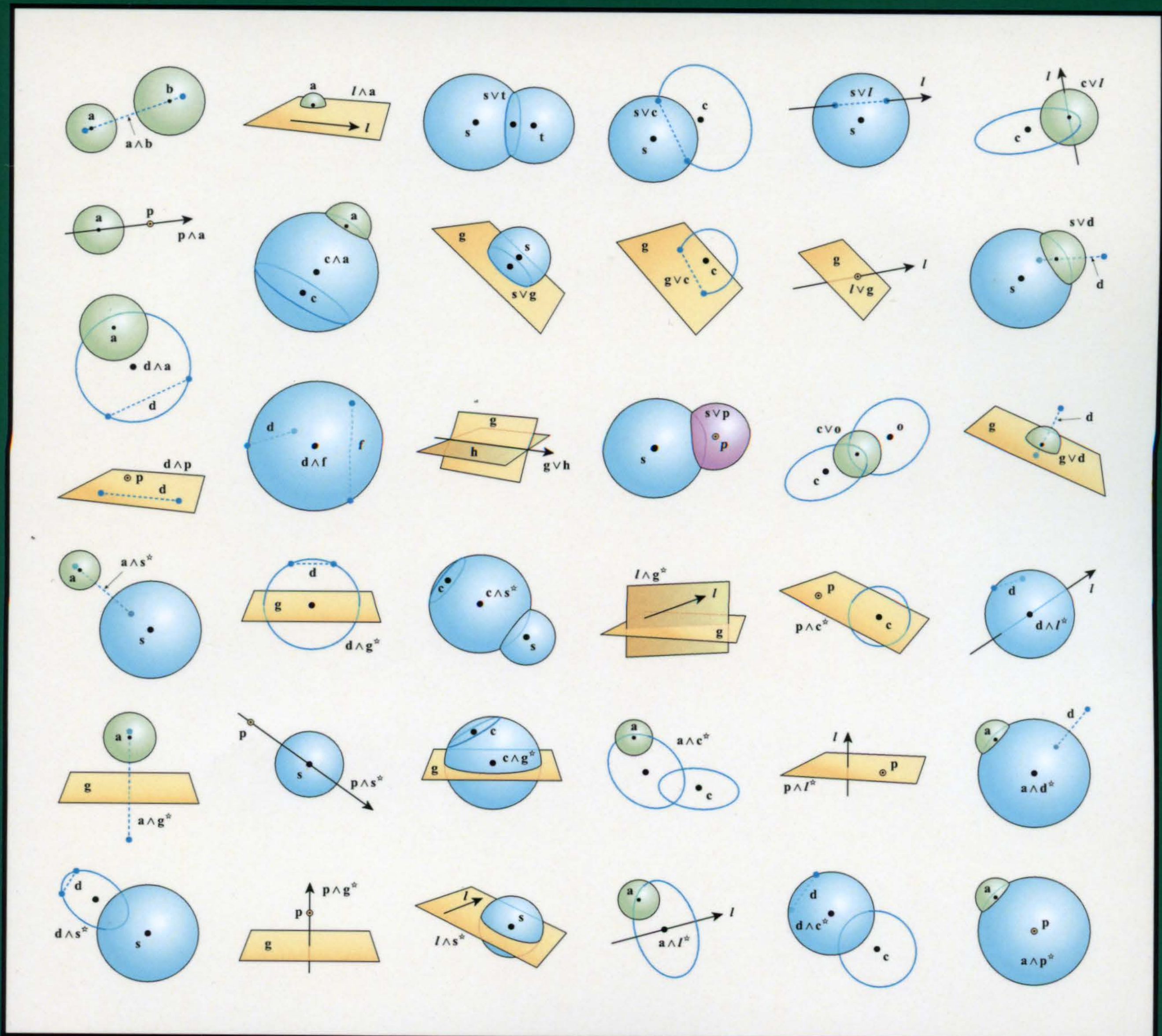
Projective Geometric Algebra
ILLUMINATED



Eric Lengyel

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ISBN-13: 979-8-9853582-5-4

First printing

Published by Terathon Software LLC
terathon.com

Book website: projectivegeometricalgebra.org

About the cover: The 37 diagrams illustrate the join operations (7), meet operations (12), and expansion operations (18) in the five-dimensional conformal geometric algebra describing flat points, lines, planes, round points, dipoles, circles, and spheres in three-dimensional space.

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Preface

This book is an exploration of projective geometric algebra that emerged from more than a decade of personal research that can be described as a quest for a complete picture of the underlying mathematics. While that quest has been successful in many ways, the subject has also proven itself to be an unending source of new knowledge and understanding that can never be fully conquered. What's written in the chapters that follow is a detailed description of the author's understanding at the present time, and it is sure to be exceeded by new discoveries in the future.

The term *geometric algebra* is used more broadly in this book compared to most other publications, and it includes the *exterior algebra* first developed by Hermann Grassmann. The word *projective* in the name of the subject means that the vector spaces on which our algebras are based have more dimensions than the ordinary Euclidean spaces that they model. For example, the projective exterior algebra that models geometric objects in three-dimensional Euclidean space has four dimensions. That increases to five dimensions in the conformal algebra.

The methods through which geometric algebra is developed in this book differ from those of its predecessors. The theoretical foundations of the subject are rebuilt in a new way that occasionally diverges from established practices. This is not done with reckless abandon but with careful consideration and lucid recognition of superior results. There are many concepts presented in this book that cannot currently be found elsewhere, and new terminology comes with them. However, one of our goals is to uphold the established meanings of conventional terminology as much as possible and to maintain bridges to familiar concepts in linear algebra.

In general, this book focuses on accessibility and intuitive understanding. It is not intended to be excessively theoretical, and it does not follow a definition-theorem-proof format. The primary aim is to provide the information necessary to put the subject of projective geometric algebra to practical use in the best way possible. Employing geometric algebra is not always a win, though, and we certainly don't want to suggest that it should be regarded as a wholesale replacement for more conventional applications of linear algebra. When geometric algebra methods come with a step down in performance, for example, we are sure to point that out.

There are several full-page comparison charts throughout the book that show solutions to a specific problem in a side-by-side layout. One solution uses conventional methods and the other uses techniques from geometric algebra. Sometimes, geometric algebra will demonstrate a clear advantage, but in other cases, conventional mathematics may appear more attractive. It is not our intention to assert that one solution is superior to the other, but to give the reader enough information to make up their own mind.

Definitions and equations of key importance are boxed and often identified by annotations in the left margin. There is a color coding scheme that applies here that we also use in table headings. The color **green** indicates that a boxed equation, left-margin annotation, or table pertains in a general manner, meaning that it's valid in spaces of all dimensionalities. The color **blue** indicates that the information pertains specifically to three-dimensional space and is not valid elsewhere. And finally, the color **red** indicates that the information pertains specifically to two-dimensional space.

Website

The official website for *Projective Geometric Algebra Illuminated* can be found at the following address:

projectivegeometricalgebra.org

The website contains reference materials, exercises that accompany this book, source code for the math library, and additional resources.

Prerequisites

This book assumes a working knowledge of basic linear algebra, including vectors, matrices, dot products, cross products, and linear transformations. Some knowledge of group theory may be useful but is not required.

The accompanying math library is written in C++, but there is no code in the book itself.

Appendices

There are three appendices that provide useful information.

- Appendix A contains multiplication tables for the geometric product and antiproduct in conformal algebras that are too large to include in the main text.
- Appendix B contains tables of geometric properties for the various objects that arise in three-dimensional and two-dimensional rigid and conformal algebras.
- Appendix C is a notation reference that includes the page numbers where each specific type of notation is defined.

Notation

Any notation specific to geometric algebra or unique to this book is explained when it is first used, and these cases are also indexed in Appendix C. The general notational conventions for scalars, vectors, and matrices are as follows.

- Scalars are written in italics.
- Vectors and matrices are written in bold, but components of a vector and entries of a matrix are written in italics. For example, $\mathbf{v} = (v_x, v_y, v_z)$.
- For a vector $\mathbf{v}$ having more than four components, the notation $\mathbf{v}_{xyz}$ means the 3D vector consisting of only the x, y, and z components.
- The notation $\mathbf{M}_{[j]}$ refers to the j-th column of the matrix $\mathbf{M}$. (This matches the meaning of the bracket operator in the math library.)
- $\mathbf{M}^{\mathrm{T}}$ means the transpose of the matrix $\mathbf{M}$.
- Vectors are treated as column matrices and thus appear on the right in a matrix-vector product such as $\mathbf{Mv}$.
- For a vector $\mathbf{v}$, the notation $\mathbf{v}^2$ means the dot product $\mathbf{v} \cdot \mathbf{v}$.

Math Library

An extensive C++ math library accompanies this book, and it implements classes for conventional linear algebra as well as geometric algebra. Notes about the math library appear in the following format throughout the book to describe code that pertains to the preceding topics. The example here describes the basic vector and matrix capabilities of the library.

Math Library Notes

- The `Vector3D` class represents a generic 3D vector, and it stores three floating-point values named `x`, `y`, and `z`. The `Vector2D` class represents a generic 2D vector, and it stores only `x` and `y` components. Overloaded operators for addition, subtraction, and scalar multiplication are provided.
- The dot product and cross product are implemented by the `Dot()` and `Cross()` functions. The `*` operator is overloaded for vectors, but it performs a componentwise multiplication to be consistent with shading languages.
- The vector classes support swizzling, so it's possible to use syntax like `.yzx` to reorder the components or `.xy` to access only a subset of the components.
- The `Matrix3D` class represents a generic 3×3 matrix, and it stores nine floating-point values as an array of column vectors. The `Matrix2D` class represents a generic 2×2 matrix, and it stores only four entries. Overloaded operators for addition, subtraction, and scalar multiplication are provided. The `*` operator can be used to multiply matrices by other matrices or by vectors.
- The `[]` operator is overloaded for matrices, and `[`j`]` returns the column of a matrix specified by the zero-based index j as a reference to the vector object at the address where the column storage begins.
- The `()` operator is overloaded for matrices, and `(`i`,`j`)` returns the entry in row i and column j.
- The determinant, inverse, and adjugate of a matrix are calculated by the `Determinant()`, `Inverse()`, and `Adjugate()` functions.

Acknowledgements

Thanks are owed to the many people who supported the my work in projective geometric algebra over the past dozen years and more. Comments, suggestions, and words of encouragement have fueled the motivation to pursue this subject for so long.

Special thanks to Peeter Joot and Freya Holmér for providing feedback on draft versions of this book. Their comments helped make this a better product.

Finally, I'd like to express gratitude and appreciation for my wife, Andrea, who tirelessly provided the support I needed during the long months of labor that I poured into this work. I love you.

Eric Lengyel
March 11, 2024

Chapter 1

Conventional Mathematics

We begin with a short survey of several mathematical concepts that were developed and put to practical use without any knowledge of geometric algebra. These topics were chosen specifically because each one is really a puzzle piece containing a small part of geometric algebra expressed in its own independent form. These pieces often come with unusual quirks that can have somewhat unsatisfying explanations without the benefit of being able to see the complete picture. Assembling the rest of the puzzle in the chapters that follow will reveal a fundamental order to things and establish connections among concepts discussed here that may currently seem like they are isolated from one another. Presenting the topics of this chapter in a conventional setting will allow us to maintain bridges back to islands of familiarity as we expand our understanding.

1.1 The Cross Product

The cross product was established as a common mathematical tool in the late 19th century and is now ubiquitous in virtually all three-dimensional settings involving vector quantities. In a right-handed coordinate system, the cross product between vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ generates a new quantity with three components given by

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x). \tag{1.1}$$

This can also be expressed as a matrix product by first defining the 3×3 matrix $[\mathbf{a}]_\times$ by

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}, \tag{1.2}$$

and then computing the matrix-vector product $[\mathbf{a}]_\times \mathbf{b}$, which yields the same result. If we interpret the quantity produced by the cross product as another vector, then we find that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and its magnitude is equal to the area of a parallelogram whose sides are parallel to $\mathbf{a}$ and $\mathbf{b}$ with the same lengths as $\mathbf{a}$ and $\mathbf{b}$.

Any triplet of numbers can be considered a "vector" to the extent that it is an element of a generic three-dimensional vector space, but not all meaningful quantities having three components behave in the same way. Long known to physicists, there are two different types of vectors called *polar* vectors and *axial* vectors that require different calculations in order to be transformed from one coordinate system to another. A polar vector is an ordinary direction and magnitude such as a tangent or linear velocity that can often be derived from the difference between two positions. An axial vector is a quantity such as angular velocity or torque that is derived from a cross product between two polar vectors.

The difference in transformation properties is well demonstrated by considering vectors that represent surface normals. These vectors must remain perpendicular to the surface to which they are normal after a transformation is applied or else they would lose this inherent property that makes them the type of vector that they are. For triangulated surfaces, face normals are often generated by calculating the cross product between two sides of each triangle, as shown in Figure 1.1. Let $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ be the vertices of a triangle wound in the counterclockwise direction. An outward-facing normal vector $\mathbf{n}$ is then given by

$$\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0). \tag{1.3}$$

Any permutation of the subscripts that keeps them in the same cyclic order produces the same normal vector. It doesn't matter which vertex is chosen to be subtracted from the other two as long as the first factor in the cross product involves the next vertex in the counterclockwise order. If the order is reversed, then the calculated normal vector still lies along the same line, but it points in the opposite direction.

When a polygonal model is transformed by a 3×3 matrix $\mathbf{M}$ in order to alter its geometry, every point $\mathbf{p}$ belonging to the original model becomes a point $\mathbf{Mp}$ in the transformed model, where $\mathbf{p}$ is treated as a 3×1 column matrix. Since a tangent vector $\mathbf{t}$ can be approximated by the difference of points $\mathbf{p}$ and $\mathbf{q}$ on a surface, or is often exactly equal to such a difference, it is transformed in the same way as a point to become $\mathbf{Mt}$ because the difference between the new points $\mathbf{Mp}$ and $\mathbf{Mq}$ is tangent to the new surface. Problems arise, however, if we attempt to apply the same transformation to normal vectors.

Consider the shape shown in Figure 1.2 that has a normal vector $\mathbf{n}$ on its slanted side. Let $\mathbf{M}$ be the transformation matrix that scales by a factor of two in the horizontal direction but does not scale in the vertical direction. When we align the x axis with the horizontal direction and the y axis with the vertical direction, $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.4}$$

If the matrix $\mathbf{M}$ is multiplied by $\mathbf{n}$, then the resulting vector $\mathbf{Mn}$ is stretched horizontally and, as clearly visible in the figure, is no longer perpendicular to the surface. This indicates that something

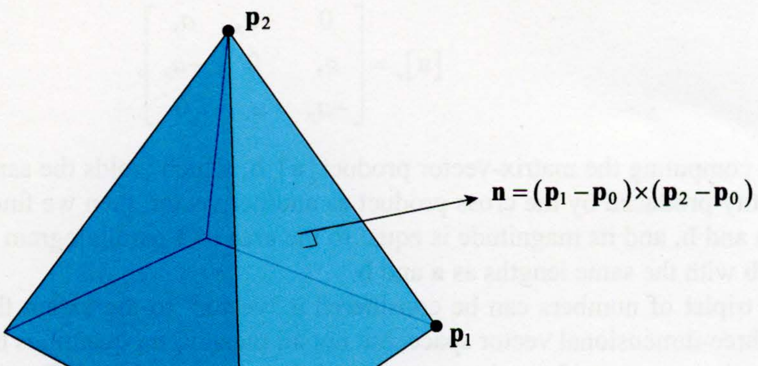


Figure 1.1. The normal vector $\mathbf{n}$ for a triangular face having vertices $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ is given by the cross product between vectors corresponding to two edges of the triangle.

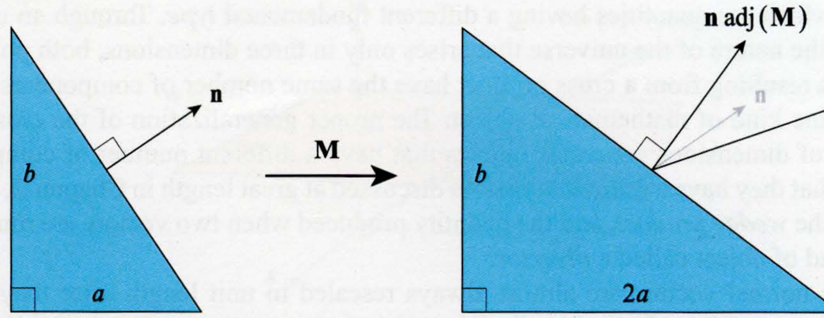


Figure 1.2. A shape is transformed by a matrix **M** that scales by a factor of two only in the horizontal direction. The normal vector **n** is perpendicular to the original surface, but if it is treated as a column vector and transformed by the matrix **M**, then it is not perpendicular to the transformed surface. The normal vector is correctly transformed by treating it as a row vector and multiplying by $\operatorname{adj}(\mathbf{M})$. (The original normal vector is shown in light gray on the transformed surface.)

is inherently different about normal vectors, and if we want to preserve perpendicularity, then we must find another way to transform them that produces the correct results.

Suppose that the matrix **M** transforms ordinary polar vectors from coordinate system A to coordinate system B. Let $\mathbf{n}_A$ represent the normal vector in coordinate system A, and let $\mathbf{n}_B$ represent the same normal vector after it has been transformed into coordinate system B. If the normal vector $\mathbf{n}_A$ is calculated as the cross product $\mathbf{s} \times \mathbf{t}$ between two tangent vectors **s** and **t**, then the transformed normal vector $\mathbf{n}_B$ should be equal to the cross product between the transformed tangent vectors. This means it must be true that $\mathbf{n}_B = (\mathbf{Ms}) \times (\mathbf{Mt})$, but we need to be able to calculate $\mathbf{n}_B$ without any knowledge of the vectors **s** and **t**. Expanding the matrix-vector products **Ms** and **Mt** by columns, we can write

$$\mathbf{n}_B = (s_x\mathbf{M}_{[0]} + s_y\mathbf{M}_{[1]} + s_z\mathbf{M}_{[2]}) \times (t_x\mathbf{M}_{[0]} + t_y\mathbf{M}_{[1]} + t_z\mathbf{M}_{[2]}), \tag{1.5}$$

where we are using the notation $\mathbf{M}_{[j]}$ to mean column j of the matrix **M** (matching the meaning of the [ ] operator in the math library's matrix classes). After distributing the cross product to all of these terms and simplifying, we arrive at

$$\begin{aligned} \mathbf{n}_B = {} & (s_y t_z - s_z t_y)(\mathbf{M}_{[1]} \times \mathbf{M}_{[2]}) \\ & + (s_z t_x - s_x t_z)(\mathbf{M}_{[2]} \times \mathbf{M}_{[0]}) \\ & + (s_x t_y - s_y t_x)(\mathbf{M}_{[0]} \times \mathbf{M}_{[1]}). \end{aligned} \tag{1.6}$$

The cross product $\mathbf{n}_A = \mathbf{s} \times \mathbf{t}$ is clearly visible here, but it may be a little less obvious that the three cross products of the matrix columns form the three rows of the adjugate of **M**. (To see why, recall that the product $\operatorname{adj}(\mathbf{M})\,\mathbf{M}$ is the diagonal matrix with all diagonal entries equal to $\det(\mathbf{M})$, and consider the scalar triple products $(\mathbf{M}_{[i]} \times \mathbf{M}_{[j]}) \cdot \mathbf{M}_{[k]}$ for $0 \le k \le 2$. If $k \ne i$ and $k \ne j$, then we get $\det(\mathbf{M})$. If $k = i$ or $k = j$, then we get zero.) We conclude that a normal vector calculated with a cross product is correctly transformed according to

Normal vector transformation

$$\boxed{\mathbf{n}_B = \mathbf{n}_A \operatorname{adj}(\mathbf{M}),} \tag{1.7}$$

where we are now treating $\mathbf{n}_A$ and $\mathbf{n}_B$ as 1×3 row matrices.

Equation (1.7) tells us not only how normal vectors transform, but how *any* vector resulting from a cross product transforms. This happens because the cross product between two vectors doesn't actually produce another vector in the strict meaning of the term. Normal vectors are axial vectors, which are quantities having a different fundamental *type*. Through an unfortunate coincidence in the nature of the universe that arises only in three dimensions, both polar vectors and the quantities resulting from a cross product have the same number of components, so they appear to be the same kind of mathematical object. The proper generalization of the cross product to other numbers of dimensions generates objects that have a different number of components, making it obvious that they have a different type. As discussed at great length in Chapter 2, this generalization is called the *wedge product*, and the quantity produced when two vectors are multiplied together is a new kind of object called a *bivector*.

Since normal vectors are almost always rescaled to unit length after they're calculated, the implicit factor of $\det(\mathbf{M})$ in the adjugate matrix is inconsequential and often ignored in practice. However, there is one situation in which $\det(\mathbf{M})$ may have an impact, and that is the case when the transform performed by $\mathbf{M}$ contains a reflection. When the vertices of a triangle are reflected in a mirror, their winding orientation is reversed, and this causes a normal vector calculated with the cross product of the triangle's edges to reverse direction as well. This is exactly the effect that a negative determinant of $\mathbf{M}$ would have on a normal vector that is transformed by Equation (1.7).

Returning to the example in Figure 1.2, we can take the normal vector before the transformation to be $\mathbf{n}_A = [b \;\; a \;\; 0]$. Applying Equation (1.7) gives us a new normal vector $\mathbf{n}_B = [b \;\; 2a \;\; 0]$, which is perpendicular to the transformed surface. Had we transformed the normal vector in the same way as an ordinary vector by treating it as a column matrix and calculating $\mathbf{n}_B = \mathbf{M}\mathbf{n}_A$, then we would have obtained $\mathbf{n}_B = (2b, a, 0)$. As noted earlier, this is an incorrect result that is not perpendicular to the transformed surface. There is a situation in which it is safe to transform a normal vector like a polar vector, however, and that's when the matrix $\mathbf{M}$ is orthogonal. In that case, we have $\mathbf{M}^{-1} = \mathbf{M}^{\mathrm{T}}$ and $\det(\mathbf{M}) = \pm 1$, so the transformed normal vector as a row matrix is given by $\mathbf{n}_B = \pm\mathbf{n}_A\mathbf{M}^{\mathrm{T}}$. But this is equivalent to calculating $\mathbf{n}_B = \pm\mathbf{M}\mathbf{n}_A$ if we treat the normal vector as a column matrix, and only need to worry about the minus sign if $\mathbf{M}$ could contain a reflection. If we're working only with rotation matrices, then the difference in transformation properties between polar vectors and axial vectors disappears. This contributes to the frequency with which the distinction between the two types of vectors is overlooked.

1.2 Homogeneous Coordinates

An n-dimensional vector represents nothing more than a magnitude and direction. The invertible $n \times n$ matrices that operate on a vector all perform some kind of transformation that simply replaces one set of basis directions with another. We can rotate, scale, and skew our vectors, but they always represent a relative change of some kind. Even a position vector represents only a relative offset from some chosen origin, and it cannot be distinguished from the difference between two positions elsewhere in space. If we wanted to translate our coordinate system in order to move the origin to a new location, we would have no way of knowing whether a vector corresponds to a positional quantity that should be affected by the translation or whether it's a directional quantity that should not be affected. This ambiguity is eliminated by using a higher-dimensional representation called *homogeneous coordinates*. Homogeneous coordinates are a staple of 3D computer graphics and other practical disciplines, and they give us our first taste of a projective vector space. We limit our discussion here to three dimensions, but it should be understood that homogeneous coordinates can be applied to any number of dimensions.

In homogeneous coordinates over a three-dimensional vector space, we append a fourth number called the w coordinate to every vector so that an arbitrary vector $\mathbf{v}$ is written as (v_x, v_y, v_z, v_w). A point in 3D space is associated with each 4D vector $\mathbf{v}$ by considering a line of infinite extent that passes through the origin in 4D space and is parallel to $\mathbf{v}$. The 3D point corresponding to $\mathbf{v}$ is given by the x, y, and z coordinates at the unique location where a point on the associated line has a w coordinate equal to one. Because all scalar multiples of $\mathbf{v}$ correspond to offsets from the origin to

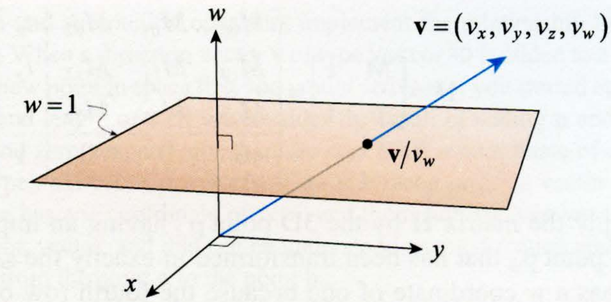


Figure 1.3. A homogeneous vector v is projected into 3D space by dividing by its w coordinate to determine the point where it intersects the subspace where $w = 1$. The z axis is omitted from the figure due to the difficulties inherent in drawing a four-dimensional diagram on a two-dimensional page, but it should be understood that the subspace for which $w = 1$ is three-dimensional and extends in the z direction as well as the x and y directions.

points on the line parallel to **v**, we can simply divide all of the components of **v** by the coordinate v_w to find the location where the line intersects the subspace where $w = 1$, as shown in Figure 1.3. Homogeneous coordinates are so named because any nonzero scalar multiple of a 4D vector **v** produces the same 3D point after dividing by the w coordinate. This is a projection of an intrinsically one-dimensional object, a line, to an intrinsically zero-dimensional object, a point, accomplished by viewing only one 3D slice of the larger 4D space.

If $v_w = 0$, we clearly cannot divide by the w coordinate of **v** to produce a 3D point. A line running in the direction of the vector $(x, y, z, 0)$ is parallel to the subspace where $w = 1$, so there is no intersection at any finite location. Thus, the vector $(x, y, z, 0)$, having a w coordinate of zero, is considered to be the point at infinity in the direction (x, y, z) when projected into 3D space. In graphical applications, such a point is sometimes used to describe the location of an object like the sun that can be treated as being infinitely far away within all practical limits. In these cases, we are describing the location of the object not by providing its absolute position, but by providing the direction that points toward the object.

Generally, 4D homogeneous vectors fall into two classes determined by whether the w coordinate is zero or nonzero. This lets us make an important distinction between 3D vectors that are intended to represent directions and 3D vectors that are intended to represent positions. It is often unnecessary to carry around a fourth coordinate in memory when computing with either type of vector because we can design our data structures in such a way that the value of the w coordinate is implied. We continue using 3D vectors for both directions and positions, but we establish a rule for converting each type to a 4D homogeneous vector wherever it's necessary. A 3D vector **v** is converted to a 4D vector by appending a w coordinate equal to zero, and a 3D point **p** is converted to a 4D vector by appending a w coordinate equal to one, as in the example shown by

$$\begin{aligned}\mathbf{v} &= (v_x, v_y, v_z, 0)\\ \mathbf{p} &= (p_x, p_y, p_z, 1).\end{aligned} \tag{1.8}$$

One of the main advantages to using homogeneous coordinates is the ability to incorporate translations into coordinate transformations by using 4×4 matrices. A general affine transformation from coordinate system A to coordinate system B is given by

$$\mathbf{p}_B = \mathbf{M}\mathbf{p}_A + \mathbf{t}, \tag{1.9}$$

where $\mathbf{M}$ is a 3×3 transformation matrix and $\mathbf{t}$ is a 3D translation vector. These can be combined into a single 4×4 transformation matrix $\mathbf{H}$ having the form

$$\mathbf{H} = \begin{bmatrix} \mathbf{M} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \left[\begin{array}{ccc:c} M_{00} & M_{01} & M_{02} & t_x \\ M_{10} & M_{11} & M_{12} & t_y \\ M_{20} & M_{21} & M_{22} & t_z \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \tag{1.10}$$

When we multiply the matrix $\mathbf{H}$ by the 3D point $\mathbf{p}_A$ having an implicit w coordinate of one, the product is a 3D point $\mathbf{p}_B$ that has been transformed in exactly the same way as in Equation (1.9). The result still has a w coordinate of one because the fourth row of the matrix $\mathbf{H}$ is $[0\ \ 0\ \ 0\ \ 1]$, which preserves the w coordinate of any 4D vector that it multiplies.

When the matrix $\mathbf{H}$ is used to transform a direction vector $\mathbf{v}$ having an implicit w coordinate of zero, the translation in the fourth column of $\mathbf{H}$ has no effect because those entries are always multiplied by the fourth coordinate of $\mathbf{v}$. A direction vector carries no information about position and is not altered by a translation of the coordinate system. Only the upper-left 3×3 portion of $\mathbf{H}$ containing the matrix $\mathbf{M}$ participates in the transformation of a direction vector.

We can accumulate transforms by multiplying as many matrices like $\mathbf{H}$ together as we want, and we will always have a matrix with a fourth row equal to $[0\ \ 0\ \ 0\ \ 1]$. Matrices of the form shown in Equation (1.10) belong to a multiplicatively closed subset of the entire set of 4×4 matrices. It is this type of matrix that is used by applications dealing with 3D space to represent a general linear transformation from one coordinate space to another. Each object in the world typically has such a transform associated with it that describes how the object's local coordinate system is embedded within some higher space in a model hierarchy or within the global coordinate system. The first three columns of the 4×4 matrix correspond to the directions in which the object's local x, y, and z axes point in the higher coordinate system, and the fourth column of the 4×4 matrix corresponds to the position of the object's local origin in the higher coordinate system.

We would expect that the matrix $\mathbf{H}$ could be inverted as long as the matrix $\mathbf{M}$ occupying the upper-left 3×3 portion of $\mathbf{H}$ represented some kind of invertible transform because the translation in the fourth column of $\mathbf{H}$ is something that can always be reversed. If we calculate the determinant of $\mathbf{H}$ in Equation (1.10) by expanding minors along the fourth row, then it becomes apparent that it's the same as the determinant of the matrix $\mathbf{M}$. This makes sense because solving Equation (1.9) for $\mathbf{p}_A$ gives us

$$\mathbf{p}_A = \mathbf{M}^{-1}(\mathbf{p}_B - \mathbf{t}), \tag{1.11}$$

which requires only that we can invert $\mathbf{M}$. Using this equation for transforming in the reverse direction from coordinate system B back into coordinate system A, the inverse of the 4×4 matrix $\mathbf{H}$ should be given by

$$\boxed{\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}}, \tag{1.12}$$

and this is easily verified to be correct.

Math Library Notes

- In order to support two types of three-component vectors with different transformation properties, the math library includes a second class to complement the `Vector3D` class. The `Vector3D` class represents a direction vector, and it possesses an implicit w coordinate of zero when necessary. A class named `Point3D` represents a position vector, and it possesses an implicit w coordinate of one. The `Point3D` class a subclass of the `Vector3D` class so it inherits the same data members and so an object of type `Point3D` can be accepted anywhere that an object of type `Vector3D` is expected.

