the <rectangle contained> by ΚΙ, ΓΛ <is equal> to the <rectangle contained> by ΚΕ, ΙΛ; (4) for ΙΚΛ is a triangle, (5) and ΕΓ has been drawn parallel to one <side>; (6) therefore it is proportionally: as ΙΚ to ΚΕ, ΙΛ to ΛΓ,[4] (7) and through this the <rectangle contained> by the extremes is equal to the <rectangle contained> by the means.[5]

/PROPOSITION 10/



The reading of the inserted numerals is almost impossible for codex C, and what little is seen gives grave grounds for suspicion that while the series from 16 down was correctly copied, the squares were mangled and perhaps misunderstood completely. It also appears almost certain that the marginalia to the diagram, extant in codex A, are not in codex C.

(The following three notes are marginalia to the figure:)

The <squares> on the <lines> equal to ΙΑ, together with the <square> on Α, comes to be 2304.[6]

The <rectangle contained> by Θ and by all the <lines> exceeding each other by an equal <difference> comes to be 144.[7]

The <squares> on all the <lines> exceeding each other by an equal <difference> comes to be 816.[8]

/SCHOLION TO PROPOSITION 10/

In order that the theorem would be made clear by numbers, too, agreeing with its text, let the same be assumed, and let the numbers be drawn in a figure, as has been assumed, together with their multiplications. Now, it is possible to find, doing the additions, that the <squares> on the <lines> exceeding each other by an equal <difference> together with the <squares> on Ι, Κ, Λ, Μ, Ν, Ξ, Ο and the <square> on Α <are> 1632, while the <squares> on the <lines>

[2] *Elements* V.8. [3] *Elements* III.35. [4] *Elements* VI.2. [5] *Elements* VI.16.
[6] This will be: (8*256)+256=2304. [7] This will be: 2*(2+4+ . . . +16)=144.
[8] This seems to be the point of the inserted numerals. 22+42+ . . . +162=816.

exceeding each other by an equal <difference are> 816 (indeed, that which is said is clear).[9] It remains needed to display that twice the <rectangle contained> by all the <lines> together with the <squares> on Ι, Κ, Λ, Μ, Ν, Ξ together with the <rectangle contained> by Θ and by all the <lines> exceeding each other by an equal <difference> is equal to the <squares> on all the <lines> exceeding each other by an equal <difference>. And this is right there, of itself, with the numbers assumed, but Archimedes, wanting to show this geometrically, transforms the idea and says:[10] "since two <rectangles>, the <rectangles> contained by Β, Ι are equal to two <rectangles>, the <rectangles> contained by Β, Θ" and so on. These are obvious once you write down the areas <=rectangles>, and it is clear that the numbers, too, shall turn out <the same>. For as in the case of twice the <rectangle contained> by Β, Ι: Β is 14 while Ι is 2. So that <twice> the <rectangle contained> by them comes to be 56. (And it is: B is 14; therefore the double is 28; <multiplied> by 2 – through Ι being equal to Θ – comes to be 56.) So, similarly, four times Γ comes to be 48, six times Δ comes to be 60, eight times Ε comes to be 64, etc., per assumptions; so that the <result> on all – twice and four times and six times and the rest in the diagram – comes to be 336.[11]

Then again:[12] "Now, all the <twice rectangles> taken together, adding on the <rectangle> contained by both: Θ, and the <line> equal to all the <lines> Α, Β, Γ, Δ, Ε, Ζ, Η, Θ shall be equal to the <rectangle> contained by both: Θ, and the <line> equal to all: Α, as well as three times Β" and so on. For since saying twice the <rectangle contained> by Β and four times by Γ and so on he brings in with them the <rectangle contained> both by Θ, as well as by the <line> equal to all the <lines> exceeding each other by an equal <difference>, it is no different from that, to the double of Β, he brings in yet another, so that it comes to be three times, and to four times of Γ, he brings in yet a fifth, and so on similarly;[13] now it comes to be, as has been said: that twice of Β and four times of Γ and so on of the remainder – 336. And Θ is 2. So clearly the <rectangle contained> by <the two terms> – 672. And the <line> equal to all the <lines> exceeding each other by an equal <difference> comes to be 72 which, <multiplied> by 2 comes to be 144. Together with 672 it comes to be 816. And the three times Β comes to be 42, and so on, per assumptions; now, the <line> composed of Α as well as three times Β as well as five times . . . and so on, is 408[14] <multiplied> by two, comes to be 816, which is also the <squares> on Α, Β, Γ, Δ, Ε, Ζ, Η, Θ.