- Overloaded addition and subtraction operators implement the relationship between direction vectors and position vectors. When a direction vector $\mathbf{v}$ of type `Vector3D` is added to a position vector $\mathbf{p}$ of type `Point3D`, it yields a new point in space that you would arrive at if you started at the point $\mathbf{p}$ and travelled along the direction and length of $\mathbf{v}$. If we consider the result of adding $\mathbf{p}$ and $\mathbf{v}$ as 4D vectors with w coordinates of one and zero, respectively, then the sum has a w coordinate of one, indicating that it is a position vector of type `Point3D`. Conversely, if we subtract a position vector $\mathbf{b}$ from a position vector $\mathbf{a}$, then the difference has a w coordinate of zero, and the result has type `Vector3D`. This indicates that the result is a direction vector, and this can be understood as the direction and distance that you would need to travel to go from the point $\mathbf{a}$ to the point $\mathbf{b}$.
- The `Transform3D` class represents a 4×4 matrix having the form shown in Equation (1.10). This class is a subclass of the `Matrix4D` data structure so that it can be used wherever a general 4×4 matrix is expected, but its fourth row is always assumed to be equal to $[0 \;\; 0 \;\; 0 \;\; 1]$ whenever calculations are performed with it. The constructors for the `Transform3D` class take data only for the first three rows and always set the fourth row to $[0 \;\; 0 \;\; 0 \;\; 1]$. The first three columns are treated as 3D direction vectors due to the fact that each column has a zero in its fourth entry. Likewise, the fourth column is treated as a 3D position vector due to the fact that it has a one in its fourth entry. This behavior is implemented by the overridden bracket operators and the `GetTranslation()` and `SetTranslation()` member functions.
- The `Inverse()` function that operates on a `Transform3D` object is a simplified version of a full 4×4 inversion that accounts for the constant values in the fourth row of the matrix. The matrix-matrix multiplication operator for `Transform3D` objects also takes advantage of the known values in the fourth row. Functions that multiply a `Transform3D` object by `Vector3D` and `Point3D` objects account for the w coordinates implied by each type of vector.

1.3 Lines and Planes

It is extremely common for 3D applications to make use of lines and planes, which we can describe as one-dimensional and two-dimensional flat surfaces having infinite extent. Both of these types of geometries have parametric representations that are often adequate for particular tasks, especially when lines are used as unidirectional rays. They also have implicit forms composed of different kinds of homogeneous coordinates, linking them in a way to the points and directions discussed in the previous section. Here, we present the mathematics in a conventional setting where some important relationships will not be apparent. An elegant unification of point, line, and plane representations that explains how they arise from a single concept is a central topic in Chapter 2.

1.3.1 Parametric Forms

To express a line parametrically, we can begin by considering two points $\mathbf{p}_1$ and $\mathbf{p}_2$ contained by the line and define the function

$$\boldsymbol{l}(t) = (1-t)\,\mathbf{p}_1 + t\mathbf{p}_2. \tag{1.13}$$

This function produces all points on the line passing through $\mathbf{p}_1$ and $\mathbf{p}_2$ in terms of a single parameter t that ranges over all real numbers. When $0 \leq t \leq 1$, the points fall inside the segment connecting $\mathbf{p}_1$ and $\mathbf{p}_2$. Otherwise, the points fall elsewhere on the line extending to infinity in both directions.

The function $\boldsymbol{l}(t)$ can be rewritten as

$$\boldsymbol{l}(t) = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1), \tag{1.14}$$

which is equivalent to Equation (1.13) but makes it clear that a line can be expressed in terms of a point and a direction. We can now express lines with the parametric function

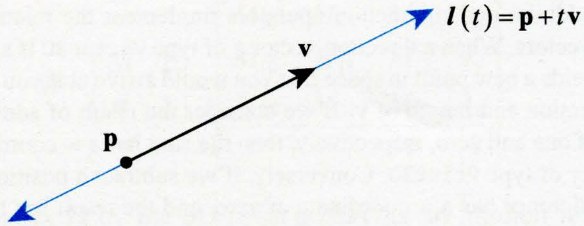


Figure 1.4. A parametric line is defined by a point **p** on the line and a direction vector **v** parallel to the line.

Parametric line

$$\boxed{l(t) = \mathbf{p} + t\mathbf{v},} \tag{1.15}$$

where **p** is a point on the line, and **v** is a direction parallel to the line, as shown in Figure 1.4. It is often the case that **v** is normalized to unit length so that the parameter t corresponds to the actual distance from the starting point **p**.

Because a function of one parameter can describe the intrinsically one-dimensional geometry of a line, it is logical to expect that a function of two parameters can describe the intrinsically two-dimensional geometry of a plane. Indeed, given three points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ that lie in a plane and are not collinear, the function

$$\mathbf{g}(s, t) = \mathbf{p}_1 + s(\mathbf{p}_2 - \mathbf{p}_1) + t(\mathbf{p}_3 - \mathbf{p}_1) \tag{1.16}$$

produces all of the points in the entire plane as the parameters s and t each range over all real numbers. As with lines, we can replace the differences between points with two direction vectors **u** and **v**, and that gives us the parametric form of a plane

$$\mathbf{g}(s, t) = \mathbf{p} + s\mathbf{u} + t\mathbf{v}. \tag{1.17}$$

However, a function of this type is not typically used in practice to represent planes. The implicit form described below provides a superior alternative and is the preferred representation in virtually all applications.

There is another way to express a plane parametrically that involves the use of a vector parameter instead of a scalar parameter. Given a point **p** lying in the plane and a vector **n** normal to the plane, the function

Parametric plane

$$\boxed{\mathbf{g}(\mathbf{v}) = \mathbf{p} + \mathbf{v} \times \mathbf{n}} \tag{1.18}$$

is also a point in the plane because the direction $\mathbf{v} \times \mathbf{n}$ must be perpendicular to **n**, as shown in Figure 1.5. All of the points in the plane are produced as **v** ranges over all possible 3D vectors. Of course, this is highly redundant since any point in the plane is produced by many different choices of the vector **v**. Nevertheless, Equation (1.18) corresponds to the natural generalization of parametric functions to higher-dimensional objects, which is a topic discussed further in Section 2.13.4.

1.3.2 Implicit Forms

The parametric form of a line requires that we know a specific point lying on the line, and the parametric form of a plane requires that we know a specific point lying in the plane. These points can be arbitrary, and that means that the actual coordinates used to represent a parametric line or plane are also arbitrary. We can normalize the vectors that the parameters multiply, but there are still an infinite number of choices for the anchor point contained in the geometry. What we would

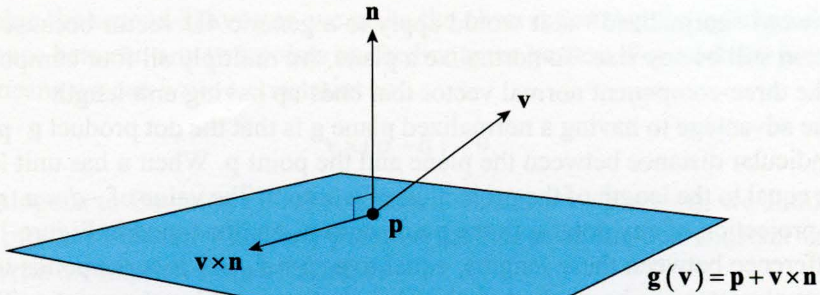


Figure 1.5. A plane is determined by a single point **p** and a normal vector **n**. Points in the plane are produced parametrically by adding **v**×**n** to **p** as the parameter **v** ranges over all 3D vectors.

like to do is come up with a standard representation that removes this choice by replacing the point with some implicit quantity that is independent of the points from which a line or plane is constructed. This is a bit easier to accomplish for planes, so we tackle them first.

A plane's normal vector **n** is perpendicular to the difference between any two distinct points **p** and **q** lying in the plane. This means we can write the equation

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{q}) = 0 \tag{1.19}$$

to describe a constraint that all pairs of points in the plane must satisfy. With a small adjustment, this equation becomes $\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{q}$, which tells us that the dot product between **n** and any point **p** in the plane has the same value as the dot product between **n** and any other point **q** in the plane. This constant dot product is an implicit property of the plane that allows us to throw away knowledge of any particular point and rewrite Equation (1.19) as

$$\mathbf{n} \cdot \mathbf{p} + d = 0, \tag{1.20}$$

where $d = -\mathbf{n} \cdot \mathbf{q}$ can be calculated from any point **q** known to lie in the plane. This equation must be satisfied by every point **p** that the plane contains.

A plane is implicitly described by the four numbers n_x, n_y, n_z, and d constituting the components of a four-dimensional row vector $\mathbf{g} = [\,n_x \;\; n_y \;\; n_z \;\; d\,]$, which we write using the shorthand notation $[\,\mathbf{n} \mid d\,]$. With this, Equation (1.20) becomes the more compact

Implicit plane

$$\boxed{\mathbf{g} \cdot \mathbf{p} = 0,} \tag{1.21}$$

where we extend the point **p** to four dimensions by using its implicit w coordinate of one. All points **p** satisfying this equation lie in the plane $\mathbf{g} = [\,\mathbf{n} \mid d\,]$.

As with normal vectors earlier in this chapter, we have defined **g** as a row vector. Since **p** is a column vector, the matrix product **gp** actually gives us the left side of Equation (1.21), but the dot is still included by convention to make it clear that we are calculating $g_x p_x + g_y p_y + g_z p_z + g_w p_w$ (and we are setting $p_w = 1$). In Chapter 2, we will replace the dot with a different symbol when the algebraic nature of planes is discussed more thoroughly.

Multiplying both sides of Equation (1.21) by a nonzero scalar quantity s has no effect on the set of points that satisfy the equation. The implicit form of a plane is *homogeneous* such that $\mathbf{g} = [\,\mathbf{n} \mid d\,]$ and $s\mathbf{g} = [\,s\mathbf{n} \mid sd\,]$ both represent the same geometric plane in space. This motivates us ask whether there is a value of s that gives the representation any appealing qualities. The answer is yes, and it is the value $s = 1/\lVert\mathbf{n}\rVert$. In this case, the plane $s\mathbf{g}$ is said to be *normalized* because its normal vector has unit length. However, it's important to realize that this is not the same meaning of the word "normalized" as it would apply to a generic 4D vector because the value of d for the plane can still be any size. To normalize a plane, we multiply all four components by $1/\|\mathbf{n}\|$, but it's only the three-component normal vector that ends up having unit length.

The advantage to having a normalized plane $\mathbf{g}$ is that the dot product $\mathbf{g} \cdot \mathbf{p}$ is equal to the signed perpendicular distance between the plane and the point $\mathbf{p}$. When $\mathbf{n}$ has unit length, the dot product $\mathbf{n} \cdot \mathbf{p}$ is equal to the length of the projection of $\mathbf{p}$ onto $\mathbf{n}$. The value of $-d = \mathbf{n} \cdot \mathbf{q}$ is equal to the length of the projection of any point $\mathbf{q}$ in the plane onto $\mathbf{n}$. As illustrated in Figure 1.6, the value of $\mathbf{g} \cdot \mathbf{p}$ is the difference between these lengths, equal to $\mathbf{n} \cdot \mathbf{p} + d$, and it corresponds to the number of times the normal vector can be stacked on the plane before reaching the point $\mathbf{p}$. The value of d is sometimes called the distance to the origin because it's what you get when you evaluate the dot product $\mathbf{g} \cdot \mathbf{o}$ for the origin $\mathbf{o}$.

The sign of the distance given by $\mathbf{g} \cdot \mathbf{p}$ depends on which side of the plane $\mathbf{p}$ lies. If the normal vector $\mathbf{n}$ were to be drawn so that its arrow starts on the plane, then it points away from the *front side* of the plane. This is also called the *positive side* of the plane because for any points lying on this side, $\mathbf{g} \cdot \mathbf{p}$ is a positive value. Naturally, the other side of the plane is called the *back side* or *negative side* of the plane, and for points lying on that side, $\mathbf{g} \cdot \mathbf{p}$ is a negative value. The meaning of front and back can be reversed by simply negating the plane $\mathbf{g}$ because $-\mathbf{g}$ represents the same set of points, but with a reversed normal vector.

As with planes, is it possible to describe a line in three dimensions using an implicit form that requires no knowledge of any particular point it contains. The components for this form of a line are called *Plücker coordinates*. Despite lines having lower dimensionality than planes, the implicit form for a line in 3D space is more complicated and utilizes six components instead of four. The reason for this is that a line has one more degree of freedom when it comes to its position and attitude in space.

The six Plücker coordinates for a line can be grouped as two 3D vectors. Given a line that passes through the distinct points $\mathbf{p}_1$ and $\mathbf{p}_2$, one of those 3D vectors is the difference $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$, which is simply the direction parallel to the line. Once $\mathbf{v}$ has been calculated, it no longer contains any information about specific points on the line. The difference between any two points on the line separated by the same distance produces the same value of $\mathbf{v}$.

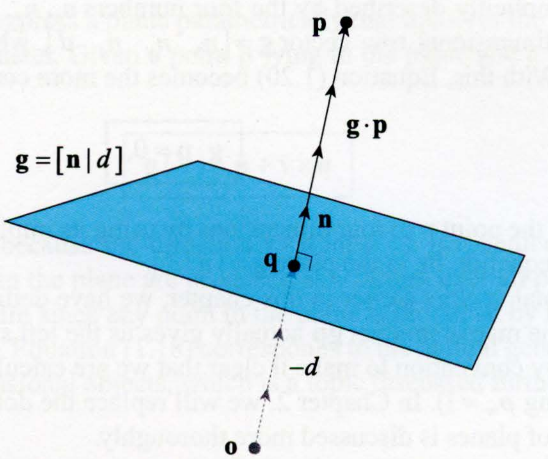


Figure 1.6. The signed perpendicular distance between a point $\mathbf{p}$ and a normalized plane $\mathbf{g} = [\mathbf{n} \mid d]$ is given by the dot product $\mathbf{g} \cdot \mathbf{p}$. This can be understood as the difference between the distances measured perpendicular to the plane from $\mathbf{p}$ to the origin $\mathbf{o}$ and from a point $\mathbf{q}$ in the plane to the origin. The perpendicular distances are calculated by projecting onto the normal vector so that the difference becomes $\mathbf{n} \cdot \mathbf{p} - \mathbf{n} \cdot \mathbf{q} = \mathbf{n} \cdot \mathbf{p} + d$. As illustrated, the value of $\mathbf{g} \cdot \mathbf{p}$ corresponds to the number of normal vectors that fit between the plane and the point $\mathbf{p}$, and the value of $-d$ is the number of normal vectors needed to reach the plane itself.

To determine the second 3D vector, we make the observation that the cross product between $\mathbf{v}$ and the difference between any two points on the line must be zero. For arbitrary points $\mathbf{p}$ and $\mathbf{q}$ on the line, this means we can write a constraint as

$$\mathbf{v} \times (\mathbf{p} - \mathbf{q}) = \mathbf{0}, \tag{1.22}$$

and this is analogous to Equation (1.19) for planes. Because $\mathbf{v} \times \mathbf{p} = \mathbf{v} \times \mathbf{q}$, we similarly conclude that the cross product between $\mathbf{v}$ and any point on the line is an implicit constant vector quantity. We can now write the implicit form of a line as

Implicit line

$$\boxed{\mathbf{v} \times \mathbf{p} + \mathbf{m} = \mathbf{0},} \tag{1.23}$$

which is the analog of Equation (1.21). The quantity $\mathbf{m} = -\mathbf{v} \times \mathbf{q}$, calculated with any point $\mathbf{q}$ known to be on the line, is called the *moment* of the line. We can plug either of the two points that participated in the calculation of $\mathbf{v}$ into the value of $\mathbf{q}$, and doing so with $\mathbf{q} = \mathbf{p}_1$ reveals

$$\mathbf{m} = -\mathbf{v} \times \mathbf{p}_1 = (\mathbf{p}_1 - \mathbf{p}_2) \times \mathbf{p}_1 = \mathbf{p}_1 \times \mathbf{p}_2. \tag{1.24}$$

So a line is implicitly described by a direction vector $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$ and a moment vector $\mathbf{m} = \mathbf{p}_1 \times \mathbf{p}_2$, both of which discard information that would allow recovery of the specific points $\mathbf{p}_1$ and $\mathbf{p}_2$.

The direction $\mathbf{v}$ and moment $\mathbf{m}$, calculated with the same pair of points $\mathbf{p}_1$ and $\mathbf{p}_2$, constitute the six Plücker coordinates for a line, and we write this using the notation $\{\mathbf{v} \mid \mathbf{m}\}$. A line specified with Plücker coordinates is *homogeneous*, again meaning that any nonzero scalar multiple of the components, applied to both $\mathbf{v}$ and $\mathbf{m}$, represents the same line. Multiplying by a scalar is equivalent to moving the points $\mathbf{p}_1$ and $\mathbf{p}_2$ on the line closer together or farther apart. A line is considered normalized when its direction $\mathbf{v}$ has unit length, which is accomplished by dividing all six components of the line by $\|\mathbf{v}\|$.

The moment vector $\mathbf{m}$ of a line is always perpendicular to its direction vector $\mathbf{v}$, and it's trivial to verify that $\mathbf{v} \cdot \mathbf{m} = 0$. As illustrated in Figure 1.7, the direction in which the moment vector points is determined by a right-hand rule. When the fingers of the right hand point in the direction vector $\mathbf{v}$, and the palm faces the origin, the right thumb points in the direction of the moment vector $\mathbf{m}$. For a normalized line, the magnitude of $\mathbf{m}$ is equal to the distance between the origin and the closest point on the line. This means that for any line $\{\mathbf{v} \mid \mathbf{m}\}$, normalized or not, the perpendicular distance to the origin is $\|\mathbf{m}\|/\|\mathbf{v}\|$.

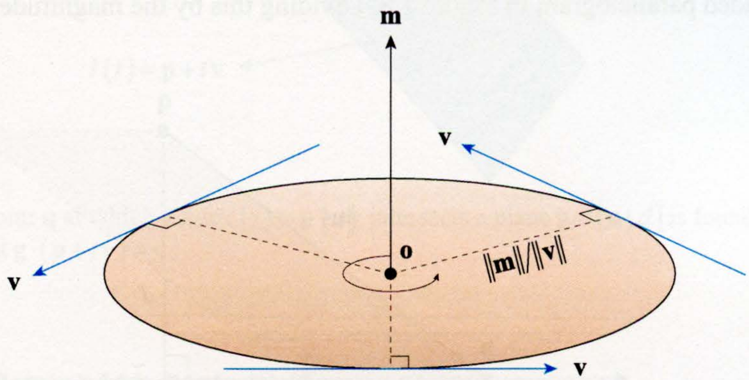


Figure 1.7. The moment vector $\mathbf{m}$ of a line $\{\mathbf{v} \mid \mathbf{m}\}$ is perpendicular to the direction vector $\mathbf{v}$, and the direction in which it points is determined by a right-hand rule. The moment vector is the same for any line tangent to the circle in this example. When the fingers of the right hand point in the direction vector $\mathbf{v}$, and the palm faces the origin $\mathbf{o}$, the right thumb points in the direction of the moment vector $\mathbf{m}$. The perpendicular distance between the origin and the line is equal to $\|\mathbf{m}\|/\|\mathbf{v}\|$.

Math Library Notes

- The `Plane3D` class represents a plane in 3D space. It has floating-point members named `x`, `y`, `z`, and `w` that can be accessed directly, and they reflect the fact that a plane can be treated as a generic 4D row vector. The first three components correspond to the normal vector $\mathbf{n}$, and the w component corresponds to the value of d. The normal vector can be retrieved as a `Vector3D` object by accessing the special `xyz` member.
- The `Line3D` class represents a line in 3D space using Plücker coordinates. The direction vector is stored as a `Vector3D` member `v`, and the moment is stored as `Bivector3D` member `m`. (The `Bivector3D` type is introduced in Chapter 2.) These members can be accessed directly.
- Overloaded operators for performing operations with `Plane3D` and `Line3D` objects with the wedge and antiwedge products are discussed in Chapter 2.

1.3.3 Distance Between a Point and a Line

Suppose that we want to find the distance d between a point $\mathbf{q}$ and the nearest point on the line given by $\boldsymbol{l}(t) = \mathbf{p} + t\mathbf{v}$. As shown in Figure 1.8, we can set up a right triangle in which one side is parallel to the line's direction $\mathbf{v}$, and the length of the hypotenuse is the distance between $\mathbf{p}$ and $\mathbf{q}$. The distance d that we want to find is the length of the third side, which is perpendicular to the line. For convenience, we define $\mathbf{u} = \mathbf{q} - \mathbf{p}$. The length of the side parallel to the line's direction is the magnitude of the projection of $\mathbf{u}$ onto the line, which is given by $|\mathbf{u} \cdot \mathbf{v}| / \mathbf{v}^2$. This means that we can express d as

Distance from point to line

$$\boxed{d = \sqrt{\mathbf{u}^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v}^2}}.} \tag{1.25}$$

If $\mathbf{v}$ is known to have unit length, then the division by $\mathbf{v}^2$ can be omitted.

When the points $\mathbf{p}$ and $\mathbf{q}$ are far apart, which doesn't necessarily mean that $\mathbf{q}$ is far from the line, the sizes of $\|\mathbf{u}\|$ and $\mathbf{u} \cdot \mathbf{v}$ can become very large. Squaring these quantities makes them even larger, and subtracting two large floating-point numbers as we have done in Equation (1.25) results in a loss of precision that can be quite severe. Fortunately, this problem can be mitigated to a degree by using an alternative method to calculate the distance. The magnitude of $\mathbf{u} \times \mathbf{v}$ is equal to the area of the shaded parallelogram in Figure 1.8. Dividing this by the magnitude of $\mathbf{v}$, which corresponds

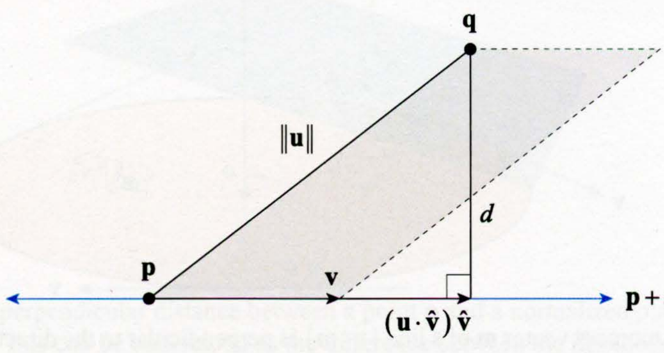


Figure 1.8. The distance d from a point $\mathbf{q}$ to the line $\mathbf{p} + t\mathbf{v}$ can be calculated by constructing a right triangle in which one side is parallel to the line and the hypotenuse has length $\|\mathbf{u}\|$, where $\mathbf{u} = \mathbf{q} - \mathbf{p}$. The length of the side parallel to the line is the magnitude of the projection of $\mathbf{u}$ onto $\mathbf{v}$. The shaded parallelogram has an area equal to $\|\mathbf{u} \times \mathbf{v}\|$, so d is also given by this area divided by the length of $\mathbf{v}$.

to the base of the parallelogram, gives us the value of d, which corresponds to the height of the parallelogram. Thus, we can also express the distance from a point to a line as

Distance from point to line

$$\boxed{d = \sqrt{\frac{(\mathbf{u} \times \mathbf{v})^2}{\mathbf{v}^2}}}, \tag{1.26}$$

and as before, the division by $\mathbf{v}^2$ can be avoided if we know that $\|\mathbf{v}\| = 1$. In the case that $\mathbf{u}$ has a large magnitude, there is still a subtraction of two large numbers happening inside the cross product, but we are not squaring them first, so they are much smaller in size than the numbers arising in Equation (1.25).

1.3.4 Intersection of a Line and a Plane

Let $\mathbf{g} = [\mathbf{n} \mid d]$ be the plane shown in Figure 1.9, and let $\boldsymbol{l}(t) = \mathbf{p} + t\mathbf{v}$ be a line such that $\mathbf{n} \cdot \mathbf{v} \neq 0$, meaning that the line is not parallel to the plane. We can find the point $\mathbf{q}$ at which the line intersects the plane by solving for the value of t that satisfies $\mathbf{g} \cdot \boldsymbol{l}(t) = 0$. A little algebra gives us

$$t = -\frac{\mathbf{g} \cdot \mathbf{p}}{\mathbf{n} \cdot \mathbf{v}}, \tag{1.27}$$

where it should be understood that the numerator is calculated using a 4D dot product in which p_w is implicitly one, but the denominator is calculated using a 3D dot product. Plugging this value of t back into $\boldsymbol{l}(t)$ tells us that the point of intersection $\mathbf{q}$ is given by

Intersection of line and plane

$$\boxed{\mathbf{q} = \mathbf{p} - \frac{\mathbf{g} \cdot \mathbf{p}}{\mathbf{n} \cdot \mathbf{v}}\mathbf{v}.} \tag{1.28}$$

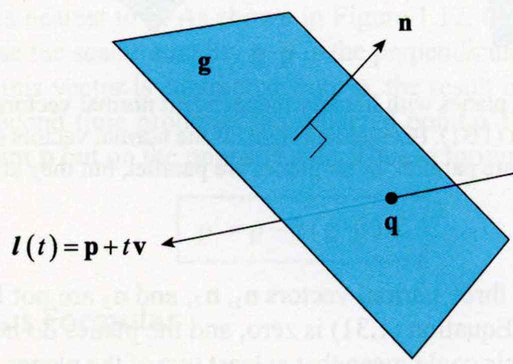


Figure 1.9. The point $\mathbf{q}$ at which a line $\boldsymbol{l}(t) = \mathbf{p} + t\mathbf{v}$ intersects a plane $\mathbf{g} = [\mathbf{n} \mid d]$ is found by solving for the value of t such that $\mathbf{g} \cdot (\mathbf{p} + t\mathbf{v}) = 0$.

1.3.5 Intersection of Multiple Planes

Let $[\mathbf{n}_1 \mid d_1]$, $[\mathbf{n}_2 \mid d_2]$, and $[\mathbf{n}_3 \mid d_3]$ be planes. As long as the normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are linearly independent, the planes intersect at a single point $\mathbf{p}$ in space, as illustrated in Figure 1.10. Since this point lies in all three planes, we know that $[\mathbf{n}_i \mid d_i] \cdot \mathbf{p} = 0$ for $i = 1, 2, 3$, and this can be expressed as the linear system

$$\begin{bmatrix} \leftarrow & \mathbf{n}_1 & \rightarrow \\ \leftarrow & \mathbf{n}_2 & \rightarrow \\ \leftarrow & \mathbf{n}_3 & \rightarrow \end{bmatrix} \mathbf{p} = \begin{bmatrix} -d_1 \\ -d_2 \\ -d_3 \end{bmatrix} \tag{1.29}$$

in which the normal vectors compose the rows of a 3×3 matrix. Solving for $\mathbf{p}$ is a simple matter of multiplying both sides by the inverse of this matrix on the left. Because $(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3$ is the determinant of the matrix in Equation (1.29), its inverse must be given by

$$\begin{bmatrix} \leftarrow & \mathbf{n}_1 & \rightarrow \\ \leftarrow & \mathbf{n}_2 & \rightarrow \\ \leftarrow & \mathbf{n}_3 & \rightarrow \end{bmatrix}^{-1} = \frac{1}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{n}_2 \times \mathbf{n}_3 & \mathbf{n}_3 \times \mathbf{n}_1 & \mathbf{n}_1 \times \mathbf{n}_2 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}. \tag{1.30}$$

Multiplying by the constant vector $(-d_1, -d_2, -d_3)$ yields the intersection point

Intersection of three planes

$$\boxed{\mathbf{p} = \frac{d_1 (\mathbf{n}_3 \times \mathbf{n}_2) + d_2 (\mathbf{n}_1 \times \mathbf{n}_3) + d_3 (\mathbf{n}_2 \times \mathbf{n}_1)}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3},} \tag{1.31}$$

where the order of the factors in each cross product has been reversed to cancel the minus signs.

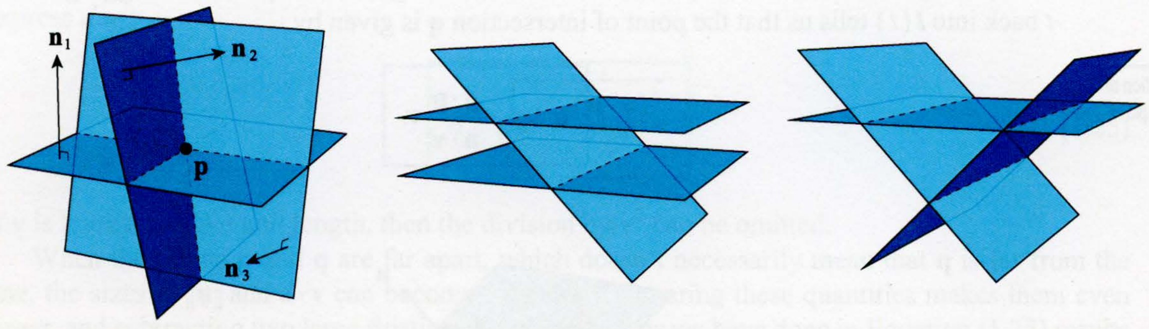


Figure 1.10. (Left) Three planes with linearly independent normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ intersect at a single point $\mathbf{p}$ given by Equation (1.31). (Center and right) If the normal vectors are not linearly independent, either at least two of the planes are parallel, or no planes are parallel, but they all intersect at parallel lines.

In the case that the three normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are not linearly independent, the determinant $(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3$ in Equation (1.31) is zero, and the planes do not intersect at a single point. As shown in Figure 1.10, this could mean that at least two of the planes are parallel to each other. There is also the possibility that no two planes are parallel, but the intersections of each pair of planes occur along parallel lines.

Two nonparallel planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect at a line that must be contained in both planes. To express this line in the parametric form $\boldsymbol{l}(t) = \mathbf{p} + t\mathbf{v}$, we need to find any starting point $\mathbf{p}$ on the line and the direction $\mathbf{v}$ in which the line runs. The direction $\mathbf{v}$ is easily calculated as

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 \tag{1.32}$$

because it must be perpendicular to both normal vectors. The point $\mathbf{p}$ can be calculated by introducing a third plane $[\mathbf{v} \mid 0]$ containing the origin $\mathbf{o}$, as shown in Figure 1.11, and then solving the problem of a three-plane intersection. In this case, Equation (1.29) becomes

$$\begin{bmatrix} \leftarrow & \mathbf{n}_1 & \rightarrow \\ \leftarrow & \mathbf{n}_2 & \rightarrow \\ \leftarrow & \mathbf{v} & \rightarrow \end{bmatrix} \mathbf{p} = \begin{bmatrix} -d_1 \\ -d_2 \\ 0 \end{bmatrix}, \tag{1.33}$$

and the solution for $\mathbf{p}$ given by Equation (1.31) is

$$\boxed{\mathbf{p} = \frac{d_1(\mathbf{v} \times \mathbf{n}_2) + d_2(\mathbf{n}_1 \times \mathbf{v})}{\mathbf{v}^2}.} \tag{1.34}$$

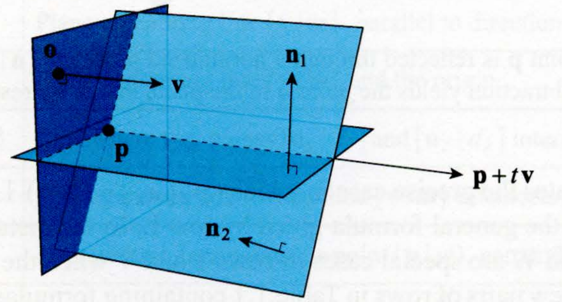


Figure 1.11. Two planes with nonparallel normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ intersect at a line $\mathbf{p} + t\mathbf{v}$ for which $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. A point $\mathbf{p}$ can be found by calculating the point where the two planes intersect the third plane $[\mathbf{v} \mid 0]$.