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[9] This follows from the (geometrically inspired) understanding that the series from Ι to Ο is the same as the series from Β to Θ; so 816*2=1632.

[10] Step 6. [11] 28+48+60+64+60+48+28. [12] Step 12.

[13] This comment, per se, is not numerical: the scholion momentarily has the feel of fully-fledged "commentary."

[14] 16+(3*14)+(5*12)+(7*10)+(9*8)+(11*6)+(13*4)+(15*2).

Α	Β	Ι	Γ	Κ	Δ	Λ	Ε	Μ	Ζ	Ν	Η	Ξ	Θ	Ο
(units) 16[15]	(units) 14	(units) 2	(units) 12	(units) 4	(units) 10	(units) 6	(units) 8	(units) 8	(units) 6	(units) 10	(units) 4	(units) 12	(units) 2	(units) 14
(on)[16] (units) 256	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)	(on) (units)
	196	4	144	16	100	36	64	64	36	100	16	144	4	196
	twice (by)	(units) 56	twice (by)	(units) 96	twice (by)	(units) 120	twice (by)	(units) 128	twice (by)	(units) 120	twice (by)	(units) 96	twice (by)	(units) 56
	The two times of		The four times of		The six times of		The eight times of		The ten times of		The twelve times of		The fourteen times of	
	Β (units) 28		Γ (units) 48		Δ (units) 60		Ε (units) 64		Ζ (units) 60		Η (units) 48		Θ (units) 28	

All together it adds up to 336. <multiplied> by two together with the <rectangle contained> by Θ, and by the <line> equal to all the <lines> exceeding each other by an equal <difference>, that is 144, it comes to be (units) 816.

[15] The Greek expression is literally "16 units." The term "units" is formulaically abbreviated, though, and in this table becomes a sheer element of tabulation – so that I prefer to keep the formulaic, non-syntactic Greek order.

[16] "on" is, in this case, an abbreviation for "the square on."

The 3 <times> Β	The 5 <times> Γ	The 7 <times> Δ	The 9 <times> Ε	The 11 <times> Ζ	The 13 <times> Η	The 15 <times> Θ
(units) 42	(units) 60	(units) 70	(units) 72	(units) 66	(units) 52	(units) 30

Together, the <line> composed of all <lines>, three times as well as five times and the rest, (units) 408, <multiplied> by two comes to be 816.

/PROPOSITION 16, STEP 13/

In 14.

/PROPOSITION 27, STEP 2/

That the 2nd spiral has to the 2nd circle a ratio which 7 <has> to 12 has been proved above and no less shall be learned now through numbers, so that what follows, too, shall be made easy to follow. For since the 2nd spiral was proved to have to the second circle a ratio which, taken together, both: the <rectangle contained> by ΒΘ, ΘΑ as well as the third part of the <square> on ΑΒ <have> to the <square> on ΘΒ, let us assume ΑΘ to be 16 units, as clearly ΒΘ comes to be 32 (units). Now, the <rectangle contained> by ΒΘ, ΘΑ comes to be 512, and the third of the <square> on ΒΑ – 85⅓ (units);[17] together they come to be 597⅓. And the <square> on ΒΘ – 1024 (units); and 597⅓ has to 1024 a ratio, which seven has to 12.[18]

And the <square> on the radius of the 2nd circle is 1024 (units), while the <square> on the radius of the 1st circle is 256 (units); and they have a ratio to each other, which 12 <has> to 3. So, the remainder is clear; for it results in the "through the equality"[19] and the "dividedly."[20]

/PROPOSITION 27, STEP 8/

It is clear that one must supply in thought; and through the inequality, the <rectangle contained> by ΓΘ, ΘΒ together with the ⅓ part of the <square> on ΓΒ, has to the <rectangle contained> by ΒΘ, ΘΑ together with the ⅓ part of the <square> on ΒΑ a ratio, which the areas ΚΛΜ <have> to the <areas> ΚΛ.[21] And these, he says, the <rectangle contained> by ΓΘ, ΘΒ together with the ⅓ part of the <square> on ΓΒ, has to the <rectangle contained> by ΒΘ, ΘΑ together with the ⅓ part of the <square> on ΒΑ a ratio, which 19 <has> to 7.