1.3.6 Reflection Across a Plane

When a point $\mathbf{p}$ is reflected across a plane, the new point $\mathbf{p}'$ lies at the same distance from the plane but on the opposite side. The line segment connecting the original point and the new point is parallel to the plane's normal vector. Let $\mathbf{g} = [\mathbf{n} \mid d]$ be a plane such that $\mathbf{n}$ has unit length, and let $\mathbf{q}$ be the point in the plane that is nearest to $\mathbf{p}$. As shown in Figure 1.12, the difference between $\mathbf{p}$ and $\mathbf{q}$ is equal to $(\mathbf{g} \cdot \mathbf{p})\,\mathbf{n}$ because the scalar quantity $\mathbf{g} \cdot \mathbf{p}$ is the perpendicular distance between the plane $\mathbf{g}$ and the point $\mathbf{p}$. When this vector is subtracted from $\mathbf{p}$, the result is the point $\mathbf{q}$ in the plane. Subtracting this vector a second time produces the reflected point $\mathbf{p}'$ that's just as far away from the plane as the original point $\mathbf{p}$ but on the opposite side. Thus, a formula for calculating $\mathbf{p}'$ is given by

Reflection of point across plane

$$\boxed{\mathbf{p}' = \mathbf{p} - 2(\mathbf{g} \cdot \mathbf{p})\,\mathbf{n}.} \tag{1.35}$$

1.3.7 Homogeneous Formulas

Table 1.1 contains many formulas that can be used to calculate interesting quantities involving points, lines, and planes. We continue using the notation $[\mathbf{n} \mid d]$ for a plane and the notation $\{\mathbf{v} \mid \mathbf{m}\}$ for a line. We also introduce the similar notation $(\mathbf{p} \mid w)$ for a 4D vector composed of a 3D vector $\mathbf{p}$ and a scalar w. It's important to realize that all three of these representations of geometric entities are homogeneous. Multiplying any of them by a nonzero scalar, and in particular negating any of them, has no effect on their geometric meaning. As they appear in the table, the signs of the formulas have been chosen to be consistent whenever there is a relationship among multiple formulas. For example, the planes given by rows P and Q are oriented so that the origin is on the positive side.

Several of the formulas in Table 1.1 show instances in which specific values are plugged into more general expressions in order to explicitly highlight some common cases. Rows B and C give special formulas for a line when $w = 1$ and $w = 0$ are plugged into the general formula given by row

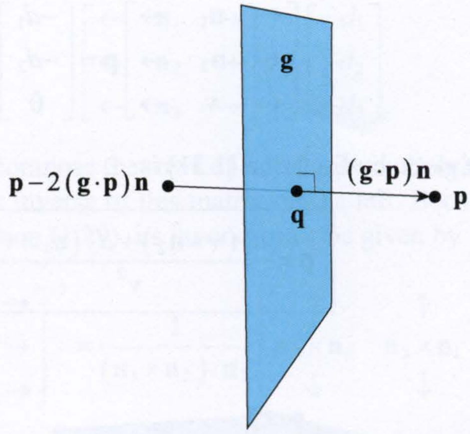


Figure 1.12. A point $\mathbf{p}$ is reflected through a normalized plane $\mathbf{g} = [\mathbf{n} \mid d]$ by subtracting the vector $(\mathbf{g} \cdot \mathbf{p})\,\mathbf{n}$ twice. The first subtraction yields the point $\mathbf{q}$ in the plane that is nearest to $\mathbf{p}$.

A, and row D states the precise case in which $(\mathbf{p}_2 \mid w_2) = (\mathbf{0} \mid 1)$. Likewise, rows F, G, and H contain special cases of the general formula given by row E. In the distance formulas at the bottom of the table, rows U and W are special cases of rows T and V when the point $\mathbf{p}$ is taken to be the origin.

There are a few pairs of rows in Table 1.1 containing formulas that are related through a concept known as *duality*. Two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$ are said to be *dual* to each other when their direction vectors and moment vectors are swapped so that $\mathbf{v}_2 = \mathbf{m}_1$ and $\mathbf{m}_2 = \mathbf{v}_1$. A point $(\mathbf{p} \mid w)$ and a plane $[\mathbf{n} \mid d]$ are dual to each other when the four components belonging to one are simply reinterpreted as the four components of the other so that $\mathbf{p} = \mathbf{n}$ and $w = d$. These relationships are exemplified by rows A and I, where the line in row A passes through two points, but when we swap its direction and moment, the line in row I represents the intersection between two planes having the same components as the two points. Another example of duality is demonstrated by the formulas in rows E and J, which both involve a line $\{\mathbf{v} \mid \mathbf{m}\}$. The formula in row J gives the point at which the line intersects a plane $[\mathbf{n} \mid d]$, and the formula in row E gives the dual plane containing the line and a point having the same components as the plane in row J. The two formulas are exactly the same after swapping the meanings of $\mathbf{v}$ and $\mathbf{m}$ for the line.

The geometric symmetry of the duality between points and planes is perhaps best exhibited by the pair of rows N and P and the pair of rows O and Q. The first pair shows the relationship between the point closest to the origin on a line $\{\mathbf{v} \mid \mathbf{m}\}$ and the plane farthest from the origin containing the same line. The formulas are the same except for the fact that $\mathbf{v}$ and $\mathbf{m}$ are swapped. The second pair shows that the point closest to the origin on a plane is related to the plane farthest from the origin containing a point by the simple reinterpretation of the four components making up each element.

Keep in mind that geometric entities calculated with the formulas in Table 1.1 are not generally normalized, and we extend this term to include homogeneous points, which would usually end up not having a w coordinate of one. To put each type of element into normalized form, which may simplify later calculations, a point $(\mathbf{p} \mid w)$ needs to be divided by its w coordinate, a plane $[\mathbf{n} \mid d]$ needs to be divided by $\|\mathbf{n}\|$, and a line $\{\mathbf{v} \mid \mathbf{m}\}$ needs to be divided by $\|\mathbf{v}\|$.

In the case of a degeneracy, each formula in Table 1.1 produces a geometric element that cannot be normalized. The simplest example is attempting to construct a line from two points that are exactly the same, in which case the formula in row A produces $\{\mathbf{0} \mid \mathbf{0}\}$. In the more complex case arising in row F, if the point $\mathbf{p}$ lies on the line $\{\mathbf{v} \mid \mathbf{m}\}$, then there is not enough information to construct a plane, and the result is $[\mathbf{0} \mid 0]$. Something a little different happens in row J if we attempt to find the intersection of a plane $[\mathbf{n} \mid d]$ and a line $\{\mathbf{v} \mid \mathbf{m}\}$ that is parallel to the plane. In this case, there can be no intersection, but the formula in row J produces the point $(\mathbf{m} \times \mathbf{n} + d\mathbf{v} \mid 0)$ with a zero in the w coordinate. This can be interpreted as a point at infinity in the direction that the line runs.

	Formula	Description
A	$\{w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
B	$\{\mathbf{p}_2 - \mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two points $\mathbf{p}_1$ and $\mathbf{p}_2$.
C	$\{\mathbf{v} \mid \mathbf{p} \times \mathbf{v}\}$	Line through point $\mathbf{p}$ with direction $\mathbf{v}$.
D	$\{\mathbf{p} \mid \mathbf{0}\}$	Line through point $\mathbf{p}$ and the origin.
E	$[\mathbf{v} \times \mathbf{p} + w\mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and homogeneous point $(\mathbf{p} \mid w)$.
F	$[\mathbf{v} \times \mathbf{p} + \mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and point $\mathbf{p}$.
G	$[\mathbf{v} \times \mathbf{u} \mid -\mathbf{u} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, parallel to direction $\mathbf{u}$.
H	$[\mathbf{m} \mid 0]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and the origin.
I	$\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$	Line where two planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect.
J	$(\mathbf{m} \times \mathbf{n} + d\mathbf{v} \mid -\mathbf{n} \cdot \mathbf{v})$	Homogeneous point where line $\{\mathbf{v} \mid \mathbf{m}\}$ intersects plane $[\mathbf{n} \mid d]$.
K	$\{w\mathbf{n} \mid \mathbf{p} \times \mathbf{n}\}$	Line through homogeneous point $(\mathbf{p} \mid w)$, perpendicular to plane $[\mathbf{n} \mid d]$.
L	$[\mathbf{v} \times \mathbf{n} \mid -\mathbf{n} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$, perpendicular to plane $[\mathbf{n} \mid d]$.
M	$[w\mathbf{v} \mid -\mathbf{p} \cdot \mathbf{v}]$	Plane containing homogeneous point $(\mathbf{p} \mid w)$, perpendicular to line $\{\mathbf{v} \mid \mathbf{m}\}$.
N	$(\mathbf{v} \times \mathbf{m} \mid \mathbf{v}^2)$	Homogeneous point closest to the origin on line $\{\mathbf{v} \mid \mathbf{m}\}$.
O	$(-d\mathbf{n} \mid \mathbf{n}^2)$	Homogeneous point closest to the origin on plane $[\mathbf{n} \mid d]$.
P	$[\mathbf{m} \times \mathbf{v} \mid \mathbf{m}^2]$	Plane farthest from the origin containing line $\{\mathbf{v} \mid \mathbf{m}\}$.
Q	$[-w\mathbf{p} \mid \mathbf{p}^2]$	Plane farthest from the origin containing point $(\mathbf{p} \mid w)$.
R	$\dfrac{\lVert w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \rVert}{\lvert w_1 w_2 \rvert}$	Distance between two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
S	$\dfrac{\lvert \mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1 \rvert}{\lVert \mathbf{v}_1 \times \mathbf{v}_2 \rVert}$	Distance between two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$.
T	$\dfrac{\lVert \mathbf{v} \times \mathbf{p} + \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to point $\mathbf{p}$.
U	$\dfrac{\lVert \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to the origin.
V	$\dfrac{\lvert \mathbf{n} \cdot \mathbf{p} + d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to point $\mathbf{p}$.
W	$\dfrac{\lvert d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to the origin.

Table 1.1. This table contains various formulas involving homogeneous points, planes, and lines described by Plücker coordinates. The notation $(\mathbf{p} \mid w)$ represents a homogeneous point with $\mathbf{p} = (x, y, z)$, the notation $[\mathbf{n} \mid d]$ represents a plane with normal direction $\mathbf{n}$ and distance to origin d, and the notation $\{\mathbf{v} \mid \mathbf{m}\}$ represents a line with direction $\mathbf{v}$ and moment $\mathbf{m}$.

The distance between two skew lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$, stated in row S of the table, can be derived by considering the distance between parallel planes constructed to contain each line and the direction $\mathbf{v}$ of the other line, as shown in Figure 1.13. Using row G in Table 1.1, these two planes are given by $[\mathbf{v}_1 \times \mathbf{v}_2 \mid -\mathbf{v}_2 \cdot \mathbf{m}_1]$ and $[\mathbf{v}_1 \times \mathbf{v}_2 \mid \mathbf{v}_1 \cdot \mathbf{m}_2]$, where the second plane has been negated so that the normal vectors point in the same direction. These planes are both normalized by dividing by $\|\mathbf{v}_1 \times \mathbf{v}_2\|$, after which their fourth coordinates correspond to the perpendicular distances between the planes and the origin. Subtracting these gives us

Distance between skew lines

$$\boxed{d = \frac{|\mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}} \tag{1.36}$$

as the distance d between the planes, which is also the distance between the original lines. If this distance is zero, then the lines are coplanar and intersect at a point.

1.3.8 Plane Transformation

Determining the correct way to transform a plane from one coordinate system to another requires a little care. Let $\mathbf{g}_A = [\mathbf{n}_A \mid d_A]$ be a plane that contains the point $\mathbf{p}_A$ in coordinate system A, and let $\mathbf{H}$ be a 4×4 matrix having the form

$$\mathbf{H} = \begin{bmatrix} \mathbf{M} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{1.37}$$

where $\mathbf{M}$ is an invertible 3×3 matrix and $\mathbf{t}$ is a 3D translation vector, that performs an affine transformation of points from coordinate system A to coordinate system B. Our goal is to find the correct method for transforming $\mathbf{g}_A$ into $\mathbf{g}_B = [\mathbf{n}_B \mid d_B]$ using the matrix $\mathbf{H}$. We know that the normal vector must transform as $\mathbf{n}_B = \mathbf{n}_A \operatorname{adj}(\mathbf{M})$ according to Equation (1.7), so the only question is what to do with d_A to transform it into d_B.

Because the original plane contains $\mathbf{p}_A$, we know that $d_A = -\mathbf{n}_A \cdot \mathbf{p}_A$. For the transformed plane, we must have $d_B = -\mathbf{n}_B \cdot \mathbf{p}_B$, where $\mathbf{p}_B$ is the transformed point. We can transform each part of this dot product independently to get

$$\begin{aligned} d_B &= -\mathbf{n}_A \operatorname{adj}(\mathbf{M})(\mathbf{M}\mathbf{p}_A + \mathbf{t}) \\ &= -\mathbf{n}_A \det(\mathbf{M})\mathbf{p}_A - \mathbf{n}_A \det(\mathbf{M})\mathbf{M}^{-1}\mathbf{t} \\ &= \det(\mathbf{M})\left(d_A - \mathbf{n}_A\mathbf{M}^{-1}\mathbf{t}\right). \end{aligned} \tag{1.38}$$

Except for the extra factor of $\det(\mathbf{M})$, this is exactly the value produced by multiplying the plane $\mathbf{g}_A$ by the fourth column of $\mathbf{H}^{-1}$, which is given by Equation (1.12). Using the fact that $\det(\mathbf{H}) = \det(\mathbf{M})$ due to the specific form of $\mathbf{H}$, we come to the conclusion that a plane is transformed as

Plane transformation

$$\boxed{\mathbf{g}_B = \mathbf{g}_A \operatorname{adj}(\mathbf{H}),} \tag{1.39}$$

and this is the four-dimensional analog of Equation (1.7). As with normal vectors, the determinant in the adjugate matrix is usually ignored because planes are typically normalized so that the first three coordinates (g_x, g_y, g_z) have unit length as a whole. Normal vectors in three dimensions and planes in four dimensions both transform in the above manner because they are each an example of a mathematical element called an *antivector*, which is an important topic in Chapter 2.

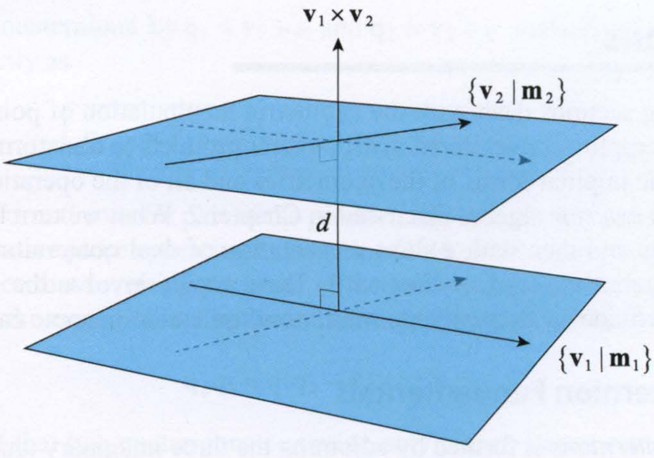


Figure 1.13. The distance d between two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$ can be calculated by considering the parallel planes containing each line and the direction of the other.

1.3.9 Line Transformation

Because Plücker coordinates contain both a difference of two points and the cross product between two points, the transformation of a line from one coordinate system to another can be somewhat tricky. Again, let $\mathbf{H}$ be a 4×4 matrix representing an affine transformation from coordinate system A to coordinate system B consisting of a 3×3 invertible matrix $\mathbf{M}$ and a 3D translation vector $\mathbf{t}$. Suppose $\{\mathbf{v}_A \mid \mathbf{m}_A\}$ is a line in coordinate system A with $\mathbf{v}_A = \mathbf{p}_2 - \mathbf{p}_1$ and $\mathbf{m}_A = \mathbf{p}_1 \times \mathbf{p}_2$. Clearly, the transformed direction vector is simply given by $\mathbf{v}_B = \mathbf{M}\mathbf{v}_A$, but the transformed moment vector requires a closer look. Applying the matrix $\mathbf{H}$ to each of the points $\mathbf{p}_1$ and $\mathbf{p}_2$, the transformed points are equal to $\mathbf{M}\mathbf{p}_1 + \mathbf{t}$ and $\mathbf{M}\mathbf{p}_2 + \mathbf{t}$. The moment of the transformed line must be the cross product between these, so we have

$$\begin{aligned}\mathbf{m}_B &= (\mathbf{M}\mathbf{p}_1 + \mathbf{t}) \times (\mathbf{M}\mathbf{p}_2 + \mathbf{t}) \\ &= (\mathbf{M}\mathbf{p}_1) \times (\mathbf{M}\mathbf{p}_2) + \mathbf{t} \times (\mathbf{M}\mathbf{p}_2) - \mathbf{t} \times (\mathbf{M}\mathbf{p}_1).\end{aligned} \tag{1.40}$$

The cross product $(\mathbf{M}\mathbf{p}_1) \times (\mathbf{M}\mathbf{p}_2)$ transforms under a 3×3 matrix $\mathbf{M}$ according to Equation (1.7), and the cross products involving the translation $\mathbf{t}$ can be combined into one cross product that operates on $\mathbf{v}_A$. This lets us write

$$\mathbf{m}_B = \mathbf{m}_A \operatorname{adj}(\mathbf{M}) + \mathbf{t} \times (\mathbf{M}\mathbf{v}_A), \tag{1.41}$$

where we are treating $\mathbf{m}_A$ and $\mathbf{m}_B$ as row vectors. The complete affine transformation of a line from coordinate system A to coordinate system B is thus given by

Line transformation

$$\boxed{\{\mathbf{v}_B \mid \mathbf{m}_B\} = \{\mathbf{M}\mathbf{v}_A \mid \mathbf{m}_A \operatorname{adj}(\mathbf{M}) + \mathbf{t} \times (\mathbf{M}\mathbf{v}_A)\}.} \tag{1.42}$$

As usual, the calculation of the adjugate of $\mathbf{M}$ can be avoided if we know that $\mathbf{M}$ is special orthogonal, in which case we can treat $\mathbf{m}_A$ and $\mathbf{m}_B$ as column vectors and replace $\mathbf{m}_A \operatorname{adj}(\mathbf{M})$ with $\mathbf{M}\mathbf{m}_A$.

1.4 Quaternions

The preceding sections dealt with the geometric manipulation of points, lines, and planes, but we continued to employ conventional matrix-vector products to transform from one coordinate system to another. The implicit forms of the geometries and all of the operations listed in Table 1.1 belong to the general *exterior* algebra discussed in Chapter 2. What we turn to now, first with a discussion of quaternions and then with a short presentation of dual quaternions, is our first glimpse of the *geometric* algebra covered in Chapter 3. These topics involve the transformation of geometric objects and provide an alternative to matrix multiplication in some cases.

1.4.1 Quaternion Fundamentals

The set of *quaternions* is formed by adjoining the three imaginary units i, j, and k, to the set of real numbers. A typical quaternion $\mathbf{q}$ has four components that can be written as

Quaternion

$$\boxed{\mathbf{q} = xi + yj + zk + w,} \tag{1.43}$$

where x, y, z, and w are real numbers. It doesn't matter what order these components are written in because multiplication by i, j, and k provide all the necessary identification for the imaginary terms. Many textbooks write the real w component first, but we choose to write it last to be consistent with the general convention used throughout computing that places the w coordinate last in a 4D vector (x, y, z, w). This is particularly useful for avoiding confusion when a quaternion is stored in a variable having a generic vector type.

Although quaternions are sometimes treated as if they were 4D vectors, and they are even written in bold to reflect their multicomponent nature, it is important to realize that they are not actually 4D vectors. A quaternion is more properly understood as the sum of a scalar and a 3D vector. It is often convenient to write a quaternion in the form $\mathbf{q} = \mathbf{v} + s$, where $\mathbf{v}$, called the *vector part* of the quaternion, corresponds to the imaginary triplet (x, y, z) in Equation (1.43), and s, called the *scalar part*, corresponds to the real component w. Note, however, that calling $\mathbf{v}$ a vector still isn't quite correct, but this terminology will suffice until we reach the more precise discussion of quaternions within the larger context of geometric algebra in Chapter 3.

As with ordinary vectors and complex numbers, quaternion addition is performed componentwise. Multiplication, however, follows the rules given by

$$\boxed{\begin{array}{cc} & ij = -ji = k \\ i^2 = j^2 = k^2 = -1 & jk = -kj = i \\ & ki = -ik = j. \end{array}} \tag{1.44}$$

This summarization provides an immediate guide for multiplication between any two of the imaginary units i, j, and k. Equation (1.44) also illustrates the fact that quaternions do not possess the commutative property. Reversing the order in which any two imaginary units are multiplied negates their product.

By following the rules given above, we can calculate the general product of two quaternions $\mathbf{q}_1 = x_1 i + y_1 j + z_1 k + w_1$ and $\mathbf{q}_2 = x_2 i + y_2 j + z_2 k + w_2$ to obtain

$$\begin{aligned} \mathbf{q}_1\mathbf{q}_2 = {} & (x_1 w_2 + y_1 z_2 - z_1 y_2 + w_1 x_2)\, i \\ & + (y_1 w_2 + z_1 x_2 + w_1 y_2 - x_1 z_2)\, j \\ & + (z_1 w_2 + w_1 z_2 + x_1 y_2 - y_1 x_2)\, k \\ & + (w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2). \end{aligned} \tag{1.45}$$

If we represent the quaternions by $\mathbf{q}_1 = \mathbf{v}_1 + s_1$ and $\mathbf{q}_2 = \mathbf{v}_2 + s_2$ instead, then the product can be written more compactly as

Quaternion product

$$\boxed{\mathbf{q}_1\mathbf{q}_2 = \mathbf{v}_1 \times \mathbf{v}_2 + s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2.} \tag{1.46}$$

The first three terms form the vector part of the product, and the last two terms form the scalar part. The only noncommutative piece appearing in Equation (1.46) is the cross product, a fact from which we can quickly deduce that reversing the order of the factors in quaternion multiplication changes the product by twice the cross product between the vector parts, as stated by

$$\mathbf{q}_2\mathbf{q}_1 = \mathbf{q}_1\mathbf{q}_2 - 2(\mathbf{v}_1 \times \mathbf{v}_2). \tag{1.47}$$

This exposes the fact that two quaternions commute only if their vector parts are parallel because when that is the case, the cross product $\mathbf{v}_1 \times \mathbf{v}_2$ is zero.

A quaternion $\mathbf{q}$ has a *conjugate* denoted by $\mathbf{q}^*$ that is similar to the complex conjugate except that we are now negating three imaginary components instead of just one. That is, the conjugate of a quaternion $\mathbf{q} = \mathbf{v} + s$ is given by

$$\mathbf{q}^* = -\mathbf{v} + s. \tag{1.48}$$

The product of a quaternion and its conjugate gives us

$$\mathbf{q}\mathbf{q}^* = \mathbf{q}^*\mathbf{q} = \mathbf{v}^2 + s^2, \tag{1.49}$$

which is a real number that we identify with the squared magnitude of the quaternion. We denote the magnitude of a quaternion using two vertical bars, as with ordinary vectors, and define it as

Quaternion magnitude

$$\boxed{\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^*} = \sqrt{\mathbf{v}^2 + s^2}.} \tag{1.50}$$

As with vectors, multiplying a quaternion $\mathbf{q}$ by a scalar value t has the effect of multiplying the magnitude of $\mathbf{q}$ by $|t|$. Quaternions also have the property that the magnitude of the product of two quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$ is equal to the product of their individual magnitudes, which we can state as

$$\|\mathbf{q}_1\mathbf{q}_2\| = \|\mathbf{q}_1\|\|\mathbf{q}_2\|. \tag{1.51}$$

The real numbers form a subset of the entire set of quaternions, and it consists of all the quaternions having the vector part $(0, 0, 0)$. In particular, the number one is a quaternion, and it continues to fill the role of the multiplicative identity element as it does in the sets of real numbers and complex numbers. For any quaternion $\mathbf{q} = \mathbf{v} + s$ that has a nonzero magnitude, we can divide the product shown in Equation (1.49) by the squared magnitude of $\mathbf{q}$ to obtain the identity element, and this means that $\mathbf{q}$ has a multiplicative inverse given by

Quaternion inverse

$$\boxed{\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\mathbf{q}\mathbf{q}^*} = \frac{-\mathbf{v} + s}{\mathbf{v}^2 + s^2}.} \tag{1.52}$$

The basic properties of quaternion addition and multiplication are listed in Table 1.2. They are all easy to verify, and none of them should come as any surprise. Due to the noncommutativity of quaternion multiplication, the last two properties listed in the table show that the conjugate or inverse of a product of quaternions is equal to the conjugate or inverse of each factor multiplied in reverse order. This is similar to how the transpose and inverse of matrix products work.

Property	Description
$(\mathbf{q}_1+\mathbf{q}_2)+\mathbf{q}_3=\mathbf{q}_1+(\mathbf{q}_2+\mathbf{q}_3)$	Associative law for quaternion addition.
$\mathbf{q}_1+\mathbf{q}_2=\mathbf{q}_2+\mathbf{q}_1$	Commutative law for quaternion addition.
$(st)\mathbf{q}=s(t\mathbf{q})$	Associative law for scalar-quaternion multiplication.
$t\mathbf{q}=\mathbf{q}t$	Commutative law for scalar-quaternion multiplication.
$t(\mathbf{q}_1+\mathbf{q}_2)=t\mathbf{q}_1+t\mathbf{q}_2$ $(s+t)\mathbf{q}=s\mathbf{q}+t\mathbf{q}$	Distributive laws for scalar-quaternion multiplication.
$\mathbf{q}_1(\mathbf{q}_2\mathbf{q}_3)=(\mathbf{q}_1\mathbf{q}_2)\mathbf{q}_3$	Associative law for quaternion multiplication.
$\mathbf{q}_1(\mathbf{q}_2+\mathbf{q}_3)=\mathbf{q}_1\mathbf{q}_2+\mathbf{q}_1\mathbf{q}_3$ $(\mathbf{q}_1+\mathbf{q}_2)\mathbf{q}_3=\mathbf{q}_1\mathbf{q}_3+\mathbf{q}_2\mathbf{q}_3$	Distributive laws for quaternion multiplication.
$(t\mathbf{q}_1)\mathbf{q}_2=\mathbf{q}_1(t\mathbf{q}_2)=t(\mathbf{q}_1\mathbf{q}_2)$	Scalar factorization for quaternions.
$(\mathbf{q}_1\mathbf{q}_2)^*=\mathbf{q}_2^*\mathbf{q}_1^*$	Product rule for quaternion conjugate.
$(\mathbf{q}_1\mathbf{q}_2)^{-1}=\mathbf{q}_2^{-1}\mathbf{q}_1^{-1}$	Product rule for quaternion inverse.

Table 1.2. These are the basic properties of quaternion addition and multiplication. Each letter **q**, with or without a subscript, represents a quaternion, and the letters s and t represent scalar values.

Math Library Notes

- The `Quaternion` class holds four floating-point components `x`, `y`, `z`, and `w` representing the vector and scalar parts of a quaternion, and they can be accessed directly.
- The class has a default constructor that performs no initialization and two additional constructors that take either all four components separately or a `Vector3D` object and a scalar.
- The * operator is overloaded so it calculates the product between two quaternions. Overloaded operators for addition, subtraction, and multiplication by scalar values are also implemented.

1.4.2 Rotations With Quaternions

Quaternions appear in practice because they can be used to represent rotations in a way that has several advantages over 3×3 matrices. In this section, we present a conventional description of how a quaternion corresponding to a particular rotation through any angle about any axis is constructed and how such a quaternion transforms an ordinary vector. Like 3×3 matrices, quaternions are only able to rotate about axes passing through the origin and cannot be used to encode translations. These limitations are overcome by the dual quaternions introduced in the next section. We'll be able to provide greater insight into the reasons why quaternions and dual quaternions work the way they do in Chapter 3.

Given a quaternion $\mathbf{q}=xi+yj+zk+w$ and a vector $\mathbf{v}=(v_x,v_y,v_z)$, a rotation is performed by recasting the vector to be the quaternion $v_xi+v_yj+v_zk$ and calculating a new vector $\mathbf{v}'$ with the product

$$\mathbf{v}'=\mathbf{qvq}^{-1}. \tag{1.53}$$

To be clear, each of the products in this equation is a quaternion multiplied by a quaternion. This is sometimes called the *sandwich product* because the quaternion $\mathbf{v}$ is sandwiched between the quaternion $\mathbf{q}$ and its inverse. The quaternion $\mathbf{v}$ is known as a *pure* quaternion, which is any quaternion that has a zero scalar component and is thus made up of only imaginary terms. When $\mathbf{v}$ is a pure quaternion, the sandwich product $\mathbf{qvq}^{-1}$ always yields another pure quaternion. By considering vectors and pure quaternions to be equivalent, we can say that the sandwich product transforms a vector $\mathbf{v}$ into another vector $\mathbf{v}'$.

The magnitude of $\mathbf{q}$ in Equation (1.53) doesn't matter, as long as it's nonzero, because if $\|\mathbf{q}\| = m$, then m can be factored out of $\mathbf{q}$, and $1/m$ can be factored out of $\mathbf{q}^{-1}$. These cancel each other out and leave quaternions with magnitudes of one behind. A quaternion $\mathbf{q}$ having a magnitude of one is called a *unit quaternion*, and it has the special property that its inverse is simply equal to its conjugate because $\mathbf{qq}^* = 1$. In the case that $\mathbf{q}$ is a unit quaternion, Equation (1.53) simplifies to

$$\boxed{\mathbf{v}' = \mathbf{qvq}^*} \tag{1.54}$$

The set of unit quaternions forms a multiplicatively closed subset of all quaternions because the product of any two unit quaternions is another unit quaternion. For this reason and the fact that vector transformations become simpler, only unit quaternions are typically used to represent rotations in practice.