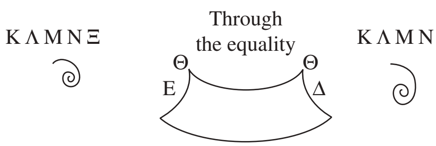
17 The scholiast did not bother to cook his example . . .

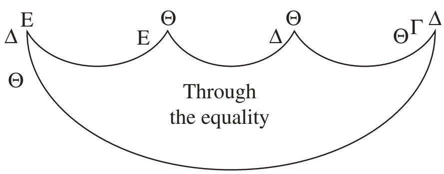
18 So that the numerical example is, in fact, not easy to follow at all.

19 *Elements* V.22. 20 *Elements* V.17. 21 *Elements* V.22.

For the <rectangle contained> by ΓΘ, ΘΒ is 1536 (units),[22] while the ⅓ of the <square> on ΒΓ <is> 85⅓ (units), since the <square> on <ΒΓ>,[23] as well, <is> 256. Together: 1621⅓. And the <rectangle contained> by ΒΘ, ΘΑ together with the ⅓ part of the <square> on ΒΑ <is> 597⅓, which have to each other a ratio which 19 <has> to 7.

/PROPOSITION 27, STEPS 15–18/





A charming visualization of the course of argument in Steps 15–18: the ratio of the spiral areas ΚΛΜΝΞ to ΚΛΜΝ is decomposed so that it may be recomposed "through the equality" in terms spiral-to-circle to circle-to-spiral (top figure) or ratios of a certain construction square-to-square to a certain construction (bottom figure), hence spiral-to-spiral as construction-to-construction.

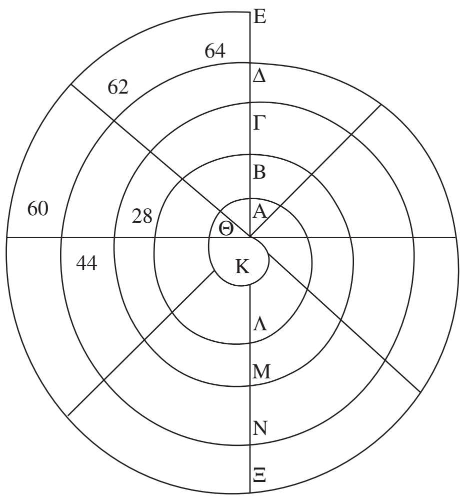
/PROPOSITION 27, STEP 24/

Clearly, also compoundedly.[24]

[22] The scholiast still operates with the preceding set of numerals, hence now he multiplies 32 by 48.

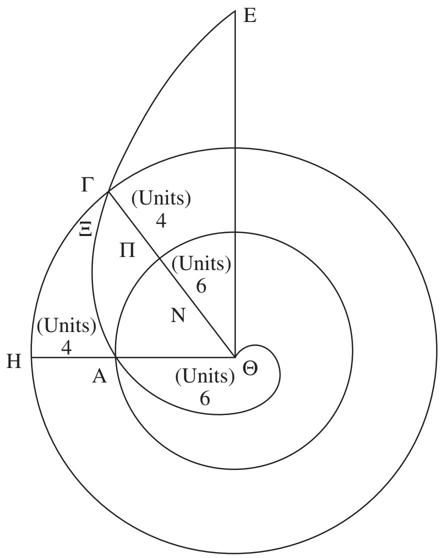
[23] The scholiast's authorial omission. [24] *Elements* V.18.

/PROPOSITION 27, FIGURE[25]/



Codices BD have the straight lines continuous through the central area.