An optimal implementation of Equation (1.54) requires some clever simplification. We can write $\mathbf{q} = \mathbf{b} + c$ and expand the quaternion products using Equation (1.46), keeping in mind that the scalar part of $\mathbf{v}$ is zero. The product $\mathbf{qv}$ is given by

$$\mathbf{qv} = (\mathbf{b} + c)\,\mathbf{v} = \mathbf{b} \times \mathbf{v} + c\mathbf{v} - \mathbf{b} \cdot \mathbf{v}. \tag{1.55}$$

When we multiply this by $\mathbf{q}^* = -\mathbf{b} + c$, we get

$$\begin{aligned} \mathbf{qvq}^* &= (\mathbf{b} \times \mathbf{v} + c\mathbf{v} - \mathbf{b} \cdot \mathbf{v})(-\mathbf{b} + c) \\ &= -\mathbf{b} \times \mathbf{v} \times \mathbf{b} + 2c\,(\mathbf{b} \times \mathbf{v}) + (\mathbf{b} \cdot \mathbf{v})\,\mathbf{b} + c^2\mathbf{v}. \end{aligned} \tag{1.56}$$

Since $\mathbf{q}$ is a unit quaternion, we know that $\|\mathbf{q}\|^2 = \mathbf{b}^2 + c^2 = 1$, so we can substitute $c^2 = 1 - \mathbf{b}^2$ in the final term of this equation. We can then apply the vector triple product identity

$$-\mathbf{b} \times \mathbf{v} \times \mathbf{b} = (\mathbf{b} \cdot \mathbf{v})\,\mathbf{b} - \mathbf{b}^2\mathbf{v} \tag{1.57}$$

to the last two terms to obtain

$$\boxed{\mathbf{qvq}^* = 2\mathbf{b} \times (\mathbf{b} \times \mathbf{v}) + 2c\,(\mathbf{b} \times \mathbf{v}) + \mathbf{v}.} \tag{1.58}$$

An implementation of this formula should take advantage of the fact that $2\mathbf{b} \times \mathbf{v}$ can be calculated once and used in two places. The actual computational cost is 18 combined multiply-add operations (or 15 multiplies and 15 separate adds). By comparison, multiplying a vector $\mathbf{v}$ by a 3×3 rotation matrix requires only 9 combined multiply-add operations (or 9 multiplies and 6 separate adds). If many vectors $\mathbf{v}$ are going to be transformed by the same quaternion $\mathbf{q}$, it is much better to convert the quaternion to a matrix first, and the process for that is described below.

To see how the sandwich product $\mathbf{qvq}^*$ performs a rotation, we first consider the rotation of a vector $\mathbf{v}$ about an arbitrary axis represented by the unit vector $\mathbf{a}$. When the vector $\mathbf{v}$ is rotated into its new orientation $\mathbf{v}'$, the component of $\mathbf{v}$ parallel to the axis $\mathbf{a}$ remains the same, and only the component of $\mathbf{v}$ perpendicular to the axis $\mathbf{a}$ actually gets modified. Thus, it makes sense for us to consider the separate components of $\mathbf{v}$ with respect to the axis $\mathbf{a}$, as illustrated in Figure 1.14.

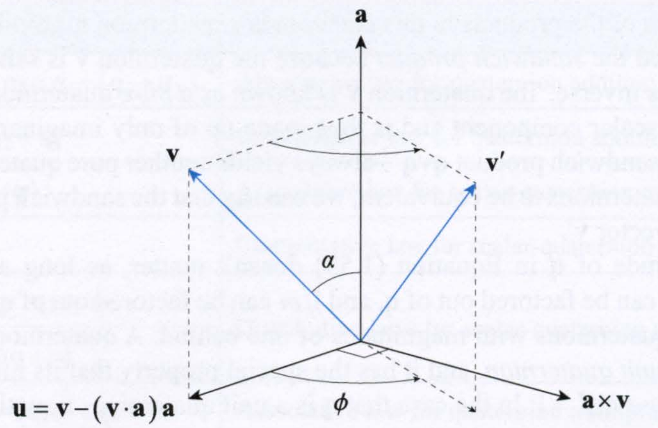


Figure 1.14. A vector $\mathbf{v}$ is rotated through an angle ϕ about an arbitrary axis $\mathbf{a}$. This is achieved by decomposing $\mathbf{v}$ into components that are parallel to and perpendicular to $\mathbf{a}$ and rotating only the perpendicular component.

The projection of $\mathbf{v}$ onto the axis $\mathbf{a}$ is given by $(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}$, and that means the remaining component perpendicular to the axis $\mathbf{a}$ must be $\mathbf{v}-(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}$, which we will call the vector $\mathbf{u}$. The length of $\mathbf{u}$ is equal to $\|\mathbf{v}\|\sin\alpha$, where α is the angle between the vectors $\mathbf{a}$ and $\mathbf{v}$, because it forms the side opposite the angle α in the right triangle shown in the figure. It's this component that we need to rotate in the plane perpendicular to the axis $\mathbf{a}$. We can perform this rotation by expressing the result as a linear combination of $\mathbf{u}$ and another vector in the plane that is the 90-degree counterclockwise rotation of $\mathbf{u}$. Fortunately, this second vector is easily obtained as $\mathbf{a}\times\mathbf{v}$, and it just happens to have the same length, $\|\mathbf{v}\|\sin\alpha$, as $\mathbf{u}$ does. This means that we can express the rotated vector $\mathbf{v}'$ as

$$\mathbf{v}'=(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}+[\mathbf{v}-(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}]\cos\phi+(\mathbf{a}\times\mathbf{v})\sin\phi, \tag{1.59}$$

where ϕ is the angle of rotation about the axis $\mathbf{a}$. This can be reorganized a little bit to obtain

$$\mathbf{v}'=\mathbf{v}\cos\phi+(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}\,(1-\cos\phi)+(\mathbf{a}\times\mathbf{v})\sin\phi. \tag{1.60}$$

We now make some modifications to Equation (1.56) so we can compare it directly to Equation (1.60) and learn something about what's happening. This time, we apply the vector triple product identity $-\mathbf{b}\times\mathbf{v}\times\mathbf{b}=(\mathbf{b}\cdot\mathbf{v})\,\mathbf{b}-\mathbf{b}^2\mathbf{v}$ to the first term, and that gives us

$$\mathbf{qvq}^*=\left(c^2-\mathbf{b}^2\right)\mathbf{v}+2(\mathbf{v}\cdot\mathbf{b})\,\mathbf{b}+2c\,(\mathbf{b}\times\mathbf{v}). \tag{1.61}$$

If we set $\mathbf{b}=s\mathbf{a}$, where $s=\|\mathbf{b}\|$ and $\mathbf{a}$ is a unit vector, then we can write this as

$$\mathbf{qvq}^*=\left(c^2-s^2\right)\mathbf{v}+2s^2\,(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}+2cs\,(\mathbf{a}\times\mathbf{v}). \tag{1.62}$$

The right side of this equation has the same three terms that appear in the formula for rotation about an arbitrary axis $\mathbf{a}$ given by Equation (1.60) except that the scalar coefficients are written in a different way. In order for Equation (1.62) to perform a rotation through an angle ϕ, the values of c and s must satisfy the equalities $c^2-s^2=\cos\phi$, $2s^2=1-\cos\phi$, and $2cs=\sin\phi$. All three of these requirements are satisfied when we choose $c=\cos(\phi/2)$ and $s=\sin(\phi/2)$ because these values produce valid trigonometric identities. (This reveals why the letters c and s were selected for this derivation.) We conclude that the quaternion

Quaternion rotation

$$\boxed{\mathbf{q}=\left(\sin\frac{\phi}{2}\right)\mathbf{a}+\cos\frac{\phi}{2}} \tag{1.63}$$

represents a rotation through the angle ϕ about the unit-length axis $\mathbf{a}$ that can be applied to a vector $\mathbf{v}$ using the sandwich product $\mathbf{qvq}^*$. A rotation through a positive angle is counterclockwise when the axis points toward the viewer.

The resemblance between Equation (1.63) and Euler's formula $e^{\phi i} = \cos\phi + i\sin\phi$ is not a coincidence. Indeed, the square of any unit-length axis $\mathbf{a}$ under the quaternion product yields −1, so it behaves just like the imaginary number i. This means that it's possible to express a quaternion as the exponential

$$\mathbf{q} = \exp(\phi\mathbf{a}) = \cos\phi + \mathbf{a}\sin\phi, \tag{1.64}$$

which is easily verified by expanding the power series for $\exp(\phi\mathbf{a})$. We just need to be careful to remember that the resulting quaternion rotates through twice the angle ϕ appearing in the exponent. We'll return to this form of a quaternion in the discussion of interpolation below.

We saw earlier that rotating a vector $\mathbf{v}$ with the sandwich product $\mathbf{qvq}^*$ is slower than the equivalent matrix-vector multiplication, but there are operations involving quaternions that are faster as well. In particular, multiple quaternion rotations can be quickly composed. To first rotate a vector $\mathbf{v}$ using a quaternion $\mathbf{q}_1$ and then rotate the result using another quaternion $\mathbf{q}_2$, we calculate the sandwich product of a sandwich product as in

$$\mathbf{v}' = \mathbf{q}_2\left(\mathbf{q}_1\mathbf{v}\mathbf{q}_1^*\right)\mathbf{q}_2^*. \tag{1.65}$$

By reassociating the factors, this can be written as

$$\mathbf{v}' = (\mathbf{q}_2\mathbf{q}_1)\,\mathbf{v}\,(\mathbf{q}_2\mathbf{q}_1)^*, \tag{1.66}$$

showing that the two successive rotations are equivalent to a single rotation using the quaternion given by the product $\mathbf{q}_2\mathbf{q}_1$. As is evident in Equation (1.45), the product of two quaternions can be calculated with 16 combined multiply-add operations (or 16 multiplies and 12 separate adds). This has a significantly lower cost than the 27 combined multiply-add operations (or 27 multiplies and 18 separate adds) that would be required to calculate the product of two 3×3 matrices.

A quaternion has the obvious advantage of requiring much less storage since it's made up of only four floating-point numbers compared to the nine floating-point entries needed by an equivalent 3×3 rotation matrix. It is often the case, however, that a quaternion needs to be converted to a matrix at some point, either because it will be used to transform many vectors or it needs to be combined with another transformation that is not a rotation. To convert a unit quaternion to a matrix, we can examine each of the terms of the sandwich product $\mathbf{qvq}^*$ in Equation (1.61), where $\mathbf{q} = \mathbf{b} + c$, and express their effects on $\mathbf{v}$ as 3×3 matrices to obtain

$$\mathbf{qvq}^* = \left(c^2 - \mathbf{b}^2\right)\mathbf{Iv} + \begin{bmatrix} 2b_x^2 & 2b_xb_y & 2b_xb_z \\ 2b_xb_y & 2b_y^2 & 2b_yb_z \\ 2b_xb_z & 2b_yb_z & 2b_z^2 \end{bmatrix}\mathbf{v} + \begin{bmatrix} 0 & -2cb_z & 2cb_y \\ 2cb_z & 0 & -2cb_x \\ -2cb_y & 2cb_x & 0 \end{bmatrix}\mathbf{v}, \tag{1.67}$$

where $\mathbf{I}$ is the identity matrix. Since $\mathbf{q}$ is a unit quaternion, we can rewrite $c^2 - \mathbf{b}^2$ as $1 - 2\mathbf{b}^2$ because $c^2 + \mathbf{b}^2 = 1$. This allows us to simplify the diagonal entries a little when we combine the three matrices because, as exemplified by the upper-left entry, we can make the replacement

$$c^2 - \mathbf{b}^2 + 2b_x^2 = 1 - 2b_y^2 - 2b_z^2. \tag{1.68}$$

For a general unit quaternion $\mathbf{q} = xi + yj + zk + w$, where we equate $\mathbf{b} = (x, y, z)$ and $c = w$, a single 3×3 matrix $\mathbf{M}_{\text{rot}}(\mathbf{q})$ corresponding to the sandwich product $\mathbf{qvq}^*$ is thus given by

Quaternion matrix

$$\mathbf{M}_{\text{rot}}(\mathbf{q}) = \begin{bmatrix} 1-2y^2-2z^2 & 2(xy-wz) & 2(zx+wy) \\ 2(xy+wz) & 1-2x^2-2z^2 & 2(yz-wx) \\ 2(zx-wy) & 2(yz+wx) & 1-2x^2-2y^2 \end{bmatrix}. \tag{1.69}$$

If we take a close look at Equation (1.61), we notice that negating both $\mathbf{b}$ and c has no effect on the transformation of $\mathbf{v}$. There would be two negations in each term that cancel each other out. The same property is also apparent in the formula for $\mathbf{M}_{\text{rot}}(\mathbf{q})$ if we were to negate all four components x, y, z, and w. This demonstrates that for any unit quaternion $\mathbf{q}$, the quaternion $-\mathbf{q}$ represents exactly the same rotation. Further insight can be gained by considering the number -1 itself as a quaternion and matching it to Equation (1.63). In this case, we must have $\cos(\phi/2) = -1$ and $\sin(\phi/2) = 0$, which are conditions satisfied when $\phi = 2\pi$, so the quaternion $\mathbf{q} = -1$ corresponds to a full revolution about any axis.

The fact that $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation can be used to reduce the amount of storage space needed by a unit quaternion to just three floating-point values. Once the components of a quaternion $\mathbf{q} = \mathbf{b} + c$ have been calculated for a particular angle and axis, we can choose whether to keep $\mathbf{q}$ or change it to $-\mathbf{q}$ based on whether the scalar part c is nonnegative. If we know that $c \geq 0$, then it can be calculated from the vector part $\mathbf{b}$ as

$$c = \sqrt{1 - b_x^2 - b_y^2 - b_z^2} \tag{1.70}$$

because the magnitude of $\mathbf{q}$ must be one. Thus, if storage space is important, then a quaternion can be negated if necessary so that the scalar part is not negative and stored as only the three components of the vector part. A short calculation is able to reconstitute the scalar part when it is needed.

Given a 3×3 matrix $\mathbf{M}$ that we know represents a rotation (because it is special orthogonal), we can go the other direction and convert to a quaternion $\mathbf{q} = xi + yj + zk + w$ by assuming that the entries of the matrix have the form shown in Equation (1.69) and solving for the individual components. We start by making an observation about the sum of the diagonal entries of $\mathbf{M}$, which is

$$M_{00} + M_{11} + M_{22} = 3 - 4(x^2 + y^2 + z^2). \tag{1.71}$$

By requiring $\mathbf{q}$ to be a unit quaternion, we can replace $x^2 + y^2 + z^2$ with $1 - w^2$ and solve for w to get

$$w = \pm\frac{1}{2}\sqrt{M_{00} + M_{11} + M_{22} + 1}, \tag{1.72}$$

where we are free to choose whether w is positive or negative. (The value under the radical is never negative because $x^2 + y^2 + z^2 \leq 1$.) Once we have calculated the value of w, we can use it to find the values of x, y, and z using the relationships

$$\begin{aligned} M_{21} - M_{12} &= 4wx \\ M_{02} - M_{20} &= 4wy \\ M_{10} - M_{01} &= 4wz, \end{aligned} \tag{1.73}$$

each of which simply requires a division by $4w$.

Unfortunately, in cases when w is very small, dividing by it can cause floating-point precision problems, so we need alternative methods that calculate the largest of x, y, or z first and then solve for the other components. If $M_{00} + M_{11} + M_{22} > 0$, then $|w|$ is guaranteed to be larger than $1/2$, and we can safely use Equations (1.72) and (1.73) to calculate $\mathbf{q}$. Otherwise, we make use of three more relationships involving the diagonal entries of $\mathbf{M}$, given by

$$\begin{aligned} M_{00} - M_{11} - M_{22} + 1 &= 4x^2 \\ -M_{00} + M_{11} - M_{22} + 1 &= 4y^2 \\ -M_{00} - M_{11} + M_{22} + 1 &= 4z^2 \end{aligned} \tag{1.74}$$

At first, it might seem like we can use these in conjunction with Equation (1.72) to calculate all four components of $\mathbf{q}$, but we do not have enough information to select the correct signs. We are able to arbitrarily choose the sign of one component, but making that choice determines the signs of the other components when they are subsequently calculated using off-diagonal entries of $\mathbf{M}$. To determine which of x, y, and z is largest, we can manipulate Equation (1.74) by replacing the negated entries of $\mathbf{M}$ with the values shown in Equation (1.69) to obtain

$$\begin{aligned} 2x^2 &= M_{00} - 2w^2 + 1 \\ 2y^2 &= M_{11} - 2w^2 + 1 \\ 2z^2 &= M_{22} - 2w^2 + 1, \end{aligned} \tag{1.75}$$

where we have used the fact that $w^2 = 1 - x^2 - y^2 - z^2$. These equations show that the sizes of x, y, and z are directly related to the sizes of M_{00}, M_{11}, and M_{22}. Once the largest diagonal entry has been identified, we calculate the corresponding component of $\mathbf{q}$ using one of the relationships in Equation (1.74) and then calculate the remaining two imaginary components of $\mathbf{q}$ using the relationships

$$\begin{aligned} M_{21} + M_{12} &= 4yz \\ M_{02} + M_{20} &= 4zx \\ M_{10} + M_{01} &= 4xy. \end{aligned} \tag{1.76}$$

The w component is always calculated using one of the relationships shown in Equation (1.73). Making an example of the case in which M_{00} is the largest diagonal entry, we calculate x with the formula

$$x = \pm\frac{1}{2}\sqrt{M_{00} - M_{11} - M_{22} + 1}. \tag{1.77}$$

The y, z, and w components are then given by

$$\begin{aligned} y &= \frac{M_{10} + M_{01}}{4x} \\ z &= \frac{M_{02} + M_{20}}{4x} \\ w &= \frac{M_{21} - M_{12}}{4x}. \end{aligned} \tag{1.78}$$

Math Library Notes

- There is a `Transform()` function that takes a `Vector3D` object and a `Quaternion` object as parameters. It performs the transformation given by Equation (1.58) and returns the `Vector3D` result.
- The `Quaternion` class has a `GetRotationMatrix()` member function that calculates a 3×3 matrix with Equation (1.69) and returns it as a `Matrix3D` object.
- The `Quaternion` class has a `SetRotationMatrix()` member function that accepts either a `Matrix3D` or `Transform3D` object as its parameter. It converts the matrix to a quaternion using the method described in this section. The code assumes that the input is a true rotation matrix, meaning that it is orthogonal and has a determinant of +1. When a `Transform3D` object is converted to a quaternion, any translation stored in the fourth column is ignored.

1.4.3 Interpolating Quaternions

Quaternions are well suited for smoothly interpolating between arbitrary spatial orientations. If we had two unit quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$, then we could calculate an intermediate unit quaternion $\mathbf{q}(t)$ as a renormalized linear interpolation of $\mathbf{q}_1$ and $\mathbf{q}_2$ using the formula

$$\mathbf{q}(t) = \frac{(1-t)\,\mathbf{q}_1 + t\mathbf{q}_2}{\|(1-t)\,\mathbf{q}_1 + t\mathbf{q}_2\|}. \tag{1.79}$$

This actually produces acceptable results in cases that don't require high accuracy. If we were to try doing the same thing with two rotation matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ by interpolating entrywise, the results would be completely unusable because $(1-t)\,\mathbf{M}_1 + t\mathbf{M}_2$ can be very nonorthogonal and may even become singular.

The only problem with Equation (1.79) is that it does not interpolate from $\mathbf{q}_1$ to $\mathbf{q}_2$ at a constant rate as measured by the angle between $\mathbf{q}(t)$ and either $\mathbf{q}_1$ or $\mathbf{q}_2$ if we treat them all as ordinary 4D vectors. The rate is slower near the endpoints where $t = 0$ and $t = 1$, and it's fastest in the middle where $t = 1/2$. We can correct this problem by using a method called *spherical linear interpolation*, or just *slerp* for short. Slerp applies to ordinary vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of any dimension, and it provides a way of generating intermediate vectors $\mathbf{v}(t)$ that sweep through the arc between $\mathbf{v}_1$ and $\mathbf{v}_2$ at a constant angular rate as t changes linearly.

To derive a formula for slerp, first let $\mathbf{v}_1$ and $\mathbf{v}_2$ be unit-length vectors, and let ϕ be the angle between them in the two-dimensional subspace that they span. Now suppose that the function $\mathbf{v}(t)$ produces unit-length vectors that interpolate from $\mathbf{v}_1$ to $\mathbf{v}_2$ at a constant angular rate. This assertion requires that the angle between $\mathbf{v}_1$ and $\mathbf{v}(t)$ is equal to $t\phi$ and that the angle between $\mathbf{v}_2$ and $\mathbf{v}(t)$ is equal to $(1-t)\,\phi$, as shown in Figure 1.15. We can write $\mathbf{v}(t)$ as

$$\mathbf{v}(t) = d_1(t)\,\mathbf{v}_1 + d_2(t)\,\mathbf{v}_2, \tag{1.80}$$

where $d_1(t)$ and $d_2(t)$ are the lengths of the components of $\mathbf{v}(t)$ lying along the directions $\mathbf{v}_1$ and $\mathbf{v}_2$. We can determine formulas for $d_1(t)$ and $d_2(t)$ by constructing similar triangles, as shown by the highlighted green lines in the figure. Keeping in mind that $\|\mathbf{v}_1\| = 1$ and $\|\mathbf{v}(t)\| = 1$, basic trigonometry tells us that the perpendicular distances from $\mathbf{v}_1$ and $\mathbf{v}(t)$ to the line connecting the origin to $\mathbf{v}_2$ are equal to $\sin\phi$ and $\sin((1-t)\,\phi)$, respectively. Using the similar triangles shown on the left side of the figure, we obtain the ratio

$$d_1(t) = \frac{\sin((1-t)\,\phi)}{\sin\phi}. \tag{1.81}$$

The same procedure is used to find the formula for $d_2(t)$. This time, we look at the distances from $\mathbf{v}_2$ and $\mathbf{v}(t)$ to the line connecting the origin to $\mathbf{v}_1$. The similar triangles shown on the right side of the figure give us the ratio

$$d_2(t) = \frac{\sin(t\phi)}{\sin\phi}. \tag{1.82}$$

Plugging these functions into Equation (1.80) produces the complete formula for spherical linear interpolation, which is given by

Spherical linear interpolation

$$\boxed{\mathbf{v}(t) = \frac{\sin((1-t)\,\phi)}{\sin\phi}\,\mathbf{v}_1 + \frac{\sin(t\phi)}{\sin\phi}\,\mathbf{v}_2.} \tag{1.83}$$

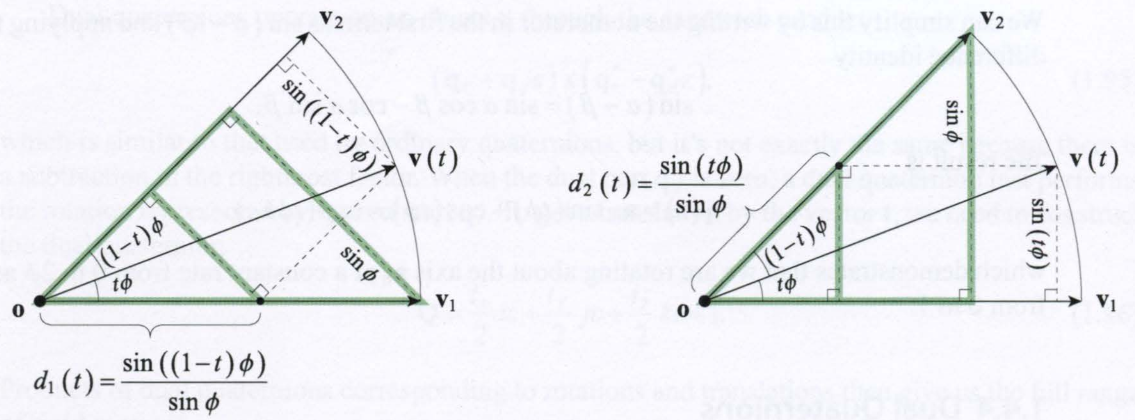


Figure 1.15. Similar triangles can be used to determine the lengths $d_1(t)$ and $d_2(t)$ needed to calculate the spherical linear interpolation of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Spherical linear interpolation between two unit quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$ is calculated with Equation (1.83) by simply treating them as four-dimensional vectors. The angle ϕ between the quaternions is determined by evaluating the dot product $\mathbf{q}_1 \cdot \mathbf{q}_2$ to obtain $\cos\phi$ and then inverting the cosine to get

$$\phi = \cos^{-1}(\mathbf{q}_1 \cdot \mathbf{q}_2). \tag{1.84}$$

To avoid calculating $\sin\phi$ directly, we can apply the identity $\sin^2\phi + \cos^2\phi = 1$ to instead calculate

$$\sin\phi = \sqrt{1-(\mathbf{q}_1 \cdot \mathbf{q}_2)^2}. \tag{1.85}$$

Since the quaternions $\mathbf{q}$ and $-\mathbf{q}$ correspond to the same rotation, it is common practice to flip the sign of either $\mathbf{q}_1$ or $\mathbf{q}_2$ as necessary so that $\mathbf{q}_1 \cdot \mathbf{q}_2 \geq 0$. This also ensures that the interpolation from $\mathbf{q}_1$ to $\mathbf{q}_2$ follows the shortest arc between them and doesn't end up taking the long way around.

It's important to realize that the angle ϕ in (1.83) is the angle between two quaternions and not the angle through which either quaternion actually rotates. As interpolation progresses, the axis of rotation and the angle of rotation are both changing, and it can be difficult to tell what's going on. This can be cleared up by multiplying everything by $\mathbf{q}_1^*$ so that the interpolation formula becomes

$$\mathbf{q}(t) = \frac{\sin[(1-t)\phi]}{\sin\phi} + \mathbf{q}_0 \frac{\sin(t\phi)}{\sin\phi}, \tag{1.86}$$

where $\mathbf{q}_0 = \mathbf{q}_2\mathbf{q}_1^*$. This essentially realigns the coordinate axes to the directions to which they are rotated by $\mathbf{q}_1$. After calculating a value of $\mathbf{q}(t)$ to interpolate between the identity and $\mathbf{q}_0$, we multiply by $\mathbf{q}_1$ to return to the original coordinate system. The quaternion $\mathbf{q}_0$ can be expressed as

$$\mathbf{q}_0 = \mathbf{a}_0 \sin\phi + \cos\phi, \tag{1.87}$$

which rotates by the angle 2ϕ about the axis $\mathbf{a}_0$. In this case, the dot product between the identity $(0,0,0,1)$ and $\mathbf{q}_0$ is just the scalar term $\cos\phi$, so the angle between the quaternions being interpolated is half the angle by which $\mathbf{q}_0$ rotates. This means that the angle ϕ has the same interpretation in both Equations (1.86) and (1.87), and the expression for $\mathbf{q}_0$ given by Equation (1.87) can be directly substituted into Equation (1.86) to get

$$\mathbf{q}(t) = \frac{\sin[(1-t)\phi]}{\sin\phi} + \left(\mathbf{a}_0 + \frac{\cos\phi}{\sin\phi}\right)\sin(t\phi). \tag{1.88}$$

We can simplify this by writing the numerator in the first term as $\sin(\phi - t\phi)$ and applying the angle difference identity

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{1.89}$$

The result is

$$\mathbf{q}(t) = \mathbf{a}_0 \sin(t\phi) + \cos(t\phi) = \exp(t\phi\mathbf{a}_0), \tag{1.90}$$

which demonstrates that we are rotating about the axis $\mathbf{a}_0$ at a constant rate from 0 to 2ϕ as t varies from 0 to 1.

1.4.4 Dual Quaternions

Quaternions can only perform rotations about the origin. Because they are incapable of performing translations, it's common practice to combine a quaternion $\mathbf{q}$ with 3D vector $\mathbf{t}$ to form a complete rigid motion in space. A point $\mathbf{p}$ is then transformed according to

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^* + \mathbf{t}, \tag{1.91}$$

which is very similar to Equation (1.9) in Section 1.2. In our earlier discussion of homogeneous coordinates, we incorporated a translation into a transformation matrix by adding a fourth column containing a translation vector and a fourth row that is always $[0\ \ 0\ \ 0\ \ 1]$. This required 12 numbers in order to describe a rotation about an arbitrary axis followed by a translation, but the combination of a quaternion and vector requires only 7 numbers, and this can be reduced to 6 if we're willing to reconstitute the scalar part of a quaternion with Equation (1.70).

Equation (1.91) can be somewhat clunky when it comes to composing or interpolating transformations, and it does not distinguish between direction vectors and position vectors. There is a quaternion-based analog of homogeneous coordinates that makes a similar leap from 3D space to 4D space and has all the nice properties that we would like. It works through the combination of quaternions and something called *dual numbers*, and it thus has the name *dual quaternions*. We introduce the concept of dual quaternions here, but our discussion is kept very brief because they are limited and difficult to understand in the conventional setting, and they have been entirely supplanted by the motion operators that are the main topic of Chapter 3.

A dual number is a quantity of the form $a + b\varepsilon$, where a and b are real numbers, and the special nonzero value ε has the property that $\varepsilon^2 = 0$. Multiplication of two dual numbers can easily be worked out to arrive at

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon. \tag{1.92}$$

A dual quaternion is defined as $\mathbf{Q} = \mathbf{q}_r + \mathbf{q}_d\varepsilon$, where $\mathbf{q}_r$ is a quaternion called the *real* part or *rotational* part, and $\mathbf{q}_d$ is another quaternion called the *dual* part. A dual quaternion contains a mixture of the imaginary units $i, j,$ and k that each square to -1 and the special unit ε that squares to zero. We can write out all eight components of a dual quaternion $\mathbf{Q}$ as

$$\mathbf{Q} = q_{rx}i + q_{ry}j + q_{rz}k + q_{rw} + q_{dx}i\varepsilon + q_{dy}j\varepsilon + q_{dz}k\varepsilon + q_{dw}\varepsilon. \tag{1.93}$$

The products $i\varepsilon$, $j\varepsilon$, and $k\varepsilon$ between a quaternion's imaginary units and the value ε do not simplify. They can be thought of as new basis vectors that make up part of the eight-dimensional vector space of dual quaternions. The value ε does commute with each of $i, j,$ and k, however, so all three of these basis vectors square to zero, as expressed by

$$(i\varepsilon)^2 = (j\varepsilon)^2 = (k\varepsilon)^2 = 0. \tag{1.94}$$

Dual quaternions operate on an object **x** through the sandwich product

$$(\mathbf{q}_r + \mathbf{q}_d\varepsilon)\,\mathbf{x}\left(\mathbf{q}_r^* - \mathbf{q}_d^*\varepsilon\right), \tag{1.95}$$

which is similar to that used by ordinary quaternions, but it's not exactly the same because there is a subtraction in the rightmost factor. When the dual part $\mathbf{q}_d$ is zero, a dual quaternion just performs the rotation represented by the real part $\mathbf{q}_r$. To get a translation by the vector **t**, we need to construct the dual quaternion

$$\mathbf{Q} = \frac{t_x}{2}i\varepsilon + \frac{t_y}{2}j\varepsilon + \frac{t_z}{2}k\varepsilon + 1. \tag{1.96}$$

Products of dual quaternions corresponding to rotations and translations then give us the full range of rigid motions.

In order to transform a point **p** with Equation (1.95), we have to cast **p** into the form

$$\mathbf{p} = p_x i\varepsilon + p_y j\varepsilon + p_z k\varepsilon + 1. \tag{1.97}$$

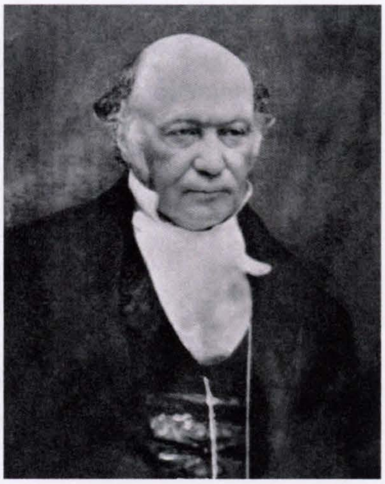
When we plug this in for **x** and perform the multiplication with the rules for quaternions and dual numbers, we get back another quantity having this form, and the components of the transformed point are given by the coefficients of its $i\varepsilon$, $j\varepsilon$, and $k\varepsilon$ terms. If we want to transform a direction vector **v** instead of a position vector, then we simply leave off the scalar term in Equation (1.97) so that **v** is encoded as

$$\mathbf{v} = v_x i\varepsilon + v_y j\varepsilon + v_z k\varepsilon. \tag{1.98}$$

This direction is affected only by the rotational part $\mathbf{q}_r$ of a dual quaternion, so we could just calculate $\mathbf{q}_r \mathbf{v} \mathbf{q}_r^*$ and avoid spending time on other terms that sum to zero.

If the above formulation of dual quaternion transformations feels like a messy hack, then your intuition is correct. We do not bother going any further in this direction because the whole thing has been superseded by the much clearer and more elegant development of motion operators in geometric algebra. In Chapter 3, we will see that a dual quaternion is really an encoding of a line in 3D space, a rotation angle about that line, and a displacement along that line. This covers all possible rigid transformations, and we will be able to use it to transform not only direction and position vectors, but also lines and planes with the same formula. This will be extended to include round objects like circles and spheres in Chapter 5.

Historical Remarks



William Rowan Hamilton
(1805–1865)

In the mid-nineteenth century, the Irish scientist William Rowan Hamilton was studying the nature of multiplication and division in the complex numbers. He attempted to extend similar algebraic principles to a three-dimensional space of real numbers, but after considerable effort, he failed to find a logically sound method for multiplication. (This would later be proven by others to be an impossible feat.) So he turned his attention to four-dimensional numbers. One morning in 1843, while contemplating four-dimensional multiplication as he walked to a meeting of the Royal Irish Academy with his wife, and he had an epiphany. He had discovered a consistent rule for multiplying "quaternions", and he excitedly etched

$$i^2 = j^2 = k^2 = ijk = -1$$

into the stones of a nearby bridge over the Royal Canal. Today, Hamilton's original carving is no longer visible, but a stone plaque adorns the Broome bridge in Dublin to mark the location where his "flash of genius" took place. Hamilton spent much of the rest of his life studying and writing about quaternions [Hami1844]. Today, the set of quaternions is denoted by the blackboard bold letter $\mathbb{H}$ in honor of Hamilton's discovery, and we continue to use the terms scalar and vector that Hamilton coined for the separate parts of a quaternion.

As discussed in Chapter 3, we now understand that the quaternions arise as part of the geometric algebra in three-dimensional space and form a rotation group. The dual quaternions are part of the *projective* geometric algebra over three-dimensional space, and they form the Euclidean group of all proper isometries, which includes rotations and translations. In two-dimensional space, the complex numbers correspond to rotations, and the quaternions are the natural analog in one dimension higher.

Julius Plücker
(1801–1868)

Julius Plücker was a German mathematician and physicist who was known for his work in analytic geometry. In late 1864, he wrote an article entitled "On a New Geometry of Space" [Pluc1865] that contained the first description of a six-component homogeneous representation of a line in three-dimensional space. His discovery provided a new way of uniquely identifying a line by its direction and moment independently of any specific points contained by the line. The six components became known as the Plücker coordinates of a line.

We now understand that Plücker coordinates arise as the six components of a constrained bivector in the projective geometric algebra over three-dimensional space. As discussed in Chapter 2, such a bivector belongs to a set of flat geometry types covering all possible dimensionalities in which vectors represent homogeneous points, bivectors represent homogeneous lines, and trivectors represent homogeneous planes.

Chapter 2

Flat Projective Geometry

The previous chapter reviewed cross product transformations, homogeneous coordinates, implicit lines and planes, and quaternions in the context of conventional mathematics that are widely known. However, for each one of these topics, the conventional approach either doesn't tell the whole story or causes us to misinterpret subtle concepts that could be understood in a more intuitive way if they were derived from the right foundations. From this point onward, our goal is to remedy this situation by revealing a complete picture that includes everything discussed in Chapter 1 and much more under the unified theories of *exterior algebra* and *geometric algebra*. Though the origins of these theories extend back to the mid 19th century, they were lost to obscurity for much of their existence, and the details weren't hammered out until well into the 20th century (and not always correctly). It has been only very recently, in the 21st century, that many aspects of *projective* exterior algebra and geometric algebra have been well understood, and those are the subjects on which this book focuses. The material will likely require some significant mental adjustments on the part of readers who have no prior familiarity with it, but the reward will be an understanding of a natural and elegant mathematical structure.

This chapter introduces projective exterior algebra and applies it to flat geometries in three-dimensional space (which are points, lines, and planes) followed by flat geometries in two-dimensional space (which are just points and lines). Exterior algebra can be thought of as an extension to linear algebra that completes the total structure of a vector space. Given n linearly independent direction vectors spanning an n-dimensional space, there are 2^n ways to combine any number of those directions to create a mathematical structure much richer than what arises from only a set of scalars and n basis vectors. In addition to its direct geometric significance, projective exterior algebra possesses intrinsic symmetries that lead to a recurring theme of *duality*. Nearly all concepts in exterior algebra come in pairs that have similar but opposite properties, and these are highlighted throughout this chapter both in the text and in special duality graphics that appear at the end of many sections.

2.1 Algebraic Structure

At the heart of exterior algebra is an operation called the *exterior product* that provides a geometrically meaningful way to multiply any two quantities together. The exterior product between **a** and **b** is denoted by $\mathbf{a} \wedge \mathbf{b}$, with an upward pointing wedge symbol, and we read it as "**a** wedge **b**". Just like the conventional dot product and cross product get their names from the symbol used to represent them, the exterior product is often called the syllabically simpler *wedge product*, and that's what we prefer in this book. An n-dimensional exterior algebra is constructed from a vector space by starting with ordinary scalars and n basis vectors and then defining how they are multiplied together using the wedge product. This ultimately leads to a larger space with 2^n basis elements arranged in a stratified formation.

2.1.1 The Wedge Product

For any product involving a scalar, which includes scalar times scalar and scalar times vector, the wedge product is no different from the scalar multiplication that we are familiar with from conventional mathematics. Beyond that, things get a little different, but the entire algebra is derived from one simple rule: any vector multiplied by itself using the wedge product is zero. That is, for any vector $\mathbf{v}$, we always have

$$\boxed{\mathbf{v} \wedge \mathbf{v} = 0.} \tag{2.1}$$

This rule has an important consequence that reveals itself when we consider the sum of two vectors $\mathbf{a}$ and $\mathbf{b}$. The wedge product of $\mathbf{a} + \mathbf{b}$ with itself gives us

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) = \mathbf{a} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{b} = 0. \tag{2.2}$$

The products $\mathbf{a} \wedge \mathbf{a}$ and $\mathbf{b} \wedge \mathbf{b}$ must both be zero, and that leaves us with

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0, \tag{2.3}$$

from which we conclude that

$$\boxed{\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.} \tag{2.4}$$

This establishes the property that multiplication of vectors with the wedge product is anticommutative. Reversing the order of the two factors negates the product. We should stress that we have only shown this to be true for vectors at this point, and it does not hold in general. In particular, the wedge product between scalars, being ordinary multiplication, is commutative. These facts and a few additional properties of the wedge product, when used to multiply scalars and vectors, are summarized in Table 2.1.

Math Library Notes

- The wedge product is implemented by the `Wedge()` function. The wedge product can also be calculated by using the ^ symbol as an infix operator.
- The ^ operator has a very low evaluation precedence among operators in C and C++, even lower than the relational operators, so it is rather ill-suited in this respect for the role of infix multiplication. It is often necessary to surround each wedge product with parentheses to prevent operations from occurring in the wrong order. For example, the expression `a ^ b < c ^ d` would be interpreted by the compiler as `a ^ (b < c) ^ d`, so it would have to be written as `(a ^ b) < (c ^ d)` to get the correct result.

2.1.2 Bivectors

The wedge product $\mathbf{a} \wedge \mathbf{b}$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ cannot be expressed in terms of scalars and vectors. It forms a new type of mathematical element called a *bivector*. Whereas a vector can be thought of as a combination of a direction and a magnitude, a bivector can be thought of as a combination of an *oriented area* and a magnitude. A bivector $\mathbf{a} \wedge \mathbf{b}$ can be visualized as a parallelogram whose sides are parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$, as shown in Figure 2.1. The parallelogram has an intrinsic winding direction that reflects the order in which the vectors $\mathbf{a}$ and $\mathbf{b}$ appear in the wedge product, and this direction can be determined by following the perimeter of the parallelogram first along the direction of $\mathbf{a}$ and then along the direction of $\mathbf{b}$. If the order of the vectors is reversed, negating the result, then the winding direction is also reversed, exchanging clockwise and counterclockwise directions around the perimeter.

Property	Description
$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}=\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$	Associative law for the wedge product.
$\mathbf{a}\wedge(\mathbf{b}+\mathbf{c})=\mathbf{a}\wedge\mathbf{b}+\mathbf{a}\wedge\mathbf{c}$ $(\mathbf{a}+\mathbf{b})\wedge\mathbf{c}=\mathbf{a}\wedge\mathbf{c}+\mathbf{b}\wedge\mathbf{c}$	Distributive laws for the wedge product.
$(s\mathbf{a})\wedge\mathbf{b}=\mathbf{a}\wedge(s\mathbf{b})=s(\mathbf{a}\wedge\mathbf{b})$	Scalar factorization for the wedge product.
$s\wedge t=t\wedge s=st$	Wedge product between scalars.
$s\wedge\mathbf{a}=\mathbf{a}\wedge s=s\mathbf{a}$	Wedge product between a scalar and a vector.
$\mathbf{a}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}$	Anticommutativity of the wedge product for vectors.

Table 2.1. These are the basic properties of the wedge product. The letters **a**, **b**, and **c** represent vectors, and the letters *s* and *t* represent scalar values.

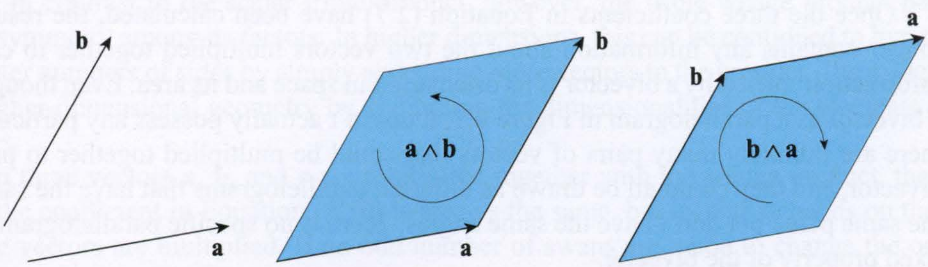


Figure 2.1. The bivector $\mathbf{a}\wedge\mathbf{b}$ can be visualized as a parallelogram whose sides are parallel to the vectors **a** and **b**. The intrinsic winding direction follows the perimeter along the first vector in the wedge product and then along the second vector. Reversing the order of the vectors in the product also reverses the winding direction.

In order to give some quantitative substance to a bivector, we can examine the effect the wedge product has on vectors that have been decomposed into components over an orthonormal basis. We'll be working in three dimensions for now, but keep in mind that a similar analysis is valid in any number of dimensions. Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ represent three mutually orthogonal unit vectors in three-dimensional space. These generic labels are intended to avoid being tied to any particular coordinate system, but we will equate them to a typical right-handed configuration of the *x*, *y*, and *z* axes. We can write an arbitrary vector $\mathbf{v}=(v_x, v_y, v_z)$ in terms of the three basis vectors as

$$\mathbf{v}=v_x\mathbf{e}_1+v_y\mathbf{e}_2+v_z\mathbf{e}_3. \tag{2.5}$$

Doing this for two vectors **a** and **b** allows us to write the bivector $\mathbf{a}\wedge\mathbf{b}$ as

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b}&=(a_x\mathbf{e}_1+a_y\mathbf{e}_2+a_z\mathbf{e}_3)\wedge(b_x\mathbf{e}_1+b_y\mathbf{e}_2+b_z\mathbf{e}_3)\\&=a_xb_y(\mathbf{e}_1\wedge\mathbf{e}_2)+a_xb_z(\mathbf{e}_1\wedge\mathbf{e}_3)+a_yb_x(\mathbf{e}_2\wedge\mathbf{e}_1)\\&\quad+a_yb_z(\mathbf{e}_2\wedge\mathbf{e}_3)+a_zb_x(\mathbf{e}_3\wedge\mathbf{e}_1)+a_zb_y(\mathbf{e}_3\wedge\mathbf{e}_2),\end{aligned} \tag{2.6}$$

where each term containing the wedge product of a basis vector with itself has been dropped because it is zero. Every term in this expression contains a wedge product whose factors appear in the reverse order of the wedge product in another term, so we can negate one half of the terms and collect over common bivectors to arrive at

$$\boxed{\mathbf{a} \wedge \mathbf{b} = (a_y b_z - a_z b_y)(\mathbf{e}_2 \wedge \mathbf{e}_3) + (a_z b_x - a_x b_z)(\mathbf{e}_3 \wedge \mathbf{e}_1) + (a_x b_y - a_y b_x)(\mathbf{e}_1 \wedge \mathbf{e}_2).} \tag{2.7}$$

Here, we have arranged terms in the order by which a basis vector does *not* appear in the wedge product, so the term missing $\mathbf{e}_1$ comes first, the term missing $\mathbf{e}_2$ comes second, and the term missing $\mathbf{e}_3$ comes third. This expression can be simplified no further, and it demonstrates that an arbitrary bivector in 3D space has three components over a *bivector basis* consisting of $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_3 \wedge \mathbf{e}_1$, and $\mathbf{e}_1 \wedge \mathbf{e}_2$.

The three coefficients in Equation (2.7) should have a familiar ring to them. They are exactly the same values that are calculated by the cross product, which is something usually defined without much explanation as to where the values come from. Now, these numbers appear as a result that was derived from a fundamental property of the wedge product. The fact that bivectors have three components is unique to three dimensions, and this similarity makes a bivector *look* like an ordinary vector, but it is indeed something different. Failing to make a distinction leads to an incomplete and inelegant picture of the mathematics. An important thing to understand is that the wedge product is defined in a manner similar to Equation (2.7) in any number of dimensions, while the cross product is confined to only three dimensions, limiting its usefulness.

Once the three coefficients in Equation (2.7) have been calculated, the resulting bivector no longer contains any information about the two vectors multiplied together to create it. The only information carried by a bivector is its orientation in space and its area. Even though we have drawn a bivector as a parallelogram in Figure 2.1, it doesn't actually possess any particular shape. In fact, there are infinitely many pairs of vectors that could be multiplied together to produce any given bivector, and they could all be drawn as different parallelograms that have the same area and lie in the same plane but don't have the same angles. There is no specific parallelogram whose shape is a fixed property of the bivector.

Math Library Notes

- The `Bivector3D` class stores the three coordinates of a 3D bivector. It has floating-point members named `x`, `y`, and `z`, and they correspond to the basis bivectors $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_3 \wedge \mathbf{e}_1$, and $\mathbf{e}_1 \wedge \mathbf{e}_2$, respectively.
- When two `Vector3D` objects are multiplied together with either the `Wedge()` function or the `^` operator, the result is a `Bivector3D` object.

2.1.3 Trivectors

We now have an algebraic system that includes scalars, vectors, and bivectors, but this is not the end of the road, at least not in three dimensions. Let's consider the wedge product among three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ given by

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3) \wedge (b_x \mathbf{e}_1 + b_y \mathbf{e}_2 + b_z \mathbf{e}_3) \wedge (c_x \mathbf{e}_1 + c_y \mathbf{e}_2 + c_z \mathbf{e}_3). \tag{2.8}$$

When multiplying all of this out, remember that any term containing a repeated factor is zero, and the only parts that remain are the terms containing all three of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in the six possible orders in which they can be multiplied. Fully written out, these six terms are

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = {} & a_x b_y c_z (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) + a_y b_z c_x (\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1) \\ & + a_z b_x c_y (\mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2) + a_z b_y c_x (\mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1) \\ & + a_y b_x c_z (\mathbf{e}_2 \wedge \mathbf{e}_1 \wedge \mathbf{e}_3) + a_x b_z c_y (\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2).\end{aligned} \tag{2.9}$$

We can swap the order of adjacent factors one or more times in each of the triple wedge products to make all of them equal to $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ as long as we negate the scalar coefficient each time we do it. For example, $\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 = -(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$ because it requires a single swap of the last two factors, and $\mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 = +(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$ because it requires two swaps. After adjusting all of the terms, we can write the complete product as

$$\boxed{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_y b_x c_z - a_z b_y c_x)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3).} \tag{2.10}$$

This is yet another new mathematical element called a *trivector*, which is distinct from a scalar, vector, and bivector. Notice that in three dimensions, the trivector has only one component, and it is associated with the basis trivector $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. A trivector combines an *oriented volume* and a magnitude, but the only choice we have about the orientation is whether the volume is positive or negative.

You may recognize the scalar coefficient in Equation (2.10) as the determinant of a 3×3 matrix whose columns or rows are the vectors **a**, **b**, and **c**. Or you may recognize it as the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, which has the same value. Just as the wedge product $\mathbf{a} \wedge \mathbf{b}$ of two vectors has a magnitude equal to the area of the parallelogram spanned by **a** and **b**, the wedge product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ of three vectors has a magnitude equal to the volume of the parallelepiped spanned by **a**, **b**, and **c**. In contrast to the scalar triple product, however, the triple wedge product possesses a pleasing symmetry among its factors. In higher dimensions, this can be continued to hypervolumes with greater numbers of sides by simply appending more vectors to the product. The wedge product builds higher-dimensional geometry by combining the dimensionalities of the elements on which it operates.

When three vectors **a**, **b**, and **c** are multiplied together with the wedge product, the absolute value of the coefficient in Equation (2.10) is always the same, but its sign depends on the order in which the vectors are multiplied. If an odd number of swaps are made to change the order from $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$, then the result is negated, but if an even number of swaps are made, then nothing happens. The six possible orderings are illustrated in Figure 2.2. The three products in the top row have one

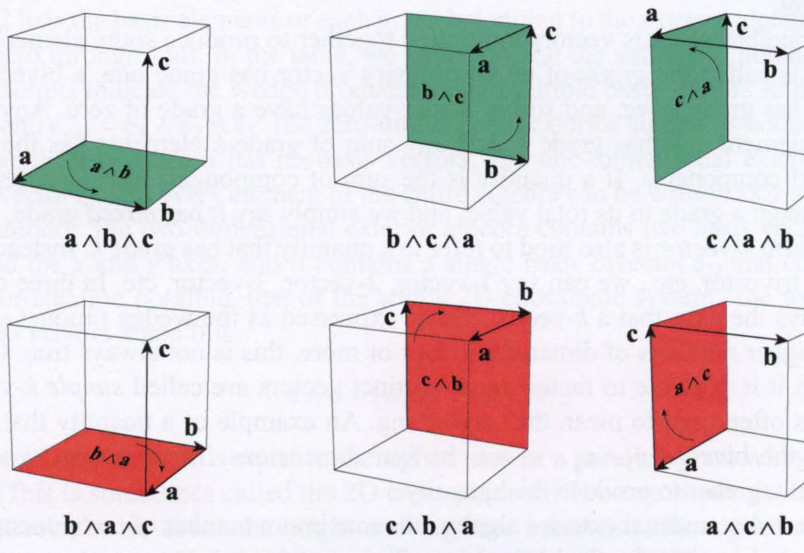


Figure 2.2. These are the six ways in which three vectors can be multiplied together with the wedge product to construct a trivector. The three trivectors in the top row are equal to each other, and the three trivectors in the bottom row are negated relative to the top row. In the top row, the third vector in the product satisfies the right-hand rule by pointing out of an area that is wound counterclockwise due to the order of the first two vectors in the product. The trivectors in the bottom row have the opposite sign because their third vectors point out of areas that are instead wound clockwise.

sign, whichever is given by Equation (2.10) for any given inputs, and the three products in the bottom row have the opposite sign. In this figure, the vectors have a right-handed configuration so that the volumes in the top row are positive and the volumes in the bottom row are negative, but reversing any one of the vectors so that it points in the opposite direction would cause these signs to be flipped. In general, when the third vector in the product follows the right-hand rule, meaning that the right thumb points in the direction of the third vector when the fingers of the right hand curl in the winding direction of the first two vectors, the volume is positive, and otherwise, the volume is negative. The universe doesn't actually have a preference for the right-hand rule over a similar left-hand rule, however, and the sign of our calculation depends on the fact that we are choosing $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ as our trivector basis element (as opposed to some other ordering of those vectors) and that $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form the axes of a right-handed coordinate system.

2.1.4 Basis Elements

In three dimensions, trivectors are the limit for new mathematical types. We cannot multiply by a fourth vector to create a *quadrivector* because the product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}$ for *any* vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ must be zero. When we expand the product, we find that every term contains a repeated factor because it's impossible to have four linearly independent vectors in three-dimensional space. This means that the complete exterior algebra in three dimensions consists of elements that are scalars, vectors having three components, bivectors having three components, and trivectors having one component.

There is a combinatorial reason why it works out this way, and it has to do with how basis vectors from the set $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are used by each type of element. Components of a vector each use one member of the set S, and there are three ways to choose one member, so vectors have three components. Components of a bivector each use two members of the set S, and there are three ways to choose two members, so bivectors also have three components. Finally, the single component of a trivector uses all three members of the set S, and because there is only one way to choose all three, trivectors have only one component. Along the same line of reasoning, we can say that scalars use *no* members of the set S, and there is only one way to choose nothing, so scalars also have only one component.

The number of basis vectors multiplied together to produce some element $\mathbf{u}$ of higher dimensionality is called the *grade* of $\mathbf{u}$. An ordinary vector has grade one, a bivector has grade two, a trivector has grade three, and so on. Scalar values have a grade of zero. Any scalar multiple of a grade-k element also has grade k, and any sum of grade-k elements has the same grade k as its individual components. If a quantity is the sum of components having different grades, then we cannot assign a grade to its total value, and we simply say it has *mixed* grade.

The term *k-vector* is also used to refer to a quantity that has grade k. Instead of the terms vector, bivector, trivector, etc., we can say 1-vector, 2-vector, 3-vector, etc. In three dimensions or fewer, it is always the case that a k-vector can be expressed as the wedge product of k vectors of grade one. In higher numbers of dimensions, four or more, this is not always true. Quantities of grade k for which it is possible to factor into k distinct vectors are called *simple* k-vectors, and the term *k-blade* is often used to mean the same thing. An example of a quantity that is not simple (not a blade) is the bivector $\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4$ in four dimensions. There are no two vectors that can be multiplied together to produce this quantity.

In the n-dimensional exterior algebra, the maximum number of components making up a quantity of grade k is given by the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.11}$$

because this gives the number of ways to choose k items from a set of n items. The binomial coefficients produce Pascal's triangle, as shown in Figure 2.3, where the row corresponding to three

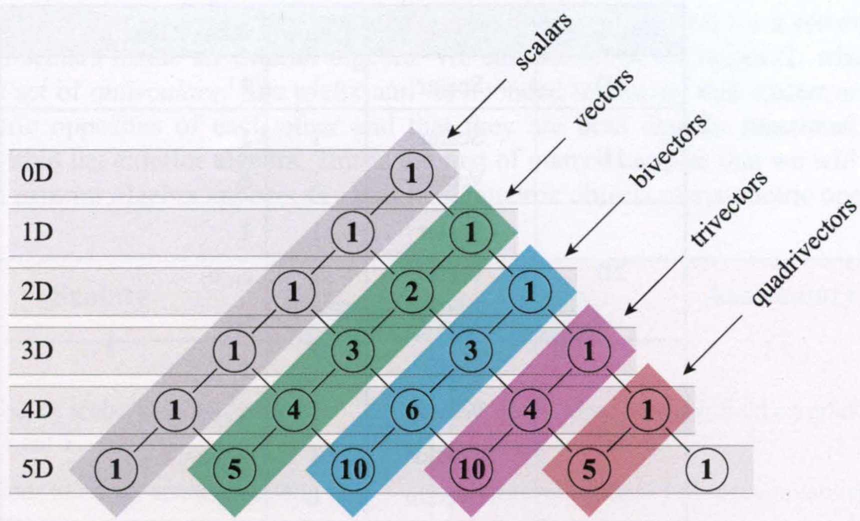


Figure 2.3. The number of components making up a quantity of grade k in the n-dimensional exterior algebra is given by the binomial coefficient $\binom{n}{k}$, which produces Pascal's triangle. Each row corresponds to a particular number of dimensions, and the numbers in each row tell how many independent basis elements exist for each grade. There is always one scalar basis element and n vector basis elements. The number of bivector basis elements is given by the third number in each row, the number of trivector basis elements is given by the fourth number, and so on.

dimensions reads 1, 3, 3, 1. The complete n-dimensional exterior algebra is constructed by choosing basis vectors from the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in every possible quantity and combining them in every possible way. For any given element in the algebra, a particular basis vector $\mathbf{e}_i$ is either included or excluded, so we can think of the inclusion status of the entire set of basis vectors as an n-bit quantity for which all 2^n values are allowed. This is reflected in the fact that each row of Pascal's triangle sums to the power of two corresponding to the number of dimensions.

Table 2.2 lists the basis elements of each grade belonging to the exterior algebras in dimensions numbering zero through four. In the table, we have adopted the simplified notation $\mathbf{e}_{ab\ldots}$ in which multiple subscripts indicate the wedge product among multiple basis vectors so that, for example, $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. The zero-dimensional exterior algebra is nothing more than the set of real numbers because it has no basis vectors. The one-dimensional exterior algebra has a single basis vector $\mathbf{e}_1$, and every element of the entire algebra can be written as $a + b\,\mathbf{e}_1$, where a and b are real numbers. The two-dimensional exterior algebra contains two basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ that correspond to the x and y axes, and it contains a single basis bivector $\mathbf{e}_{12}$ that corresponds to the only planar orientation possible, that of the whole 2D coordinate system. The wedge product between two 2D vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \wedge \mathbf{b} = (a_x b_y - a_y b_x)\,\mathbf{e}_{12}, \tag{2.12}$$

and the value of this bivector is equal to the signed area of a parallelogram whose sides are given by $\mathbf{a}$ and $\mathbf{b}$. (This is sometimes called the 2D cross product.) The area is positive when $\mathbf{a}$ and $\mathbf{b}$ are wound counterclockwise about the origin and negative otherwise. The signed area of a triangle having sides given by $\mathbf{a}$ and $\mathbf{b}$ is half the value given by Equation (2.12).

In each of the exterior algebras listed in Table 2.2, the order of vector multiplication for the basis elements of grade two and higher contains an arbitrary choice. For example, the value $\mathbf{e}_{23}$ is listed as one of the basis bivectors for the 3D and 4D algebras, but we could have instead decided to use $\mathbf{e}_{32}$, which involves the same two vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ multiplied in the opposite order. There are logical reasons why we have chosen the multiplication orders that are shown in the table, and these

Dimension	Type	Count	Basis Elements
0D	Scalar	1	$\mathbf{1}$
1D	Scalar	1	$\mathbf{1}$
	Vector	1	$\mathbf{e}_1$
2D	Scalar	1	$\mathbf{1}$
	Vector	2	$\mathbf{e}_1, \mathbf{e}_2$
	Bivector	1	$\mathbf{e}_{12}$
3D	Scalar	1	$\mathbf{1}$
	Vector	3	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
	Bivector	3	$\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}$
	Trivector	1	$\mathbf{e}_{123}$
4D	Scalar	1	$\mathbf{1}$
	Vector	4	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
	Bivector	6	$\mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}$
	Trivector	4	$\mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}, \mathbf{e}_{321}$
	Quadrivector	1	$\mathbf{e}_{1234}$

Table 2.2. This table lists the basis elements of each grade in the n-dimensional exterior algebras for $0 \le n \le 4$. The total number of basis elements is always equal to 2^n, and the number of basis elements of a particular grade k is given by $\binom{n}{k}$.

are highlighted below as they arise. In general, once we have made the decision as to which way we are multiplying vectors for each of the basis elements, we stick with it in all of our expressions and avoid mixing the two possibilities that exist for everything of grade two or higher. The only effect that making one choice over the other for any particular basis element has is that every term involving that basis element gets negated without any change to the geometric interpretation.

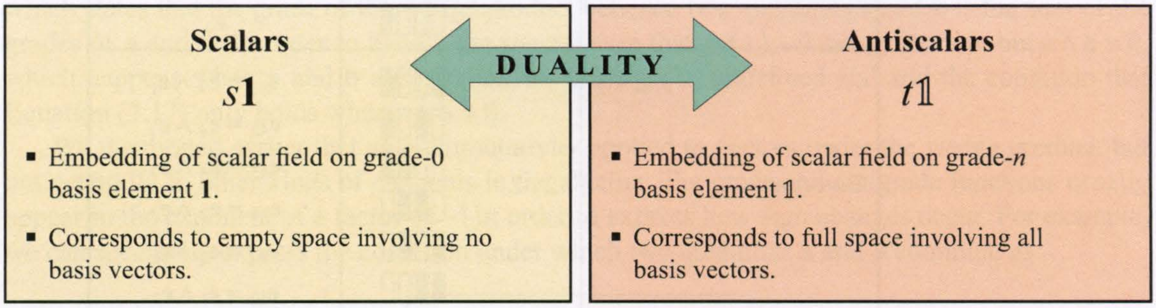
Every exterior algebra has a basis element denoted by $\mathbf{1}$, the number one written in a bold style, that represents the embedding of the set of scalars inside the algebra. For every scalar value s, the corresponding member of the exterior algebra is technically given by $s\mathbf{1}$, but we won't always write the $\mathbf{1}$ explicitly because it will be obvious what we mean when we just write s by itself. The basis element $\mathbf{1}$ is the multiplicative identity of the wedge product such that $\mathbf{u} \wedge \mathbf{1} = \mathbf{1} \wedge \mathbf{u} = \mathbf{u}$ for any value $\mathbf{u}$ of any grade.

Every n-dimensional exterior algebra with $n \ge 1$ also contains exactly one basis element of grade n given by the wedge product of all n basis vectors. This highest-grade basis element is called the *unit volume element*, and it is denoted by the special symbol $\mathbb{1}$, the number one written in a blackboard bold style. The unit volume element is defined as

$$\boxed{\mathbb{1} = \pm\mathbf{e}_{12\cdots n} = \pm(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n),} \tag{2.13}$$

and it corresponds to an n-dimensional volume of magnitude one. The undetermined sign in this definition represents a choice that we can make about the orientation of the volume element. For example, in three dimensions, we can define $\mathbb{1} = \mathbf{e}_{123}$ or $\mathbb{1} = -\mathbf{e}_{123} = \mathbf{e}_{321}$, where any other ordering of the basis vectors must be equal to one of those two possibilities. The orientation that we choose establishes a relationship between dual concepts in exterior algebra that arise due to the pronounced symmetry visible in Figure 2.3.

Because the volume element has only one component, it is the basis for a second embedding of the set of scalars inside an exterior algebra. We call the set of all values $t\mathbb{1}$, where t is a real number, the set of *antiscalars*.[1] The prefix anti- is intended to convey that scalars and antiscalars are symmetric opposites of each other and that they are both equally functional and equally important within the exterior algebra. This is the first of many examples that we will see in which a concept in exterior algebra appears as a pair of symmetric objects or symmetric operations.