/PROPOSITION 28, FIGURE[26]/



[25] The purpose of this figure is to verify the length of the spiral line at various positions, assuming it progresses by two units for each half quadrant of a rotation (this is reminiscent of the use of the series of even numbers in the numerals inserted into proposition 10). This would indeed end up with the values 16, 32 and 48 at the end of the first, second and third lines, as demanded by the scholia – numbers that, however, are not inserted into the diagram, while, at the same time, the actual numbers inserted are not used by the scholiast.

[26] The figure is made to consider the case where AΘ is 6 units to AH is 4; one wonders whether the opposite was not meant, so that the ratio of the two areas then becomes 10:8. Once again, the scholiast operates with even numbers only.

GLOSSARY

Added to Itself (*heauta suntithemena*) In proposition 4, Archimedes envisages lines being "added to themselves." Just adding a line to itself *once* would result in double the line, but this expression does not specify how many times the addition is to take place, hence the result is simply a line that is some arbitrary multiple of the original line. Why does Archimedes not specify it in this way? Because he is thinking not in terms of abstract multiplication, but of concrete geometrical operations, the line "being added to itself" actually meaning taking a copy of the original line, positioning it next to one end of the line, and in this way obtaining a line double the original one (in length, as we would say).

Assigned (*tachthen*) In the introduction, Archimedes refers to a ratio being "assigned," in the context of a geometrical task such as dividing the sphere into two segments so that they have the "ratio assigned." We can say, for instance, the assigned ratio might be 2:1, in which case the task becomes to divide the sphere into two segments, one of which is twice the other (in volume, as we would say). But this would be imprecise, as in fact a task that involves an "assigned" ratio does not really call for a specification of what this assigned ratio is. Rather, the mathematician's job is to find a general algorithm with the aid of which it would be possible, no matter which two magnitudes are proposed, such as A, B, to divide the sphere into two segments such that one stands to the other in the ratio A:B.

Compose (*suntithemi*) The verb *suntithemi* literally means "to put together," and it can simply mean "to add" (this is the translation I use in expressions such as "added to itself," *heauta suntithemena* – see the Glossary above). It has a wide range of meanings (including that of "synthesis"). In this book, it is used in two main ways. First, a magnitude may be composed of another when it is, effectively, its multiple (a line 4 meters long may be composed of a meter-long line – by the shorter line being added four times; but the same 4-meter-long line cannot be composed in the same way from a 1.10-meter-long line).

Second, a magnitude may be said to be composed of its constituents (the line ABCD is composed of three lines: AB, BC and CD).

Diameter (*diametros*) While Archimedes usually refers to the diameter of a circle in an obvious way, he also refers in the introduction to the diameter of the sphere, perhaps a slightly less familiar usage for a modern reader; it means, of course, precisely the same as the diameter of the circle (a straight line whose ends are on the surface of the sphere, passing through the sphere's center).

End (*peras*) This word for "end" or "limit" can have a range of meanings in Greek, but here it has one usage only: referring to one of the two points that are at the limits of a line segment (note that the Greek word for "line" usually means, effectively, what we call a "line segment," so that a Greek line does have two ends).

Equal in Magnitude (*isos megethei*) Usually, when a Greek mathematician says that a geometrical object is equal to another, this means in our terms: for lines, that they are equal in length; for plane figures, that they are equal in area; and for solid figures, that they are equal in volume. This is "equality in magnitude," and usually it is not even spelled out that this is the intended meaning of equality. In some contexts, however (especially when there is a need to distinguish between equality in magnitude and an equality in multitude, for which see below), then it would be explicitly said that objects are equal "in magnitude" – meaning the same, effectively, as "equal."

Equal in Multitude (*isoi plēthei*) When two collections of objects are considered, say a collection of lines a_1, a_2, a_3 and b_1, b_2, b_3, we may compare the lines as magnitudes, but we may also compare the collections in terms of the number of objects they contain. When they contain the same number of objects, they are said to be "equal in multitude": there are exactly three a's in the example above, and three b's. The distinction between "equality in magnitude" and "equality in multitude" is important, in this treatise, in propositions 10–11.