The elements of the n-dimensional exterior algebra can be generalized so that the components of every grade are mixed into a single quantity called a *multivector*. For example, a multivector belonging to the three-dimensional exterior algebra is written as

$$s\mathbf{1} + v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3 + b_x\mathbf{e}_{23} + b_y\mathbf{e}_{31} + b_z\mathbf{e}_{12} + t\mathbb{1}. \tag{2.14}$$

It would be possible to design a computational system in which all eight components of the 3D multivector in Equation (2.14) were always stored in memory and operations were always performed between two complete multivectors. However, this would be rather impractical and quite wasteful because many of the components would often be zero. This is especially true in higher dimensions due to the exponential increase in the number of components. It will always be the case in this chapter that geometric objects are represented by quantities having one specific grade. In the next chapter, quantities representing transformations will be composed of components having different grades, but those grades will always be either all even or all odd, so the maximum number of components in a single quantity will never be greater than 2^{n-1} in an n-dimensional algebra. In the math library, we define different types that allow us to store geometric objects and transformations using the smallest number of components possible.

Most of this chapter deals with the four-dimensional exterior algebra because it completes the vector space where the homogeneous coordinates described in Chapter 1 represent points in three dimensions. It is a *projective* exterior algebra because, as discussed in greater detail later, we project into a three-dimensional subspace in order to interpret a 4D object as a 3D geometry. Since this particular exterior algebra will be our main focus, it is valuable to spend a moment taking a closer look at its basis elements.

In four dimensions, the exterior algebra has 16 basis elements across five different grades, as shown in Table 2.3. As with all exterior algebras, there is a scalar basis element $\mathbf{1}$, and any real number s is mapped into the algebra as $s\mathbf{1}$. Since the algebra is four-dimensional, there are four

[1] Conventionally, a bold $\mathbf{I}$ has denoted the unit volume element, and multiples of $\mathbf{I}$ have been called *pseudoscalars*. However, the symbol $\mathbf{I}$ conflicts with the symbol commonly used to denote an identity matrix, and we use $\mathbf{I}$ for that purpose in contexts where the volume element also appears. The symbol $\mathbb{1}$ that we use for the volume element strongly reflects the symmetry with $\mathbf{1}$, and it fits consistently into a larger system of notation developed throughout this chapter. The prefix anti- is also conceptually consistent with the many examples of duality that appear in this book. We eschew the prefix pseudo- because its meaning of "false" suggests that antiscalars are somehow less important or less meaningful than scalars, but that is not true.

Type	Grade / Antigrade		Values
Scalar	0 / 4	▯▯▯▯	$\mathbf{1}$
Vectors	1 / 3	▮▯▯▯	$\mathbf{e}_1$
		▯▮▯▯	$\mathbf{e}_2$
		▯▯▮▯	$\mathbf{e}_3$
		▯▯▯▮	$\mathbf{e}_4$
Bivectors	2 / 2	▮▯▯▮	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$
		▯▮▯▮	$\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$
		▯▯▮▮	$\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$
		▯▮▮▯	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$
		▮▯▮▯	$\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$
		▮▮▯▯	$\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$
Trivectors / Antivectors	3 / 1	▯▮▮▮	$\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$
		▮▯▮▮	$\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$
		▮▮▯▮	$\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$
		▮▮▮▯	$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$
Quadrivector / Antiscalar	4 / 0	▮▮▮▮	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$

Table 2.3. These are the 16 basis elements of the 4D projective exterior algebra.

vector basis elements named $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$. A general vector $\mathbf{v} = (x, y, z, w)$ has the form

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4. \tag{2.15}$$

There are six bivector basis elements named $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, $\mathbf{e}_{43}$, $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, and $\mathbf{e}_{12}$, where we have purposely chosen the orders in which the subscripts appear so the components have the specific geometric meaning described in Section 2.4.2 below. Whenever a wedge product would result in basis vectors being multiplied in the opposite order, the term is negated so that basis elements can always be written exactly as they appear in the table. Next, there are four trivector basis elements named $\mathbf{e}_{423}$, $\mathbf{e}_{431}$, $\mathbf{e}_{412}$, and $\mathbf{e}_{321}$, where we have again purposely chosen those specific orders for the subscripts. These basis elements will always be written exactly as shown in the table, and negation will be applied for any odd permutation of the vector multiplication order. Finally, there is a single quadrivector basis element $\mathbf{e}_{1234}$, and this is the unit volume element $\mathbb{1}$. The full multiplication table for the wedge product with these 16 basis elements is shown in Table 2.4.

As shown by the solid and hollow bars in Table 2.3, each of the basis elements can be identified by which specific multiplicative combination of the four available dimensions it represents. This is essentially a four-bit code in which solid bars correspond to the dimensions that are present or *full*, and hollow bars correspond to the dimensions that are absent or *empty*. The grade of a basis element is the number of solid bars it has, which is the same as the number of vector basis elements in its factorization. A new term called the *antigrade* of a basis element corresponds to the number of hollow bars that is has, which is the same as the number of vector basis elements that do *not* participate in its factorization. For any element $\mathbf{u}$ that does not have mixed grade, we define the function $\operatorname{gr}(\mathbf{u})$ to be the grade of $\mathbf{u}$ and the function $\operatorname{ag}(\mathbf{u})$ to be the antigrade of $\mathbf{u}$. Of course, it is always the case that

$$\operatorname{gr}(\mathbf{u}) + \operatorname{ag}(\mathbf{u}) = n, \tag{2.16}$$

where n is the total number of dimensions in the algebra. Using the grade function, we can write things like

$$\boxed{\operatorname{gr}(\mathbf{a} \wedge \mathbf{b}) = \operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}),} \tag{2.17}$$

which states that the grade of the wedge product between two quantities $\mathbf{a}$ and $\mathbf{b}$ is the sum of the grades of $\mathbf{a}$ and $\mathbf{b}$. In order to handle the special case that $\operatorname{gr}(\mathbf{a}) > 0$ and $\operatorname{gr}(\mathbf{b}) > 0$, but $\mathbf{a} \wedge \mathbf{b} = 0$, which happens when $\mathbf{a}$ and $\mathbf{b}$ are parallel, we leave $\operatorname{gr}(0)$ undefined and add the condition that Equation (2.17) only holds when $\mathbf{a} \wedge \mathbf{b} \neq 0$.

We mentioned earlier that anticommutativity applied to vectors under the wedge product, but not generally to other kinds of elements in the algebra. The grade and antigrade functions usually appear in the exponent of a factor of -1 in order to express how sign changes occur. For example, we can succinctly express the condition under which two quantities $\mathbf{a}$ and $\mathbf{b}$ commute as

$$\boxed{\mathbf{a} \wedge \mathbf{b} = (-1)^{\operatorname{gr}(\mathbf{a})\operatorname{gr}(\mathbf{b})}\, \mathbf{b} \wedge \mathbf{a}.} \tag{2.18}$$

This means that if either $\operatorname{gr}(\mathbf{a})$ or $\operatorname{gr}(\mathbf{b})$ is even, then $\mathbf{a}$ and $\mathbf{b}$ commute under the wedge product. Otherwise, when both grades are odd, they anticommute. The case in which $\mathbf{a}$ and $\mathbf{b}$ commute can be understood by considering the factor with an even grade as the wedge product of an even number of basis vectors. The other factor can be moved from one side of the product to the other by making an even number of transpositions and multiplying by -1 for each one, resulting in no overall change in sign.

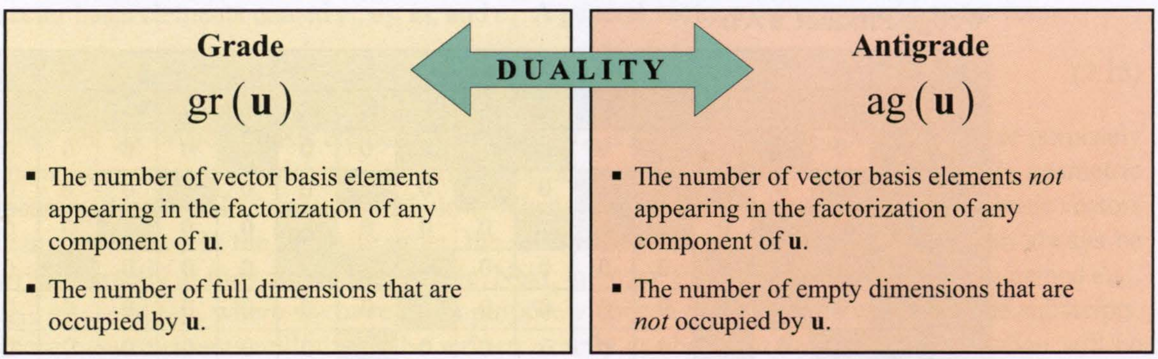
Wedge Product $\mathbf{a} \wedge \mathbf{b}$

$\mathbf{a}\backslash\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0	0
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	0	$-\mathbf{e}_{321}$	0	0	$\mathbb{1}$	0	0
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	0	0	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2.4. This is the multiplication table for the wedge product between the 16 basis elements in the 4D projective exterior algebra representing 3D Euclidean space.

For a thorough understanding of the algebraic structure and the geometric interpretations we can make, it is critically important to recognize that there is a fundamental symmetry at work. We have assigned a dimensionality to each basis element according to the number of full dimensions it occupies, but it is equally valid to assign a dimensionality according to the number of empty dimensions each one does not occupy. Vectors, bivectors, and trivectors have dimensions one, two, and three when we count the solid bars in Table 2.3. However, from the opposite perspective, vectors, bivectors, and trivectors have dimensions three, two, and one if we count the hollow bars instead. Both of these interpretations are simultaneously correct, and this pair of perspectives can be regarded as the source of the duality that is always present in an exterior algebra. Duality can be found not only in the elements of the algebra but also in the operations that act on those elements.

We have already discussed the duality between scalars and antiscalars. It should be rather obvious that for each type of basis element with k full dimensions, there is another type of basis element with k empty dimensions. (In even numbers of dimensions n, these are the same when $k = n/2$.) We can say that scalars fill no dimensions of space, but antiscalars fill all dimensions of space. A similar pairing exists for basis elements of all grades, and we could introduce special terms for all of them, but we limit ourselves to just one more. In n dimensions, we call an element having grade $n-1$ an *antivector*. Whereas a vector occupies one dimension of space, an antivector occupies all except one dimension of space. Vectors have grade one, and antivectors have antigrade one. They are opposites of each other and stand on equal ground with perfect symmetry. Because vectors and antivectors have the same numbers of components, a clear distinction between the two has not always been understood, historically speaking, and both types have been treated as the same kind of mathematical entity.[2] However, it has not gone unnoticed that the two types of vectors have different properties. In particular, antivectors transform from one coordinate system to another in a manner different from vectors, and this is highlighted in Section 2.7. In Chapter 1, we saw that 3D normal vectors and 4D planes transform in a special way, and the reason is that they are actually antivectors in three and four dimensions.



2.2 Complements

The 2^n basis elements in an n-dimensional exterior algebra correspond to all possible combinations of the n basis vectors. Each basis element $\mathbf{u}$ represents the wedge product of a unique subset S of the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, and those are what the solid bars symbolize in Table 2.3 for the four-dimensional case. There is always another basis element that represents the wedge product of all the basis vectors that are *not* in S and are thus not part of $\mathbf{u}$. We call this a *complement* of $\mathbf{u}$. The

[2] What we are now calling antivectors have been called by several different names in the past, and *pseudovector* is among them. As with pseudoscalars, we prefer the term antivector to highlight an equal symmetry with vectors and to be consistent with the entire body of terminology developed in this chapter.

binomial coefficients that make up Pascal's triangle exhibit a natural symmetry because the number of ways that we can choose k items from the set S is exactly equal to the number of ways that we can choose *all except k* items. When we take a complement, we are turning a k-dimensional element into an $(n-k)$-dimensional element by essentially inverting the spatial dimensions that are involved. In Table 2.3, this is equivalent to inverting the bars for any basis element so that solid becomes hollow and hollow becomes solid.

We define a complement operation by exchanging full and empty dimensions and determining what sign the result should have by enforcing specific multiplication rules. There are two ways this can be done called the *right complement* and *left complement*. The right complement of a basis element $\mathbf{u}$ is denoted by $\overline{\mathbf{u}}$, with a horizontal bar above it, and defined such that

Right complement

$$\boxed{\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1}.} \tag{2.19}$$

That is, the right complement of $\mathbf{u}$ must be the quantity the produces the volume element $\mathbb{1}$ when it appears as the right operand in a wedge product with $\mathbf{u}$. Similarly, the left complement of a basis element $\mathbf{u}$ is denoted by $\underline{\mathbf{u}}$, with a horizontal bar below it, and defined such that

Left complement

$$\boxed{\underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}.} \tag{2.20}$$

The left complement of $\mathbf{u}$ produces the volume element $\mathbb{1}$ when it appears as the left operand in a wedge product with $\mathbf{u}$.

The definitions of right and left complement depend on the orientation chosen for the volume element $\mathbb{1}$. If the orientation of the volume element were to be flipped, then the sign of both the right and left complement of every basis element $\mathbf{u}$ except $\mathbf{1}$ and $\mathbb{1}$ would also flip. However, because $\mathbf{1} \wedge \mathbb{1} = \mathbb{1}$ and $\mathbb{1} \wedge \mathbf{1} = \mathbb{1}$, both complements of $\mathbf{1}$ are always $\mathbb{1}$, and both complements of $\mathbb{1}$ are always $\mathbf{1}$.

The complements of all 16 basis elements in the 4D exterior algebra used throughout this chapter are listed in Table 2.5. Notice that the right and left complements differ in sign only for the basis elements having odd grade. This is due to the fact that $\mathbf{u} \wedge \overline{\mathbf{u}} = -\overline{\mathbf{u}} \wedge \mathbf{u}$ only when both $\operatorname{gr}(\mathbf{u})$ and $\operatorname{gr}(\overline{\mathbf{u}})$ are odd according to Equation (2.18). In general, right and left complements have different signs when the dimensionality n of the whole space is even and the grade of $\mathbf{u}$ is odd. Otherwise, right and left complements are equal to each other. We can summarize this relationship with the equation

$$\underline{\mathbf{u}} = (-1)^{\operatorname{gr}(\mathbf{u})\operatorname{ag}(\mathbf{u})}\,\overline{\mathbf{u}}, \tag{2.21}$$

where we have made use of the fact that $\operatorname{gr}(\overline{\mathbf{u}}) = \operatorname{ag}(\mathbf{u})$. This equation gives us a way to raise or lower the horizontal bar in order to switch between the right and left complement whenever it would be convenient. For an element $\mathbf{u}$ having k vector factors, which means its complement $\overline{\mathbf{u}}$ has $n-k$ vector factors, Equation (2.21) works because transforming $\mathbf{u} \wedge \overline{\mathbf{u}}$ into $\overline{\mathbf{u}} \wedge \mathbf{u}$ amounts to moving each of the $n-k$ factors of $\overline{\mathbf{u}}$ left through all k factors of $\mathbf{u}$, so there are a total of $k(n-k)$ transpositions that each negate the whole product. Those negations are undone by applying them to $\overline{\mathbf{u}}$ in

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$
$\underline{\mathbf{u}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{1}$

Table 2.5. For each of the 16 basis elements $\mathbf{u}$ in the 4D projective exterior algebra, this table lists the right complement $\overline{\mathbf{u}}$ and left complement $\underline{\mathbf{u}}$ with respect to the volume element $\mathbb{1} = \mathbf{e}_{1234}$.

order to obtain $\underline{\mathbf{u}}$. When n is even, whether $\overline{\mathbf{u}}$ and $\underline{\mathbf{u}}$ differ in sign ultimately depends only on $\operatorname{gr}(\mathbf{u})$ being even or odd.

In odd numbers of dimensions, one of $\operatorname{gr}(\mathbf{u})$ or $\operatorname{ag}(\mathbf{u})$ must be even for any basis element $\mathbf{u}$, and Equation (2.21) tells us that there is no difference between right and left complements. In those cases, we simply call $\overline{\mathbf{u}}$ the "complement" of $\mathbf{u}$ without stating right or left. Throughout most of this chapter, the right and left complements are different because we are modeling 3D flat geometry in a 4D projective space that has an even number of dimensions. However, there is only one complement later in Section 2.14 where we model flat 2D geometry in a 3D projective space. In Chapters 4 and 5, we model round 3D geometry in a 5D doubly projective space, and there is only one complement in that setting as well.

The right and left complement operations are inverses of each other, as expressed by

$$\underline{\overline{\mathbf{u}}} = \mathbf{u}. \tag{2.22}$$

This is true for any basis element $\mathbf{u}$ of any grade in any number of dimensions, and it doesn't matter in which order the two complements are applied, so the notation is intentionally ambiguous about whether the right or left complement is taken first. This allows us to do things like take the right or left complement of both sides of Equation (2.21) in order to eliminate the complement operation on one side. When we do this, we find that

$$\overline{\overline{\mathbf{u}}} = \underline{\underline{\mathbf{u}}} = (-1)^{\operatorname{gr}(\mathbf{u})\operatorname{ag}(\mathbf{u})}\,\mathbf{u}. \tag{2.23}$$

Values of $\mathbf{u}$ for which the right and left complements have different signs also have the property that they change sign when either complement is applied twice.

So far, we have defined complements only for basis elements. We extend the complement operation to all elements of an exterior algebra by simply requiring that it is a linear operation. That is, for any scalar s and basis elements $\mathbf{a}$ and $\mathbf{b}$, we have

$$\overline{s\mathbf{a}} = s\,\overline{\mathbf{a}} \quad \text{and} \quad \overline{\mathbf{a}+\mathbf{b}} = \overline{\mathbf{a}} + \overline{\mathbf{b}}. \tag{2.24}$$

For example, if we apply these rules to an arbitrary vector $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4$, then we can calculate its right complement as $\overline{\mathbf{v}} = x\mathbf{e}_{423} + y\mathbf{e}_{431} + z\mathbf{e}_{412} + w\mathbf{e}_{321}$. We can think of this as a reinterpretation of the original components of $\mathbf{v}$ in terms of the complementary basis elements, a notion that will be discussed further in Section 2.6.

Math Library Notes

- The right complement and left complement operations are implemented by the `RightComplement()` and `LeftComplement()` functions.

2.3 Antiproducts

The wedge product $\mathbf{a} \wedge \mathbf{b}$ combines the dimensions that are present in the factors $\mathbf{a}$ and $\mathbf{b}$. Referring once again to Table 2.3, the wedge product between any two basis elements $\mathbf{a}$ and $\mathbf{b}$ that don't have any vector factors in common produces a result containing all of the factors from both $\mathbf{a}$ and $\mathbf{b}$. The solid bars represent the vector factors that are present, so the set of solid bars corresponding to $\mathbf{a} \wedge \mathbf{b}$ is the union of the sets of solid bars contributed separately by $\mathbf{a}$ and $\mathbf{b}$. It is possible to define a different operation called the *antiwedge product* that is dual to the wedge product and takes the union of the *hollow* bars representing the vector factors that are *absent* in exactly the same way. The antiwedge product is denoted by $\mathbf{a} \vee \mathbf{b}$, with a downward pointing wedge, and we read it as "**a** antiwedge **b**". The full multiplication table for the antiwedge product with the 16 basis elements of the 4D projective algebra is shown in Table 2.6. This is our first encounter with the general concept of an *antiproduct*, which can always be constructed by a simple procedure from a given product. Because the wedge product is also known as the exterior product, the antiwedge product is also known as the *exterior antiproduct*.

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

$\mathbf{a}$ \ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	0	$\mathbf{-1}$	0	0	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	0	$\mathbf{-1}$	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	0	$\mathbf{-1}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	0	$\mathbf{-1}$	0	0	0	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	0	$\mathbf{-1}$	0	0	0	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	$\mathbf{-1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	0	$\mathbf{-1}$	0	0	0	$-\mathbf{e}_4$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	$\mathbf{-1}$	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	$\mathbf{-1}$	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$\mathbf{-1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Table 2.6. This is the multiplication table for the antiwedge product in the 4D projective algebra representing 3D Euclidean space.

Since the complement operations exchange the meanings of full and empty dimensions, we can use them to give an explicit definition of the antiwedge product in terms of the wedge product. To calculate the antiwedge product $\mathbf{a} \vee \mathbf{b}$, we first take either the left or right complement of both **a** and **b**, then calculate the wedge product of those complements, and finally take the opposite complement of the result to undo the complements that were taken to begin with. If we choose to take the right complements of the operands, then this gives us the definition

Antiwedge product

$$\boxed{\mathbf{a} \vee \mathbf{b} = \underline{\overline{\mathbf{a}} \wedge \overline{\mathbf{b}}}.} \tag{2.25}$$

The result is the same if we instead take the left complements of the operands and then the right complement of the wedge product. Either definition is fine as long as the complement operations are opposites of each other.

If we take the right complement of both sides of Equation (2.25) and similarly take the left complement of both sides in the case that we had instead chosen to take left complements of the operands **a** and **b**, then we can write both of the relationships

$$\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}} \quad \text{and} \quad \underline{\mathbf{a} \vee \mathbf{b}} = \underline{\mathbf{a}} \wedge \underline{\mathbf{b}}. \tag{2.26}$$

These correspond to De Morgan's laws from logic and set theory.[3] They also work in reverse when we exchange the wedge and antiwedge products to give us two more relationships

$$\overline{\mathbf{a} \wedge \mathbf{b}} = \overline{\mathbf{a}} \vee \overline{\mathbf{b}} \quad \text{and} \quad \underline{\mathbf{a} \wedge \mathbf{b}} = \underline{\mathbf{a}} \vee \underline{\mathbf{b}}. \tag{2.27}$$

It's a simple matter to transform Equations (2.26) and (2.27) into each other by replacing **a** and **b** by their left complements in the first relationship and by their right complements in the second relationship.

Just as we have the shorthand notation $\mathbf{e}_{ab}$ for the wedge product $\mathbf{e}_a \wedge \mathbf{e}_b$ between vectors, we can use a complement to make a shorthand notation for the antiwedge product between antivectors. When we write $\overline{\mathbf{e}}_{ab}$, the meaning is $\overline{\mathbf{e}}_a \vee \overline{\mathbf{e}}_b$, which is the antiwedge product between the right complements of $\mathbf{e}_a$ and $\mathbf{e}_b$. Due to Equation (2.27), we can also interpret $\overline{\mathbf{e}}_{ab}$ as the right complement of the wedge product $\mathbf{e}_a \wedge \mathbf{e}_b$, and it has exactly the same meaning. The same is true for left complements. The notation $\underline{\mathbf{e}}_{ab}$ means both the antiwedge product of left complements $\underline{\mathbf{e}}_a \vee \underline{\mathbf{e}}_b$ and the left complement of the wedge product $\mathbf{e}_a \wedge \mathbf{e}_b$.

The antiwedge product operates on empty dimensions in exactly the same way that the wedge product operates on full dimensions. The wedge product and antiwedge product are analogous to the union and intersection of spatial dimensions, and this feature is put to practical use in Section 2.5 below. Compared to the wedge product, the roles of the scalar unit **1** and the antiscalar unit $\mathbb{1}$ are reversed under the antiwedge product. In particular, $\mathbb{1}$ is the multiplicative identity for the antiwedge product such that $\mathbf{u} \vee \mathbb{1} = \mathbb{1} \vee \mathbf{u} = \mathbf{u}$ for any value **u**. Whereas the antiscalar unit $\mathbb{1}$ corresponds to the wedge product of all basis vectors, the scalar unit **1** corresponds to the antiwedge product of all basis antivectors. In the 4D exterior algebra, we can thus write

$$\mathbf{1} = \overline{\mathbf{e}}_1 \vee \overline{\mathbf{e}}_2 \vee \overline{\mathbf{e}}_3 \vee \overline{\mathbf{e}}_4. \tag{2.28}$$

This highlights how the roles of scalars and antiscalars, vectors and antivectors, and every other pair of k-vectors and $(n-k)$-vectors are reversed when we switch between the wedge product and antiwedge product. In general, for any specific manner in which the wedge product operates on elements of grades k and m, the antiwedge product operates on elements of antigrades k and m in a symmetric manner. In any case that the results are nonzero, the wedge product adds the grades of its operands, and the antiwedge product adds the antigrades of its operands. Thus, the antiwedge analog of Equation (2.17) is

$$\boxed{\operatorname{ag}(\mathbf{a} \vee \mathbf{b}) = \operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b}).} \tag{2.29}$$

To see what the effect on grades is, we can subtract each antigrade from the dimension n of the algebra and simplify to get

$$\operatorname{gr}(\mathbf{a} \vee \mathbf{b}) = \operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) - n. \tag{2.30}$$

Just as the wedge product is zero whenever the grade given by Equation (2.17) would be greater than n, the antiwedge product is zero whenever the antigrade given by Equation (2.29) would be greater than n. Equivalently, the antiwedge product is zero whenever the grade given by Equation (2.30) would be less than zero.

In a manner symmetric to the wedge product between vectors, the antiwedge product between antivectors is anticommutative. For any antivectors **a** and **b**, this means that

[3] Unfortunately, the established meanings of the symbols $\wedge$ and $\vee$ in exterior algebra are opposite to the meanings of the same symbols when used for the AND and OR operations in logic or the similar symbols $\cap$ and $\cup$ when used for intersection and union operations in set theory.

$$\boxed{\mathbf{a} \vee \mathbf{b} = -\mathbf{b} \vee \mathbf{a},} \tag{2.31}$$

and whenever an antivector $\mathbf{v}$ is multiplied by itself, we must have $\mathbf{v} \vee \mathbf{v} = 0$. The general rule for the antiwedge product between any quantities $\mathbf{a}$ and $\mathbf{b}$ is

$$\boxed{\mathbf{a} \vee \mathbf{b} = (-1)^{\operatorname{ag}(\mathbf{a})\operatorname{ag}(\mathbf{b})}\, \mathbf{b} \vee \mathbf{a}.} \tag{2.32}$$

Comparing this to the rule for the wedge product given by Equation (2.18), we see that two elements commute under the antiwedge product precisely when their complements commute under the wedge product. Specifically, if either $\operatorname{ag}(\mathbf{a})$ or $\operatorname{ag}(\mathbf{b})$ is even, then $\mathbf{a} \vee \mathbf{b} = \mathbf{b} \vee \mathbf{a}$.

The procedure that we used to construct the antiwedge product from the wedge product can be used to construct a dual operation from *any* operation, and this is not limited to binary operations. In general, if we have some function f that takes one or more inputs and generates a single output, then we can construct the corresponding dual function f^* by taking either the right or left complement of all the inputs, applying the original function f, and then taking the opposite complement of the output. That is,

$$f^*(\mathbf{a},\mathbf{b},\mathbf{c},\ldots) = \underline{f(\overline{\mathbf{a}},\overline{\mathbf{b}},\overline{\mathbf{c}},\ldots)} = \overline{f(\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},\ldots)}. \tag{2.33}$$

For whatever operation f performs, f^* performs the corresponding *anti-operation*. Whenever we use the prefix anti- in the name of an operation, it means that it is related to the original operation, without the anti- prefix, in exactly the way stated by Equation (2.33). This relationship is symmetric, meaning that Equation (2.33) still holds if we can exchange f and f^*, so it makes sense to say that two functions related by Equation (2.33) are anti-operations of each other. In addition to the wedge product and antiwedge product, we will encounter several more unary and binary pairs of operations and anti-operations throughout this chapter and the next.

Wedge Product $\mathbf{a} \wedge \mathbf{b}$	DUALITY	**Antiwedge Product** $\mathbf{a} \vee \mathbf{b}$
▪ Combines the dimensions that are present in **a** and **b**.		▪ Combines the dimensions that are absent in **a** and **b**.
▪ Result has grade $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b})$.		▪ Result has grade $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) - n$.
▪ Result has antigrade $\operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b}) - n$.		▪ Result has antigrade $\operatorname{ag}(\mathbf{a}) + \operatorname{ag}(\mathbf{b})$.

2.4 3D Flat Geometry

The utility of the 4D projective exterior algebra lies mainly in its ability to represent 3D points, lines, and planes in a consistent manner and to perform fundamental geometric operations among them. The concepts of homogeneous coordinates, Plücker coordinates, and implicit planes that were discussed with separate formulations in Chapter 1 are now subsumed by a single mathematical framework in which each type of geometry simply corresponds to a different grade in the exterior algebra. In the next section, we show how these geometries can all be combined by using the wedge and antiwedge products to perform operations similar to union and intersection.

2.4.1 Points

In Section 1.2, we introduced the concept of homogeneous coordinates and explained how a 4D vector corresponds to a 3D point by means of its projection into the subspace where $w = 1$. Points work exactly the same way here in the 4D exterior algebra. A general 3D point $\mathbf{p}$ is written as

Point (3D)

$$\mathbf{p} = \underbrace{p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\mathbf{e}_4}_{\text{Weight}} \tag{2.34}$$

with the four coordinates p_x, p_y, p_z, and p_w assigned to the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$. The first three coordinates correspond to the weighted position of the point, and the fourth coordinate is the point's weight. When we divide all four coordinates by p_w to make the weight one, we are projecting the 4D vector $\mathbf{p}$ onto the 3D subspace where $w = 1$ as shown in Figure 2.4. As with conventional homogeneous coordinates, all nonzero scalar multiples of $\mathbf{p}$ correspond to the same 3D point, just with different weights.

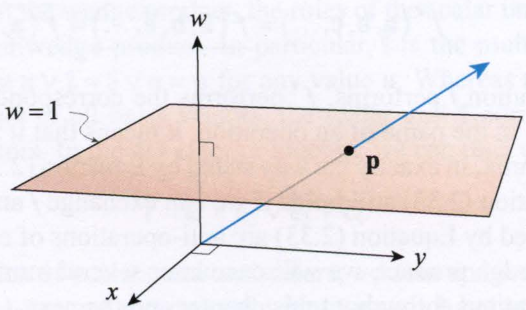


Figure 2.4. A 4D vector intersects the 3D subspace where $w = 1$ at the point $\mathbf{p}$. (The z axis is omitted from the figure, and it should be understood that the subspace for which $w = 1$ is not planar, but also extends in the z direction.)

If the p_x, p_y, and p_z coordinates are all zero, but p_w is still nonzero, then the vector $\mathbf{p} = p_w\mathbf{e}_4$ points straight up or down along the w axis in Figure 2.4, and it represents the weighted *origin* of 3D space. This gives the basis vector $\mathbf{e}_4$ a special geometric interpretation as the point at the origin with unit weight. In general, when we talk about the "origin" in an n-dimensional projective algebra corresponding to $(n-1)$-dimensional space, we are referring to the point $\mathbf{e}_n$.

If the weight p_w is zero, but at least one of the coordinates p_x, p_y, and p_z is nonzero, then the vector $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is parallel to the subspace $w = 1$ and cannot intersect it at any finite distance from the origin. In this case, $\mathbf{p}$ can be interpreted as the point at infinity in the particular direction (p_x, p_y, p_z). Because points are homogeneous and all nonzero scalar multiples of $\mathbf{p}$ are equivalent, it is always true that $\mathbf{p}$ and $-\mathbf{p}$ represent the same point in 3D space. This means that points at infinity in exactly opposite directions are equivalent as if space has a circular nature in every direction.

A point with zero weight can also be interpreted as a direction vector with no position, just as such 4D vectors were interpreted in conventional homogeneous coordinates. The difference $\mathbf{q} - \mathbf{p}$ between two points $\mathbf{p}$ and $\mathbf{q}$ with the same weight yields a direction vector pointing from $\mathbf{p}$ to $\mathbf{q}$ having a magnitude equal to the distance between $\mathbf{p}$ and $\mathbf{q}$ multiplied by their common weight. Adding a direction vector $\mathbf{v}$ to a point $\mathbf{p}$ yields a new point that has been offset from $\mathbf{p}$ by the magnitude of $\mathbf{v}$ divided by the weight of $\mathbf{p}$.

2.4.2 Lines

The wedge product between two nonparallel 4D vectors **p** and **q** generates the 4D bivector shown in Figure 2.5. This bivector is an oriented 2D span inside the 4D space, and it contains both of the 4D direction vectors **p** and **q**. We projected a 4D vector into 3D space by finding its intersection with the subspace where $w = 1$, and we do exactly the same thing to a 4D bivector to obtain the meaning of its projection into 3D space. As illustrated in the figure, the intersection of the bivector $\mathbf{p} \wedge \mathbf{q}$ with the subspace where $w = 1$ is a line that contains both of the points **p** and **q**.

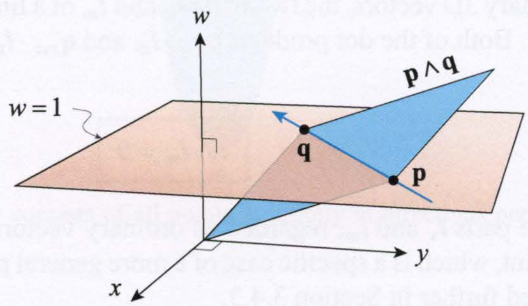


Figure 2.5. The 4D bivector $\mathbf{p} \wedge \mathbf{q}$ intersects the 3D subspace where $w = 1$ at the line determined by the homogeneous points **p** and **q**. (The z axis is omitted from the figure, and it should be understood that the subspace for which $w = 1$ is not planar, but also extends in the z direction.)

The wedge product of two arbitrary points $\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4$ and $\mathbf{q} = q_x \mathbf{e}_1 + q_y \mathbf{e}_2 + q_z \mathbf{e}_3 + q_w \mathbf{e}_4$ is given by

$$\begin{aligned} \mathbf{p} \wedge \mathbf{q} = {} & (q_x p_w - p_x q_w)\,\mathbf{e}_{41} + (q_y p_w - p_y q_w)\,\mathbf{e}_{42} + (q_z p_w - p_z q_w)\,\mathbf{e}_{43} \\ & + (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}. \end{aligned} \tag{2.35}$$

When the weights p_w and q_w are one, the six components of this bivector are precisely the Plücker coordinates that were introduced as the implicit form of a line in Section 1.3.2, but now these components emerge from the fundamental multiplication rules of the exterior algebra. The $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, and $\mathbf{e}_{43}$ components correspond to the difference between **p** and **q**, but they have been generalized a bit to allow for points with any weight. The $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, and $\mathbf{e}_{12}$ components contain the moment of the line given by the cross product between **p** and **q** without involving their weights at all. Recognizing that an arbitrary line $\boldsymbol{l}$ has two parts of three components each, we can write it as the bivector

Line (3D)

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}. \tag{2.36}$$

Here, the line's direction has been labeled l_{vx}, l_{vy}, and l_{vz}, where each two-letter subscript designates a specific coordinate. We sometimes use the notation $\boldsymbol{l}_{\mathbf{v}} = (l_{vx}, l_{vy}, l_{vz})$ to mean the direction of the line $\boldsymbol{l}$ interpreted as a generic 3D vector. Likewise, the line's moment has been labeled l_{mx}, l_{my}, and l_{mz}, and the notation $\boldsymbol{l}_{\mathbf{m}} = (l_{mx}, l_{my}, l_{mz})$ means the moment of the line $\boldsymbol{l}$ interpreted as a generic 3D vector. These labels assign meaning to the components of a bivector and allow us to continue regarding lines as having two three-dimensional parts that correspond to the $\{\mathbf{v} \mid \mathbf{m}\}$ notation used in Sections 1.3.2 and 1.3.7. As with points, lines are homogeneous, so multiplying all six components by the same nonzero scalar value does not change the geometric meaning of the bivector in Equation (2.36).

If one of the points $\mathbf{p}$ or $\mathbf{q}$ in Equation (2.35) lies at infinity because its weight is zero, then we still get a valid line. Assuming that $\mathbf{q}$ lies at infinity with $q_w = 0$, the line resulting from the wedge product $\mathbf{p} \wedge \mathbf{q}$ contains the point $\mathbf{p}$ and runs in the direction given by the x, y, and z components of $\mathbf{q}$. It would also be correct to say that the line contains $\mathbf{q}$, and that would be the point where the line intersects the set of all points at infinity.

A line is always a simple bivector because it can be expressed as the wedge product of two vectors representing points $\mathbf{p}$ and $\mathbf{q}$. In the 4D algebra, there are bivectors that are not simple, so not all bivectors correspond to valid lines in 3D space. Bivectors that do correspond to valid lines satisfy an internal constraint that becomes clear when we take a closer look at Equation (2.35). Interpreted as ordinary 3D vectors, the two parts $\boldsymbol{l}_\mathbf{v}$ and $\boldsymbol{l}_\mathbf{m}$ of a line are given by $\boldsymbol{l}_\mathbf{v} = p_w \mathbf{q}_{xyz} - q_w \mathbf{p}_{xyz}$ and $\boldsymbol{l}_\mathbf{m} = \mathbf{p}_{xyz} \times \mathbf{q}_{xyz}$. Both of the dot products $\mathbf{p}_{xyz} \cdot \boldsymbol{l}_\mathbf{m}$ and $\mathbf{q}_{xyz} \cdot \boldsymbol{l}_\mathbf{m}$ must be zero, so it is always true that the equation

Line constraint

$$\boxed{\boldsymbol{l}_\mathbf{v} \cdot \boldsymbol{l}_\mathbf{m} = 0} \tag{2.37}$$

is satisfied, and the parts $\boldsymbol{l}_\mathbf{v}$ and $\boldsymbol{l}_\mathbf{m}$, regarded as ordinary vectors, are always orthogonal. All lines satisfy this constraint, which is a specific case of a more general property possessed by all geometric objects, as discussed further in Section 3.4.3.

Even though all six components of a line are bivector values, our interpretation of the 4D algebra in which the $\mathbf{e}_4$ direction is eliminated when we project into 3D space causes the parts $\boldsymbol{l}_\mathbf{v}$ and $\boldsymbol{l}_\mathbf{m}$ to behave in different ways. The direction $\boldsymbol{l}_\mathbf{v}$ behaves like a vector or length-like quantity, and the moment $\boldsymbol{l}_\mathbf{m}$ behaves like a bivector or area-like quantity. This becomes apparent when we apply certain transformations to a line such as the scale example in Section 2.7 below. In general, any part of the representation of a geometric object behaves as if it has the dimensionality of its components after disregarding any factor of $\mathbf{e}_4$.

If the moment of a line is zero, meaning that $l_{mx} = l_{my} = l_{mz} = 0$, then the line passes through the origin. General distance to the origin is discussed in Section 2.10.3, but for now, we can say that the magnitude of the moment is the distance between the origin and the closest point on the line, but it is multiplied by the magnitude of the line's direction. This gives lines represented in the direction-moment form a practical advantage over the equivalent parametric form when it comes to precision. Whenever a line is subjected to one or more transformations, the three components making up the moment always have the same magnitude as the smallest magnitude possible for any particular point on the line. This can lead to better floating-point precision compared to the reference point in the parametric form because that point could end up being moved far from the origin even if the actual distance between the line and the origin remains small.

If the weight of a line is zero, meaning that $l_{vx} = l_{vy} = l_{vz} = 0$, then the entire line lies at infinity in all directions perpendicular to $\boldsymbol{l}_\mathbf{m}$, regarded as a vector, as shown in Figure 2.6. When the moment is regarded as a bivector $\mathbf{m} = l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$, a line at infinity can be thought of as containing points at infinity in all directions $\mathbf{v}$ parallel to the moment, which satisfy $\mathbf{m} \wedge \mathbf{v} = 0$. If we construct a line by taking the wedge product between two direction vectors with zero weights, then that line lies at infinity in the two-dimensional subspace spanned by the two vectors. Despite the intuition that a line at infinity is round because it encircles a specific direction, its infinite size means that it is actually flat everywhere.

2.4.3 Planes

When projected into 3D space, we've seen that a grade-one vector in the 4D exterior algebra corresponds to a zero-dimensional point, and we've seen that a grade-two bivector corresponds to a one-dimensional line. Continuing to the next higher grade of the exterior algebra, it should come as no surprise that a grade-three trivector corresponds to a two-dimensional plane. The triple wedge

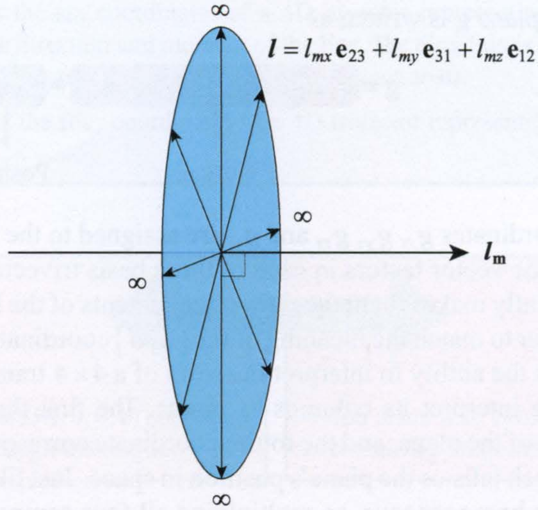


Figure 2.6. A line at infinity consists of all points at infinity in directions perpendicular to the moment $\boldsymbol{l}_{\mathbf{m}}$, regarded as a 3D vector.

product among three 4D vectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$, all pointing in different directions, generates the 4D trivector shown in Figure 2.7. As illustrated in the figure, the intersection of the trivector $\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$ with the subspace where $w = 1$ is a plane that contains all three of the points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. Keep in mind that the subspace onto which we're projecting is three-dimensional despite being drawn as a two-dimensional slice, so the plane can have any arbitrary orientation in space.

Writing the exact calculations for the four components of $\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$ gets a little messy. Things get much nicer when we realize that the first wedge product constructs a line, and the second wedge product multiplies that line by another point. This lets us express the calculations for a plane's components in terms of the components of a line $\boldsymbol{l}$ and point $\mathbf{p}$, from which we obtain

$$\begin{aligned} \boldsymbol{l} \wedge \mathbf{p} = {} & (l_{vy} p_z - l_{vz} p_y + l_{mx})\,\overline{\mathbf{e}}_1 + (l_{vz} p_x - l_{vx} p_z + l_{my})\,\overline{\mathbf{e}}_2 \\ & + (l_{vx} p_y - l_{vy} p_x + l_{mz})\,\overline{\mathbf{e}}_3 - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\overline{\mathbf{e}}_4. \end{aligned} \tag{2.38}$$

We have expressed the components in terms of the complements of the basis vectors to highlight the fact that a plane is a 4D antivector, the significance of which is discussed further in Section 2.6. Because $\boldsymbol{l}$ has an even grade, it commutes with everything under the wedge product according to Equation (2.18), so it is always true that $\boldsymbol{l} \wedge \mathbf{p} = \mathbf{p} \wedge \boldsymbol{l}$, and it doesn't matter in which order a line and point are multiplied together.

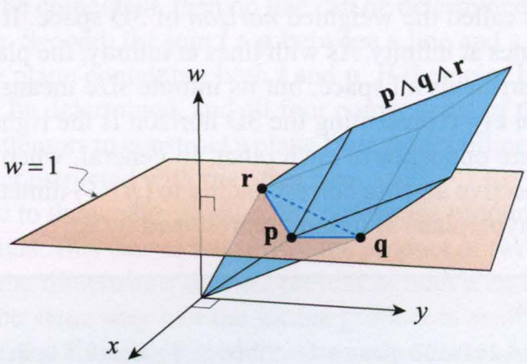


Figure 2.7. The 4D trivector $\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$ intersects the 3D subspace where $w = 1$ at the plane determined by the homogeneous points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. (The z axis is omitted from the figure, and it should be understood that the subspace for which $w = 1$ is not planar, but also extends in the z direction.)

A general 3D plane $\mathbf{g}$ is written as

Plane (3D)

$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w \mathbf{e}_{321}}_{\text{Position}}, \tag{2.39}$$

where the four coordinates g_x, g_y, g_z, and g_w are assigned to the basis trivectors $\mathbf{e}_{423}$, $\mathbf{e}_{431}$, $\mathbf{e}_{412}$, and $\mathbf{e}_{321}$. The order of the vector factors in each of these basis trivectors, in particular $\mathbf{e}_{321}$, were chosen because it consistently makes them the right complements of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$. This is necessary in order to match the meaning of the $[\,\mathbf{n} \mid d\,]$ coordinates used in Sections 1.3.2 and 1.3.7 as well as to retain the ability to interpret the rows of a 4×4 transformation matrix as planes at the same time that we interpret its columns as points. The first three coordinates correspond to the normal direction $\mathbf{n}$ of the plane, and the fourth coordinate corresponds to the plane's signed distance d to the origin, which tells us the plane's position in space. Just like everything else in the projective algebra, planes are homogeneous, so multiplying all four components by the same nonzero scalar value does not change the geometric meaning of the trivector in Equation (2.39).

When the w coordinates of the points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are all one, the normal direction and distance to origin for the plane containing all three points can be calculated with the formulas

$$\begin{aligned} \mathbf{g}_{xyz} &= \mathbf{n} = \mathbf{p}_{xyz} \wedge \mathbf{q}_{xyz} + \mathbf{q}_{xyz} \wedge \mathbf{r}_{xyz} + \mathbf{r}_{xyz} \wedge \mathbf{p}_{xyz} \\ g_w &= d = -\mathbf{p}_{xyz} \wedge \mathbf{q}_{xyz} \wedge \mathbf{r}_{xyz}, \end{aligned} \tag{2.40}$$

where these wedge products occur in three dimensions, and the complement is taken with respect to the volume element $\mathbf{e}_{123}$. Both pieces of Equation (2.40) possess an elegant three-way symmetry, but neither those nor the formula in Equation (2.38) represents the most computationally efficient way of constructing a plane if we are starting with three points. The conventional method in which the normal is calculated by first subtracting one point from the other two and then taking a cross product is still the best. In this case, the basic operations for calculating a plane containing the points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are given by

$$\begin{aligned} \mathbf{g}_{xyz} &= \mathbf{n} = (\mathbf{q}_{xyz} - \mathbf{p}_{xyz}) \times (\mathbf{r}_{xyz} - \mathbf{p}_{xyz}) \\ g_w &= d = -\mathbf{n} \cdot \mathbf{p}_{xyz} = -\mathbf{n} \cdot \mathbf{q}_{xyz} = -\mathbf{n} \cdot \mathbf{r}_{xyz}, \end{aligned} \tag{2.41}$$

where any one of the three points may be used in the calculation of the distance d.

Of course, if a plane's w coordinate is zero, then the plane passes through the origin because its distance to the origin is zero. If a plane's normal direction is zero, leaving only the component $g_w \mathbf{e}_{321}$, then it is given the special geometric interpretation of the plane lying at infinity in all directions. This plane is called the weighted *horizon* of 3D space. It contains all points at infinity and, consequently, all lines at infinity. As with lines at infinity, the plane at infinity may seem intuitively round because it surrounds all space, but its infinite size means that it is actually flat everywhere. The basis antivector $\mathbf{e}_{321}$ representing the 3D horizon is the right complement of the origin $\mathbf{e}_4$, and these two entities are opposites of each other. In general, when we talk about the "horizon" in an n-dimensional projective algebra corresponding to $(n-1)$-dimensional Euclidean space, we are referring to the unit hyperplane at infinity represented by $\overline{\mathbf{e}}_n$.

Math Library Notes

- The `Vector3D` and `Point3D` classes described in Section 1.2 both store only the x, y, and z coordinates of a homogeneous 4D vector. The w coordinate is implicitly zero for a `Vector3D` object, and it is implicitly one for a `Point3D` object. There is also a `FlatPoint3D` class that stores all four coordinates explicitly.

- The `Line3D` class stores the six coordinates of a 4D bivector representing a flat line. There are two members that contain the direction and moment of the line. The direction is named `v`, and it has the type `Vector3D`. The moment is named `m`, and it has the type `Bivector3D`.
- The `Plane3D` class stores the four coordinates of a 4D trivector representing a flat plane, and they are named `x`, `y`, `z`, and `w`.

Origin	**DUALITY**	**Horizon**
$\mathbf{e}_4$		$\overline{\mathbf{e}}_4$
▪ The point with coordinates $(0,0,0,1)$ contained by all planes passing through the origin.		▪ The plane with coordinates $(0,0,0,1)$ containing all points at infinity.

2.5 Join and Meet

In the previous section, we built homogeneous representations of lines and planes in 3D space by taking wedge products of points that they contain. This ability to take lower-dimensional geometries and combine them into higher-dimensional geometries is an exquisitely straightforward and natural consequence of the way in which the wedge product combines the dimensions that are present in its operands. The way in which we could build geometries is similar to a union operation, but it's not quite the same because we must always obtain a result that has higher grade. This excludes situations such as a line being multiplied by a point contained in the same line. There is not enough information to construct a unique plane containing the line, but we can't just keep the original line that would result from a true union, either. Instead, the wedge product detects a linear dependence and gives us zero in those cases.

The operation of combining geometric objects **a** and **b** with the wedge product is called the *join* of **a** and **b**, and we write its definition as

Join

$$\boxed{\operatorname{join}(\mathbf{a},\mathbf{b}) = \mathbf{a} \wedge \mathbf{b}.} \tag{2.42}$$

We have already encountered the two types of join operations that arise in 3D space, and they are summarized again with illustrations in Table 2.7. First, the join $\mathbf{p} \wedge \mathbf{q}$ between two points represented by 4D vectors is given by Equation (2.35), and it produces the line containing $\mathbf{p}$ and $\mathbf{q}$. If the those points happen to be coincident, then no line can be determined, and all six components of the bivector result are zero. Second, the join $\boldsymbol{l} \wedge \mathbf{p}$ between a line and a point is given by Equation (2.38), and it produces the plane containing both $\boldsymbol{l}$ and $\mathbf{p}$. If the point happens to be contained by the line, then no plane can be determined, and all four components of the trivector result are zero. This degeneracy includes attempts to construct a plane from three collinear points because one point will always lie on the line constructed with the other two.

In a manner symmetric to the wedge product, the antiwedge product combines the dimensions that are absent in its operands. This causes the antiwedge product of two geometric objects **a** and **b** to retain in its result only the dimensions that are present in both **a** and **b**, so it is analogous to an intersection operation in the same way that the wedge product is analogous to a union operation. As with the wedge product, the antiwedge product does not perform an actual intersection because it must construct a geometric object of lower grade than one of its operands. If one object is contained in the other, then the antiwedge product gives us zero in cases where a true intersection would give us the larger object.

The operation of combining geometric objects **a** and **b** with the antiwedge product is called the *meet* of **a** and **b**, and we write its definition as

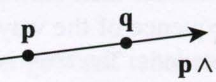
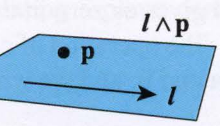
Meet

$$\boxed{\operatorname{meet}(\mathbf{a},\mathbf{b}) = \mathbf{a} \vee \mathbf{b}.} \tag{2.43}$$

There are two types of meet operations that arise in 3D space, and they are summarized in Table 2.7 along with the join operations. First, we can calculate the meet of two planes **g** and **h** represented by 4D trivectors with the antiwedge product

$$\begin{aligned}\mathbf{g} \vee \mathbf{h} &= (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43} \\ &\quad + (g_x h_w - g_w h_x)\,\mathbf{e}_{23} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}.\end{aligned} \tag{2.44}$$

This gives us the 4D bivector representing the line where the two planes **g** and **h** intersect. If the two planes happen to be coincident, then there is no unique line where they meet, and all six components of the bivector result are zero. However, if the two planes are parallel but not coincident,

Join Operation	**Illustration**
Line containing points **p** and **q**. $\begin{aligned}\mathbf{p} \wedge \mathbf{q} &= (p_w q_x - p_x q_w)\,\mathbf{e}_{41} + (p_y q_z - p_z q_y)\,\mathbf{e}_{23} \\ &\quad + (p_w q_y - p_y q_w)\,\mathbf{e}_{42} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} \\ &\quad + (p_w q_z - p_z q_w)\,\mathbf{e}_{43} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}\end{aligned}$	
Plane containing line ***l*** and point **p**. $\begin{aligned}\boldsymbol{l} \wedge \mathbf{p} &= (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\,\mathbf{e}_{423} \\ &\quad + (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\,\mathbf{e}_{431} \\ &\quad + (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\,\mathbf{e}_{412} \\ &\quad - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}\end{aligned}$	

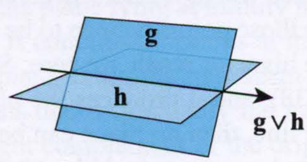
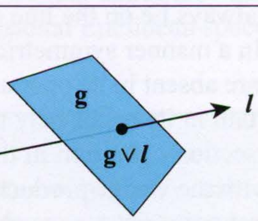
Meet Operation	**Illustration**
Line where planes **g** and **h** intersect. $\begin{aligned}\mathbf{g} \vee \mathbf{h} &= (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_w - g_w h_x)\,\mathbf{e}_{23} \\ &\quad + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} \\ &\quad + (g_y h_x - g_x h_y)\,\mathbf{e}_{43} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}\end{aligned}$	
Point where plane **g** and line ***l*** intersect. $\begin{aligned}\mathbf{g} \vee \boldsymbol{l} &= (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_1 \\ &\quad + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_2 \\ &\quad + (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_3 \\ &\quad - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_4\end{aligned}$	

Table 2.7. These are the join and meet operations among points, lines, and planes in three dimensions.

we still obtain a meaningful result. In this case, the intersection of the two planes is the line $\boldsymbol{l}$ at infinity in all directions parallel to the planes. The direction part of $\boldsymbol{l}$ is zero, but its moment part is a nonzero 3D bivector whose complement is a vector perpendicular to the planes.

The second type of meet operation is the intersection of a plane $\mathbf{g}$ and a line $\boldsymbol{l}$. The antiwedge product between these is given by

$$\begin{aligned}\mathbf{g} \vee \boldsymbol{l} = &(l_{my} g_z - l_{mz} g_y + l_{vx} g_w)\,\mathbf{e}_1 + (l_{mz} g_x - l_{mx} g_z + l_{vy} g_w)\,\mathbf{e}_2 \\ &+ (l_{mx} g_y - l_{my} g_x + l_{vz} g_w)\,\mathbf{e}_3 - (l_{vx} g_x + l_{vy} g_y + l_{vz} g_z)\,\mathbf{e}_4,\end{aligned} \tag{2.45}$$

and the resulting vector corresponds to the homogeneous point where the the line $\boldsymbol{l}$ passes through the plane $\mathbf{g}$. It's generally not the case that the w coordinate of the point produced by the meet operation is one, so it's necessary to divide by w to obtain a projected 3D point. As with the wedge product between a point and a line, the antiwedge product between a plane and a line is commutative (but this time because $\boldsymbol{l}$ has an even antigrade), so it is always true that $\boldsymbol{l} \vee \mathbf{g} = \mathbf{g} \vee \boldsymbol{l}$, and it doesn't matter in which order a plane and line are multiplied together. In the case that the line lies in the plane, there is no unique point of intersection, and all four components of the vector result are zero. However, if the line does not lie in the plane but is still parallel to the plane, a meaningful result arises just as it did for the meet of two parallel planes. In this case, the resulting intersection of the plane and line is the point at infinity in the direction of the line. The w coordinate of this point is zero, but its x, y and z coordinates point in the same direction as the direction part $\boldsymbol{l}_{\mathbf{v}}$ of the line.

A combination that is noticeably absent from Table 2.7 is the join or meet of two lines. Because lines have grade two, the wedge product between two lines must be an antiscalar, and the antiwedge product between two lines must be a scalar. In 3D space, two lines do not intersect unless they happen to lie in a common plane, though we will see that skew lines actually intersect at infinity in Chapter 4. The single number that we get from either the join or meet of any two lines contains information about the distance between the two lines, as discussed below in Section 2.11. For two lines that do happen to be coplanar, this number is zero, and for lines that are not coplanar, the sign of this number tells us something about their crossing orientation.

The *crossing orientation* of two lines, illustrated in Figure 2.8, is a property of the spatial relationship between the lines that expresses the direction in which one line passes by the other. It is determined by calculating the antiwedge product $\boldsymbol{l} \vee \mathbf{k}$ between two lines $\boldsymbol{l}$ and $\mathbf{k}$, given by

$$\boldsymbol{l} \vee \mathbf{k} = -l_{vx} k_{mx} - l_{vy} k_{my} - l_{vz} k_{mz} - l_{mx} k_{vx} - l_{my} k_{vy} - l_{mz} k_{vz}, \tag{2.46}$$

which we could also write as the expression $\boldsymbol{l} \vee \mathbf{k} = -\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{m}} - \boldsymbol{l}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{v}}$ consisting of two dot products. The wedge product $\boldsymbol{l} \wedge \mathbf{k}$ yields the same numerical value, but as an antiscalar instead of a scalar. As shown in the figure, if $\boldsymbol{l} \vee \mathbf{k}$ is positive, then the direction of each line is wound clockwise around the other. Conversely, if $\boldsymbol{l} \vee \mathbf{k}$ is negative, then the winding is counterclockwise.

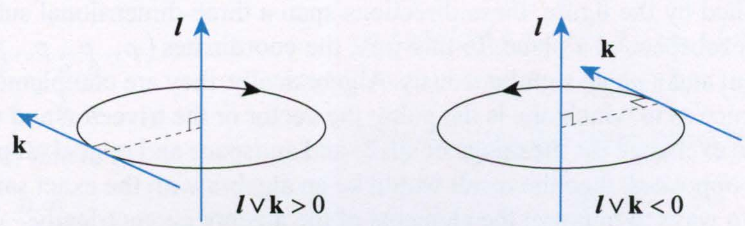


Figure 2.8. The sign of the antiwedge product $\boldsymbol{l} \vee \mathbf{k}$ corresponds to the crossing orientation of the lines $\boldsymbol{l}$ and $\mathbf{k}$. The sign is positive when the lines are wound clockwise around each other, and it is negative when they are wound counterclockwise around each other. The order of multiplication does not matter.

A practical example in which this information is useful arises when we want to determine whether a given ray passes through the interior of a triangle defined by three vertices wound counterclockwise when viewed from the front side. After calculating the antiwedge products between the ray and three lines corresponding to each of the triangle's edges, we know that the ray hits the triangle if all three products are positive and misses if any one of the products is negative. The details are laid out in Comparison Chart #1.

Math Library Notes

- The join and meet operations are implemented by the `Wedge()` and `Antiwedge()` functions.
- Any of the join and meet operations shown in Table 2.7 can also be calculated by using the ^ symbol as an infix operator. The ^ symbol is used for both the wedge and antiwedge products because there is never an ambiguity as to which produces a nonzero result.

2.6 Duality

So far in this chapter, we have encountered several examples of a precise symmetry in projective exterior algebra known generally as *duality*. Every type of mathematical element in the algebra is associated with another type of element given by its complement that is naturally opposite. Lengths, areas, and volumes are built up and torn down by two opposing fundamental operations, the wedge product and antiwedge product. The source of this symmetry is the dichotomy between dimensions that are present and absent in any geometric object, symbolized by the solid and hollow bars associated with each basis element in Table 2.3. If we abstract a little further and consider the notion that the dimensions regarded as present or absent depends on one of two possible perspectives, then we see that every quantity can actually be interpreted in two different ways, and every operation is actually doing two things at once. A complete understanding of geometric algebra requires that we acknowledge the equal importance of both parts of each such pair.

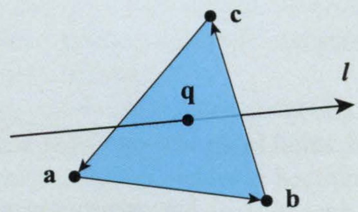
The concept of duality can be understood geometrically in an n-dimensional projective setting by considering both the subspace that an object occupies and the complementary subspace that the object concurrently does not occupy. We call these two components the *space* and *antispace* associated with an object. The dimensionality of the space is the grade of the object, and the dimensionality of the antispace is the antigrade of the object, so all n dimensions are accounted for, and the dimensionalities of the two components always sum to n. The example shown in Figure 2.9 demonstrates the duality between homogeneous points and planes in the 4D projective algebra. The quadruplet of coordinates (p_x, p_y, p_z, p_w) can be interpreted as a vector pointing from the origin toward a specific location on the 3D projection subspace $w = 1$. This vector corresponds to the one-dimensional space of the point that it represents. The geometric dual of a point materializes when we consider all of the directions of space that are orthogonal to the single direction (p_x, p_y, p_z, p_w). As illustrated by the figure, these directions span a three-dimensional subspace that intersects the projection subspace at a plane. In this way, the coordinates (p_x, p_y, p_z, p_w) can be interpreted as both a point and a plane simultaneously. Algebraically, they are complements of each other, but we have a choice as to which one is the point, the vector or the trivector, and which one is the plane. If we were to exchange the meanings of space and antispace and relabel all products and antiproducts with their opposites, then the result would be an algebra with the exact same structure, so there are always two ways to interpret the elements of the algebra geometrically.

When we express the coordinates (p_x, p_y, p_z, p_w) on the vector basis as $p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$, it explicitly states that we are working with a single spatial dimension representing a point, and the ambiguity is removed. Similarly, if we express the coordinates on the antivector basis as

Comparison Chart #1

Line-Triangle Intersection

Determine whether a line $\boldsymbol{l}$ intersects a triangle with vertices **a**, **b**, and **c**, and calculate the point of intersection **q**.