Excess, exceeds (*huperocha, huperechei*) Starting at the statements in the introduction, of the results obtained in the treatise, and then through various geometrical contexts, Archimedes returns to this expression, which is a geometrical expression of what we would think of as subtraction. When line l1 exceeds another line l2, for instance, by a certain difference D, we may say, in our own terms, that l_1-l_2=D. The Greek term implies (as usual) a more concrete understanding, where the difference is the actual going-over by which the first line is longer than the second.

Fall (*piptō*) The verb is used in a fairly non-technical sense, effectively "to be in," for instance in proposition 6, where a line "falls" outside a certain configuration. Quite often, it has a more technical sense. When lines are drawn from the center of the circle to the circumference, they are said to "fall" on that circumference, and the same meaning is here extended to the spiral, from proposition 12 onwards: lines extended from the start of the spiral, "falling" on the spiral line itself.

Given (*dotheis*) A Greek mathematical problem (such as those mentioned in Archimedes' introduction) has the form that a certain geometrical object is "given." The assignment is to find or to produce another geometrical object, somehow determined by that "given." (So, for instance, a sphere is given – and the assignment is to find an area equal to its surface.) Notice that the task is general, so that the assignment is to find a universal algorithm with which, no matter what the "given" might be, the task will be effected.

Greatest Circle (*megistos kuklos*) In the introduction, when referring to the results of *On the Sphere and the Cylinder*, Archimedes uses the expression "Greatest Circle" (specifically, "of the circles in the sphere"), a formulaic expression which nevertheless is quite transparent: there can be many circles drawn on the surface of the sphere, and the greatest of them are like the equator on the globe ("are like" – because there are in fact infinitely many such "greatest circles" on any sphere, depending on where you wish to put your "equator"). This is to be distinguished from the many circles being "greater" than others mentioned in this treatise which are no more than ordinary circles drawn on a plane.

Half-as-much-again (*hamiolos*) In many languages there are adjectives that can mean integer ratios, such as "double," "triple" etc. Greek has a word for the ratio we would express as 1.5:1 or 3:2, for which a cumbersome English expression can be found in "half-as-much-again." Usually this enters mathematical discourse through the straightforward expression that one thing stands to another in this precise numerical ratio. A very complicated special use is found in the introduction, where Archimedes recalls results having to do with a "half-as-much-again ratio," which in context meant a ratio being raised to the power 1.5, as we would put it; or the square root of the cube. (See the Glossary for "duplicate ratio.")

Join (*epizeuchthō*) When lines are drawn between two points, this is often referred to as the lines "being joined" from one point to the other; or one can refer to the line "joined" between two points (meaning, effectively, that these two points are its end).

Lemma (*lēmma*) Today, mathematicians most frequently use the word "lemma" to refer to some auxiliary argument required for the sake of a more central theorem. As is often the case with such second-order terms, the term is not entirely regimented in Greek mathematics, but it seems to have a meaning closer to our "axiom," and it is perhaps in this sense that it is evoked in the introduction (to refer to what we would call, indeed, "Archimedes' Axiom").

Magnitude (*megethos*) The word "magnitude" does not occur, in this treatise, outside of the fixed expression "equal in magnitude." I do supply it within pointed brackets in many places, and so an explanation is called for. By a "magnitude" is meant either a line, a plane or a solid. The term is typically evoked when such objects are considered purely quantitatively (that is, independently from their spatial configuration), and I supply it, for instance, when it is stated that a solid is greater than another by a given "magnitude" (however, note that one could equally supply "solid").

Perpendicular (*kathetos*) The adjectival usage of "perpendicular" is straightforward (when it is stated that a line is perpendicular to a plane, for instance, this means, well, that the angle between them is right). What is slightly confusing is that "perpendicular" can function effectively as a noun. When a line is constructed as "perpendicular, from a point X, on a line AB," this means that it is the line drawn passing through the point X, so that it is in right angles to the line AB; and within the argument (as, for instance, in proposition 6), that line might be referred to as "the perpendicular," without mentioning what it is perpendicular to.

Ratio duplicate (*diplasios logos*) The ratio of 16 to 9 is duplicate the ratio of 4 to 3. That is:

When a_1:a_2 is duplicate b_1:b_2

Then we can say in our own terms:

$$(b_1 : b_2)^2 = (a_1 : a_2)$$

However, the intuition behind the Greek term "duplicate ratio" is more straightforward. If you take the ratio 4:3, *and then apply it again*, you get the duplicate of the ratio: so, you get 16:9, duplicate 4:3.

Ratio greater (*meizōn logos*) The idea of a ratio being greater or smaller than another is very clear to us, since we can transform ratios into real numbers (thus the ratio of the diagonal to the side in the square is √2, while the ratio of the circumference to the diameter in the circle is π, and we know precisely what is meant by π>√2). For the Greeks, the idea is more mediated (ratios are greater than others dependent on certain conditions: for instance, when the consequents are the same, but the antecedent is greater). The meaning, however, is essentially the same as that intended by our comparison of the size of real numbers.

Rectangle Contained (*orthogōnion periechomenon*) The expression "rectangle contained" is in fact only rarely encountered: almost always, one of the terms is elided – or both are. The full phrase "the rectangle contained by the lines AB, BC" refers, in the strict sense, to a rectangle whose two sides AB, BC are orthogonal at the point B. The interesting complication for this expression is that one may use this when the two lines are not in fact orthogonal at that point, or even in such expressions as "the rectangle contained by the lines AB, CD" where the two lines are not contiguous at all (so for instance, "the rectangle contained by ΚΙ, ΓΛ" in proposition 8, Step 6). In such cases the expression refers to what a virtual rectangle *would* be like, if the lines *were* formed to enclose a rectangle. In practice, this is as close as standard Greek mathematical speech gets to the idea of the *multiplication* of two line segments.

Segment (*tmama*) A segment is, generally speaking, a division of a magnitude (line, plane or solid) into units that belong to the same magnitude. The segments of a line are lines; the segments of a plane are planes; the segments of solids – solids.

Similar (*homoios*) "Similarity" is defined in Greek mathematics as a matter of equality of angles and/or proportionality of sides (thus *Elements* VI def. 1; Archimedes' *Planes in Equilibrium* I def. 5). That said, its intuitive meaning, which can be easily extended to curvilinear figures (as is required by the introduction to this book), is of identity of shape: two figures are "similar" when their only difference is that of scale.

Square on . . . (*tetragōnon apo*) A square is on a line in the sense that it could be set up on it (the original expression, in fact, is "from" the line, which I change into "on" so as to make the expression somewhat less puzzling). Effectively, we do not really envisage the square being set up and consider something equivalent to the purely quantitative object of the area equal to a square on just that line, or even, if we wish to be anachronistic about, something like the line "squared."

Theorem (*theōrēma*) When Archimedes refers at the beginning of the book to "theorems," his meaning is hard to pin down (as is generally the case for second-order terminology in Greek mathematics). It cannot be the same as "theorem" in the modern sense of a fairly significant "proposition," or even in the (widespread but not universal) ancient sense of a "theorem" as opposed to a "problem." Rather, it may be closer to the literal meaning of the word in Greek: an object of contemplation; something to ponder. It is a claim put forward for discussion.

Touch (*epipsauō*) The modern word "tangent" is from a Latin root, and it literally means "the thing that touches": it is a participle form of the verb "to touch." In Greek as in Latin, the verb "to touch" has acquired the technical mathematical meaning of "to be a tangent." This is used very often in this treatise, from proposition 5 onwards. In an expression "the line touches the spiral at the point P," the point P is, obviously, the point of tangency.

Verge (*neuō*) In the group of propositions 5–9 one is sometimes asked to produce a line, that is, a line segment, so that it is "verging towards a point" P. The strict meaning of that is that the line is positioned in such a way that, if it were extended, it would pass through the point P (that is, the line segment is positioned somewhere on a line passing through P). What makes this construction unique is that the process envisaged seems to be that of taking a line segment, fixed in length, and trying to position it within certain limits so that it "verges towards the point P," by trial and error, as it were.

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