Conventional Methods	Geometric Algebra
Let $\boldsymbol{l}(t) = \mathbf{p} + t\mathbf{v}$ be a parametric line containing the point $\mathbf{p}$ and running parallel to the direction vector $\mathbf{v}$. Assume the direction is normalized so that $\lVert \mathbf{v} \rVert = 1$.	Let $\boldsymbol{l}$ be a line as defined in Equation (2.36) with direction $\boldsymbol{l}_{\mathbf{v}}$ and moment $\boldsymbol{l}_{\mathbf{m}}$. Assume the line is unitized so that $\lVert \boldsymbol{l}_{\mathbf{v}} \rVert = 1$.
Translate the line so that $\mathbf{p}$ coincides with the origin. Translate the triangle by subtracting $\mathbf{p}$ from the vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.	Translate the vertex $\mathbf{a}$ to the origin, and subtract $\mathbf{a}$ from $\mathbf{b}$ and $\mathbf{c}$. Translate the line by subtracting $\mathbf{a} \times \boldsymbol{l}_{\mathbf{v}}$ from its moment $\boldsymbol{l}_{\mathbf{m}}$.
Calculate the scalar triple products $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{v}$, $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{v}$, and $(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{v}$. If any one of these products is positive, then the line does not intersect the triangle. Otherwise, we have a hit.	Lines representing the edges of the triangle are given by $\mathbf{k}_1 = \mathbf{e}_4 \wedge \mathbf{b}$, $\mathbf{k}_2 = \mathbf{b} \wedge \mathbf{c}$, and $\mathbf{k}_3 = \mathbf{c} \wedge \mathbf{e}_4$. The moments of $\mathbf{k}_1$ and $\mathbf{k}_3$ are zero because they contain the origin. Calculate $\boldsymbol{l} \vee \mathbf{k}_1$, $\boldsymbol{l} \vee \mathbf{k}_2$, and $\boldsymbol{l} \vee \mathbf{k}_3$. If any one of these products is negative, then the line does not intersect the triangle. Otherwise, we have a hit.
Determine the plane containing the translated triangle by calculating $\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ and $d = -\mathbf{n} \cdot \mathbf{a}$.	Determine the plane $\mathbf{g}$ containing the translated triangle by calculating $\mathbf{g} = \mathbf{e}_4 \wedge \mathbf{b} \wedge \mathbf{c}$. This plane contains the origin, so $g_w = 0$.
The point of intersection $\mathbf{q}$ is found by first solving the equation $(\mathbf{n} \cdot \mathbf{v})\, t + d = 0$ for t, which gives us $$t = -\frac{d}{\mathbf{n} \cdot \mathbf{v}} = \frac{\mathbf{n} \cdot \mathbf{a}}{\mathbf{n} \cdot \mathbf{v}}.$$ Plug this into the original line to get $$\mathbf{q} = \mathbf{p} + \frac{\mathbf{n} \cdot \mathbf{a}}{\mathbf{n} \cdot \mathbf{v}} \mathbf{v}.$$	Applying the formula in Table 2.7 with $g_w = 0$, the point $\mathbf{r}$ where $\mathbf{g}$ and $\boldsymbol{l}$ meet is given by $$\mathbf{r} = \mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz})\,\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx})\,\mathbf{e}_2 + (g_y l_{mx} - g_x l_{my})\,\mathbf{e}_3 - (\mathbf{g}_{xyz} \cdot \boldsymbol{l}_{\mathbf{v}})\,\mathbf{e}_4.$$ After unitizing this point, offset by $\mathbf{a}$ to get $$\mathbf{q} = \frac{\mathbf{r}_{xyz}}{\mathbf{r}_w} + \mathbf{a}.$$

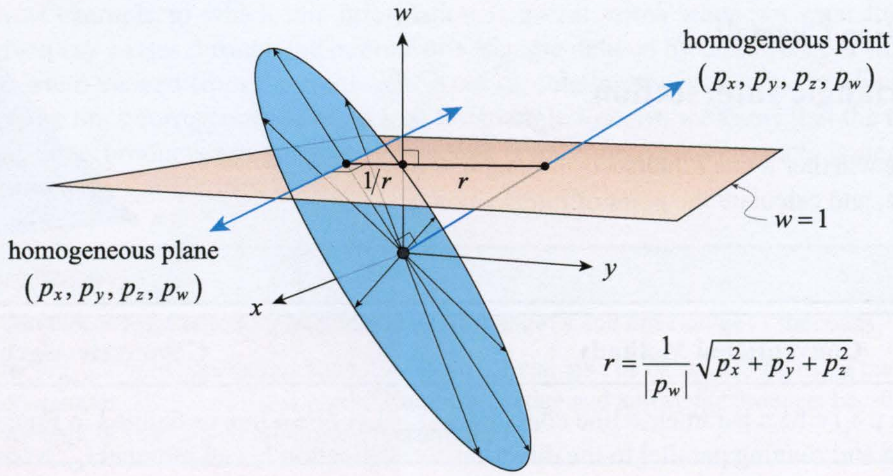


Figure 2.9. The four coordinates (p_x, p_y, p_z, p_w) can be interpreted as the one-dimensional span of a single vector representing a homogeneous point or as the three-dimensional span of all orthogonal vectors representing a homogeneous plane. Geometrically, these two interpretations are dual to each other, and the distances from the two geometric objects to the origin are reciprocals of each other. Algebraically, these two interpretations are represented by complements. (The z axis is omitted, but it should be understood that the $w = 1$ subspace is three-dimensional, and the disk of directions orthogonal to the point is really a 3D ball of vectors.)

$p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} + p_z\mathbf{e}_{412} + p_w\mathbf{e}_{321}$, then we are working with the three orthogonal spatial dimensions representing a plane. In each case, the subscripts of the basis elements tell us which basis vectors are present in the representation, and this defines the space of the object. The subscripts also tell us which basis vectors are absent in the representation, and this defines the antispace of the object. It is possible to establish a convention in which the dimensions not listed in the subscripts correspond to the dimensions that are present in each geometry, but this introduces an unnecessary extra cognitive burden, and it throws away the clear geometric intuition in Figures 2.4, 2.5, and 2.7. Nevertheless, acknowledging the existence of both the space and the antispace of any object and assigning equal meaningfulness to them allows us to explore the nature of duality to its fullest. A vector $p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$ is never only a point, but both a point and a plane simultaneously, where the point exists in space, and the plane exists in antispace. Likewise, an antivector $p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} + p_z\mathbf{e}_{412} + p_w\mathbf{e}_{321}$ is never only a plane, but both a plane and a point simultaneously, where the plane exists in space, and the point exists in antispace. If we study only one spatial facet of these objects, then we are missing half of a bigger picture.

Because each object is actually two geometries at the same time, any operations performed on them must be acting on both geometric interpretations simultaneously. Indeed, whenever we perform an operation on the space of geometric objects, the associated anti-operation is implicitly performed on the antispace of the same objects. The two operations are inextricably linked, and it's not possible to do one without the other. Whenever we take the wedge product of two objects, it computes the join of their spaces by combining the dimensions that are present in each one. At the same time, it computes the meet of their antispaces by intersecting the dimensions that are absent, decreasing the antigrade by however much the join operation increased the grade. This is a natural consequence of the complementary relationship between the wedge and antiwedge products given by Equation (2.25).

As an example, suppose that the objects on which we perform an operation are points $\mathbf{p}$ and $\mathbf{q}$ in the 4D algebra. The wedge product $\mathbf{p} \wedge \mathbf{q}$ computes the line containing $\mathbf{p}$ and $\mathbf{q}$. The complements $\overline{\mathbf{p}}$ and $\overline{\mathbf{q}}$ are planes, and the complement of the wedge product is $\overline{\mathbf{p}} \vee \overline{\mathbf{q}}$. This antiwedge product computes the line where those two planes intersect, and that meet operation is what's happening in antispace when we compute the join $\mathbf{p} \wedge \mathbf{q}$ in regular space. In this case, both operations produce a line, but the two results are complements of each other. In the 4D algebra, a bivector represents two lines at once in which the meanings of direction and moment are exchanged.

The innate duality of geometric algebra will show itself many more times throughout this book. The word "dual" has taken on a large number of meanings across many mathematical fields, but we will provide a specific definition of what "dual" means in geometric algebra below in Section 2.12. Under that definition, geometries like those shown in Figure 2.9 are not actually duals of each other, but they are complements of each other, and we will use the word "complement" to express that relationship. We will continue using the term "duality" to describe any pair of opposing concepts that are related by the complement operation such as vector and antivector or wedge product and antiwedge product.

2.7 Exomorphisms

The matrices at the heart of linear algebra perform transformations that move vectors from one coordinate system to another. In n dimensions, the columns of an $n \times n$ transformation matrix $\mathbf{m}$ are exactly the images of the n basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ after the transformation has been applied through the product $\mathbf{m}\mathbf{e}_i$. What we would like to do is extend the matrix $\mathbf{m}$ to a larger matrix in such a way that it properly transforms not only the basis vectors having grade one but all 2^n basis elements over all grades in the n-dimensional exterior algebra. We accomplish this by requiring that our extension of $\mathbf{m}$ behaves as a homomorphism with respect to the wedge product. That is, given the way in which the grade-one vectors transform, the structure of the rest of the exterior algebra must be the same before and after the extended transformation is applied. Such an extended transformation is called the *exomorphism* of the linear transformation $\mathbf{m}$, where the prefix exo- comes from its relationship to the exterior product.[4]

We use a capital $\mathbf{M}$ to denote the exomorphism matrix, which is the extension of the matrix $\mathbf{m}$ to the larger matrix needed to transform quantities of any grade. The matrix $\mathbf{M}$ has 2^n columns and 2^n rows corresponding to the 2^n basis elements in the n-dimensional exterior algebra. The generic vectors transformed by $\mathbf{M}$ are column matrices with 2^n components, one for each basis element of a complete multivector in n dimensions. For example, in three dimensions, a complete eight-component multivector $\mathbf{u}$ can be written as

$$\mathbf{u} = s\mathbf{1} + v_x \mathbf{e}_1 + v_y \mathbf{e}_2 + v_z \mathbf{e}_3 + b_x \mathbf{e}_{23} + b_y \mathbf{e}_{31} + b_z \mathbf{e}_{12} + t\mathbb{1}, \tag{2.47}$$

and its matrix representation is the column of entries s, v_x, v_y, v_z, b_x, b_y, b_z, and t in that order from top to bottom. It is sufficient to ensure that $\mathbf{M}$ is a homomorphism over the basis elements of the algebra because that property would then extend to all multivectors by the linearity of matrix multiplication. For $\mathbf{M}$ to be an exomorphism, we must have

$$\mathbf{M}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{M}\mathbf{a}) \wedge (\mathbf{M}\mathbf{b}) \tag{2.48}$$

for all basis elements $\mathbf{a}$ and $\mathbf{b}$. That is, the image of $\mathbf{a} \wedge \mathbf{b}$ under the transformation performed by $\mathbf{M}$ must be the wedge product of the images of $\mathbf{a}$ and $\mathbf{b}$. This rule allows us to build the full $2^n \times 2^n$ matrix $\mathbf{M}$ solely from the information contained in the original $n \times n$ matrix $\mathbf{m}$. We already know

[4] Early texts on geometric algebra [Hest1984] and later publications derived from them use the term *outer product* synonymously with the term *exterior product*, and that has led to the term *outermorphism* being used where we now write *exomorphism*. They are the same thing. However, the outer product has a different established meaning in linear algebra (specifically, the outer product between two n-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ is the $n \times n$ matrix $\mathbf{u}\mathbf{v}^{\mathrm{T}}$), so we avoid the ambiguity in this book.

the image of each basis vector $\mathbf{e}_i$ because it is given by $\mathbf{m}_{[i]}$, the ith column of $\mathbf{m}$, extended with zeros in the positions for basis elements having grade other than one. For any basis bivector $\mathbf{e}_{ij}$, Equation (2.48) requires that

$$\mathbf{Me}_{ij} = (\mathbf{Me}_i) \wedge (\mathbf{Me}_j), \tag{2.49}$$

but the right side of this equation is just the wedge product of $\mathbf{m}_{[i]}$ and $\mathbf{m}_{[j]}$. We calculate this wedge product by treating the column $\mathbf{m}_{[i]}$ with entries $m_{i1}, m_{i2}, m_{i3}, \ldots$ from top to bottom as the vector $m_{i1}\mathbf{e}_1 + m_{i2}\mathbf{e}_2 + m_{i3}\mathbf{e}_3 + \cdots$. When applied to all basis bivectors, Equation (2.49) produces $\binom{n}{2}$ new columns in the matrix $\mathbf{M}$, each containing the image of a single basis bivector. The entries in each column corresponding to other grades are zero, so we obtain a new $\binom{n}{2} \times \binom{n}{2}$ submatrix of $\mathbf{M}$ that tells us specifically how to transform any arbitrary bivector quantity.

Continuing this process for trivectors and higher grades by multiplying three or more columns of $\mathbf{m}$ together with the wedge product fills out the rest of the exomorphism matrix $\mathbf{M}$. The submatrix that transforms quantities of grade k has size $\binom{n}{k} \times \binom{n}{k}$, so $\mathbf{M}$ has the form of a block diagonal matrix with $n+1$ submatrices corresponding to the $n+1$ possible grades as k ranges from 0 to n. The submatrix corresponding to grade k is known as the kth *compound matrix* of $\mathbf{m}$, denoted by $C_k(\mathbf{m})$.

We define the grade-zero submatrix $C_0(\mathbf{m})$ that transforms scalars to be the 1×1 identity matrix, so the upper-left entry of $\mathbf{M}$ is always the number one. The entry in the bottom-right corner represents the 1×1 submatrix $C_n(\mathbf{m})$ containing the image of the volume element $\mathbb{1}$, and it tells us how antiscalars are transformed. Since the volume element is equal to the wedge product of all n basis vectors $\mathbf{e}_1$ through $\mathbf{e}_n$, it must be transformed by multiplying it by the wedge product of all columns of $\mathbf{m}$, which is equal to its determinant. Thus, the entry in the bottom-right corner of $\mathbf{M}$ is always equal to $\det \mathbf{m}$.

The submatrix $C_{n-1}(\mathbf{m})$ in the penultimate position along the diagonal has size $n \times n$, and it transforms antivectors. This submatrix is always the adjugate transpose of the matrix $\mathbf{m}$ because each of its columns is constructed from the wedge product of all but one column of $\mathbf{m}$.

The 3D Euclidean exterior algebra is simple enough that it enables a fully written out example of the above process. Here, the eight basis elements are $\mathbf{1}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, $\mathbf{e}_{12}$, and $\mathbb{1} = \mathbf{e}_{123}$, where it's important that we keep a consistent order. The exomorphism matrix $\mathbf{M}$ has eight rows and eight columns that correspond to the basis elements in the same order, and a multivector is expressed as an eight-component column matrix whose entries are the coefficients of the basis elements in the same order. Now suppose that we have a 3×3 matrix $\mathbf{m}$ that transforms the grade-one vectors from one coordinate system to another. We label the columns of $\mathbf{m}$ as the 3D vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ so that $\mathbf{m}$ can be written as

$$\mathbf{m} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}. \tag{2.50}$$

The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the images of the basis elements $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively, when transformed by the matrix $\mathbf{m}$.

The second compound matrix of $\mathbf{m}$ transforms bivectors, and it consists of all possible wedge products between pairs of columns of $\mathbf{m}$. We just have to be careful about the order in which we multiply the columns of $\mathbf{m}$ and the order in which we write the components of the results. The first column of $C_2(\mathbf{m})$ is given by the wedge product $\mathbf{b} \wedge \mathbf{c}$ because the first bivector basis element in the order we have imposed is $\mathbf{e}_{23}$. When we calculate $\mathbf{b} \wedge \mathbf{c}$, we get

$$\mathbf{b} \wedge \mathbf{c} = (b_y c_z - b_z c_y)\,\mathbf{e}_{23} + (b_z c_x - b_x c_z)\,\mathbf{e}_{31} + (b_x c_y - b_y c_x)\,\mathbf{e}_{12}, \tag{2.51}$$

where we have written the terms in the order imposed on the basis elements. The three coefficients on the right side are the three entries in the first column of $C_2(\mathbf{m})$. The entries of the other two columns are similarly given by $\mathbf{c} \wedge \mathbf{a}$, corresponding to the $\mathbf{e}_{31}$ basis element, and $\mathbf{a} \wedge \mathbf{b}$, corresponding to the $\mathbf{e}_{12}$ basis element. Altogether, these nine coefficients constitute the entire matrix $C_2(\mathbf{m})$, which we can now write as

$$C_2(\mathbf{m}) = \begin{bmatrix} b_y c_z - b_z c_y & c_y a_z - c_z a_y & a_y b_z - a_z b_y \\ b_z c_x - b_x c_z & c_z a_x - c_x a_z & a_z b_x - a_x b_z \\ b_x c_y - b_y c_x & c_x a_y - c_y a_x & a_x b_y - a_y b_x \end{bmatrix}. \tag{2.52}$$

The entries of this matrix are the same values appearing in Equation (1.6) that arose in the transformation of quantities generated by the cross product. The cross product is really a wedge product in disguise, and it produces bivector quantities that must transform in this manner.

The exomorphism matrix $\mathbf{M}$ is the 8×8 block diagonal matrix whose submatrices are $C_0(\mathbf{m})$, $C_1(\mathbf{m})$, $C_2(\mathbf{m})$, and $C_3(\mathbf{m})$, as illustrated in Figure 2.10. $C_0(\mathbf{m})$ is always the 1×1 identity matrix having a single entry filled with the number one. $C_1(\mathbf{m})$ is just the matrix $\mathbf{m}$ itself. $C_2(\mathbf{m})$ was derived from $\mathbf{m}$, and its entries are given by Equation (2.52). Finally, $C_3(\mathbf{m})$ is the 1×1 matrix whose single entry is the determinant of $\mathbf{m}$. With these submatrices arranged along the diagonal, the exomorphism matrix $\mathbf{M}$ correctly transforms any 3D multivector of the form shown in Equation (2.47) given that grade-one vectors are transformed by the matrix $\mathbf{m}$ and with the requirement that transformation by $\mathbf{M}$ is a homomorphism under the wedge product.

Since we have modeled homogeneous points, lines, and planes in a 4D projective space, we are interested in how 4D quantities transform from one coordinate system to another. The process for constructing the exomorphism matrix $\mathbf{M}$ is exactly the same as in the 3D case except that we now have a total of 16 basis elements, and we begin with a 4×4 matrix $\mathbf{m}$ that transforms grade-one vectors. The result is the block diagonal matrix $\mathbf{M}$ shown in Figure 2.11 containing five submatrices corresponding to the five different grades that exist in the 4D algebra.

Instead of calculating the compound matrices $C_2(\mathbf{m})$ and $C_3(\mathbf{m})$ for an arbitrary matrix $\mathbf{m}$, which becomes very tedious even if we can assume the fourth row is $[0 \;\; 0 \;\; 0 \;\; 1]$, we look at some

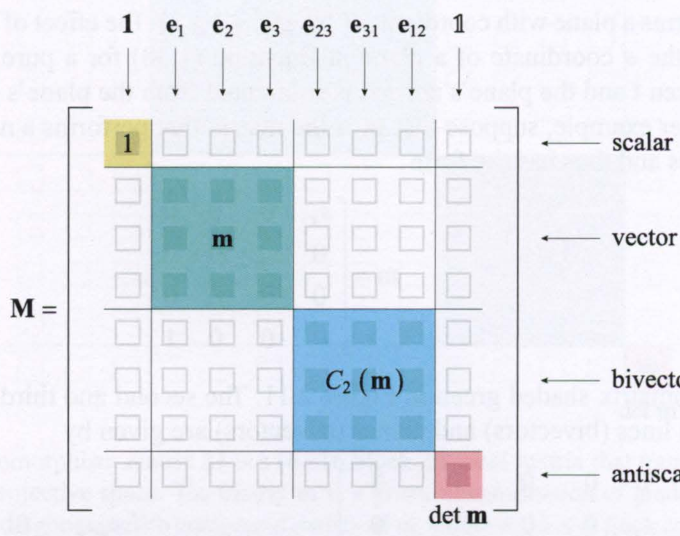


Figure 2.10. The exomorphism matrix $\mathbf{M}$ is an 8×8 block diagonal matrix that transforms eight-component 3D multivectors from one coordinate system to another. The 3×3 matrix $\mathbf{m}$ is a given transformation of grade-one vectors, and the rest of the matrix $\mathbf{M}$ is derived from it. Each submatrix along the diagonal is a kth compound matrix of $\mathbf{m}$, where $0 \le k \le 3$. Each column of $\mathbf{M}$ corresponds to the image of the basis element shown at the top under the transformation that $\mathbf{M}$ performs. Entries in unshaded areas are zero.

specific examples. First, suppose that $\mathbf{m}$ is the matrix that performs a translation by the vector $\mathbf{t}$ and thus has the form

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.53}$$

This matrix directly translates any homogeneous point (x, y, z, w). To translate a bivector representing a line, we need to calculate the 6×6 matrix $C_2(\mathbf{m})$ by taking wedge products of the columns of $\mathbf{m}$ in the appropriate order. Using the order of components appearing in the definition of a line given by Equation (2.36), we have

$$C_2(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t_z & t_y & 1 & 0 & 0 \\ t_z & 0 & -t_x & 0 & 1 & 0 \\ -t_y & t_x & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.54}$$

and this transforms a line with coordinates $(l_{vx}, l_{vy}, l_{vz}, l_{mx}, l_{my}, l_{mz})$. The effect of $C_2(\mathbf{m})$ matches the transformation of Plücker coordinates given by Equation (1.42) for a pure translation where the cross product between $\mathbf{t}$ and the line's direction is added to the line's moment. Next, to translate a trivector representing a plane, we calculate the 4×4 matrix $C_3(\mathbf{m})$ and obtain

$$C_3(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_x & -t_y & -t_z & 1 \end{bmatrix}, \tag{2.55}$$

which transforms a plane with coordinates (g_x, g_y, g_z, g_w). The effect of $C_3(\mathbf{m})$ matches the transformation of the d coordinate of a plane in Equation (1.38) for a pure translation where the dot product between $\mathbf{t}$ and the plane's normal is subtracted from the plane's distance from the origin.

For another example, suppose that $\mathbf{m}$ is the matrix that performs a nonuniform scale along the x, y, and z axes and thus has the form

$$\mathbf{m} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.56}$$

This is the submatrix shaded green in Figure 2.11. The second and third compound matrices of $\mathbf{m}$ that transform lines (bivectors) and planes (trivectors) are given by

$$C_2(\mathbf{m}) = \begin{bmatrix} s_x & 0 & 0 & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 & 0 & 0 \\ 0 & 0 & s_z & 0 & 0 & 0 \\ 0 & 0 & 0 & s_y s_z & 0 & 0 \\ 0 & 0 & 0 & 0 & s_z s_x & 0 \\ 0 & 0 & 0 & 0 & 0 & s_x s_y \end{bmatrix} \quad \text{and} \quad C_3(\mathbf{m}) = \begin{bmatrix} s_y s_z & 0 & 0 & 0 \\ 0 & s_z s_x & 0 & 0 \\ 0 & 0 & s_x s_y & 0 \\ 0 & 0 & 0 & s_x s_y s_z \end{bmatrix} \tag{2.57}$$

These are the submatrices shaded blue and purple in Figure 2.11. By examining the entries of $C_2(\mathbf{m})$, we can see that the direction of a line scales just like a vector, and this is due to the direction being a length-like quantity. The moment of a line is an area-like quantity, and its scaling transformation therefore involves two factors. The normal of a plane is also an area-like quantity, and the entries of $C_3(\mathbf{m})$ demonstrate that it scales with two factors as well. The w coordinate of a plane is the only component that scales like a volume, as shown by the product $s_x s_y s_z$ in the bottom-right corner of $C_3(\mathbf{m})$.

The construction of the exomorphism matrix **M** allows us to correctly transform any kind of object in an n-dimensional exterior algebra as long as we know how to transform grade-one vectors with the $n \times n$ matrix **m**. The generality of the matrix **M** can sometimes have its advantages, but it also makes things like parameterization and interpolation difficult. In Chapter 3, we will find that some important subsets of all possible linear transformations can be applied with products native to the projective algebra without the need for matrix multiplication.

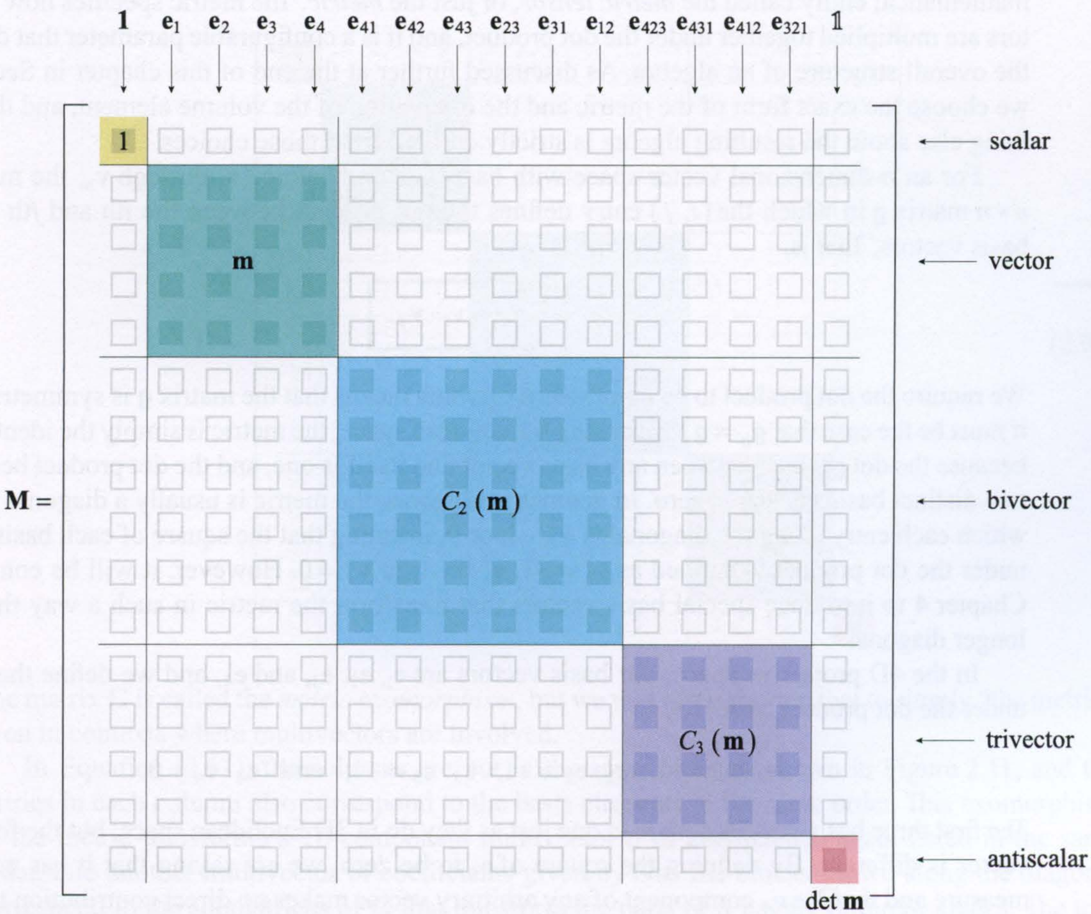


Figure 2.11. The exomorphism matrix **M** is a 16×16 block diagonal matrix that transforms multivectors belonging to the 4D projective space. The matrix **m** is a given transformation of grade-one vectors, and each submatrix along the diagonal is a kth compound matrix of **m**, where $0 \le k \le 4$. Each column of **M** corresponds to the image of the basis element shown at the top under the transformation that **M** performs. Entries in unshaded areas are zero.

2.8 Metric Transformations

The wedge and antiwedge products have allowed us to combine points, lines, and planes in various ways, but those operations do not provide methods for measuring meaningful magnitudes, distances between objects, or angles between objects. In ordinary Euclidean space, the dot product supplies essentially all we need to calculate these quantities. The squared magnitude of a vector $\mathbf{v}$ is given by $\mathbf{v} \cdot \mathbf{v}$, the distance between two points is given by the magnitude of their difference, and the cosine of the angle between two unit-length directions $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a} \cdot \mathbf{b}$. In order to obtain these kinds of measurements in the homogeneous projective space that we have been using, it is necessary to make some generalizations.

2.8.1 The Metric

In any vector space, the foundation upon which the definitions of distance and angle are built is a mathematical entity called the *metric tensor*, or just the *metric*. The metric specifies how basis vectors are multiplied together under the dot product, and it is a configurable parameter that determines the overall structure of an algebra. As discussed further at the end of this chapter in Section 2.15, we choose the exact form of the metric and the orientation of the volume element, and then everything else about the resulting algebra is strictly derived from those choices.

For an n-dimensional vector space with basis vectors denoted $\mathbf{v}_1$ through $\mathbf{v}_n$, the metric is an $n \times n$ matrix $\mathbf{g}$ in which the (i, j) entry defines the dot product between the ith and jth grade-one basis vectors. That is,

Metric

$$\boxed{\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j .} \tag{2.58}$$

We require the dot product to be commutative, which means that the matrix $\mathfrak{g}$ is symmetric because it must be the case that $\mathfrak{g}_{ij} = \mathfrak{g}_{ji}$. In ordinary Euclidean space, the metric is simply the identity matrix because the dot product between any basis vector and itself is one, and the dot product between any two distinct basis vectors is zero. In geometric algebras, the metric is usually a diagonal matrix for which each entry along the diagonal is $+1$, -1, or 0, meaning that the square of each basis vector $\mathbf{v}_i$ under the dot product is defined as $\mathbf{v}_i^2 = +1$, $\mathbf{v}_i^2 = -1$, or $\mathbf{v}_i^2 = 0$. However, it will be convenient in Chapter 4 to introduce special basis vectors that transform the metric in such a way that it is no longer diagonal.

In the 4D projective space, our basis vectors are $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$, and we define their squares under the dot product to be

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1, \quad \mathbf{e}_2 \cdot \mathbf{e}_2 = +1, \quad \mathbf{e}_3 \cdot \mathbf{e}_3 = +1, \quad \text{and} \quad \mathbf{e}_4 \cdot \mathbf{e}_4 = 0. \tag{2.59}$$

The first three basis vectors square to one just as they do in 3D Euclidean space, but the fourth basis vector is different. By defining the square of $\mathbf{e}_4$ to be zero, we are saying that it has no physical measure and that the $\mathbf{e}_4$ component of any arbitrary vector makes no direct contribution to its magnitude. The full story of $\mathbf{e}_4$ is a little more complicated than that, but the general notion that $\mathbf{e}_4$ has no size will suffice for the moment. Using the definitions given by Equation (2.59) and continuing to require that the dot product between distinct basis vectors is zero, we can write the metric $\mathfrak{g}$ as

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.60}$$

Just like any other $n \times n$ matrix, the metric $\mathbf{g}$ performs a transformation on an n-dimensional vector $\mathbf{v}$ through the matrix-vector product $\mathbf{gv}$. If $\mathbf{g}$ is diagonal and none of the basis vectors square to zero, then this transformation is a one-to-one mapping of the vector space to itself, and it is invertible. Since one of our basis vectors in the projective space does square to zero, however, the metric $\mathbf{g}$ given by Equation (2.60) is not invertible. When a metric is not invertible, is it called *degenerate* because it transforms some nontrivial subset of the vector space onto the zero vector. In the case of the 4D projective space, the product $\mathbf{gv}$ erases any $\mathbf{e}_4$ component of $\mathbf{v}$ to leave only the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ components behind. Any vector having only an $\mathbf{e}_4$ component is therefore mapped to zero. We will have more to say about this specific transformation below in Section 2.8.3.

The exterior algebra arising from a set of n basis vectors can be regarded as a larger vector space having 2^n basis elements. In order to extend the dot product and other operations to arbitrary multivectors in the exterior algebra, we construct the $2^n \times 2^n$ exomorphism matrix $\mathbf{G}$ from the metric $\mathbf{g}$ using the method described for general linear transformations in the previous section. This means that we must define $\mathbf{1} \cdot \mathbf{1} = +1$ for scalars and then require that $\mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{Ga}) \wedge (\mathbf{Gb})$ for any multivectors $\mathbf{a}$ and $\mathbf{b}$. Applying this rule to the 4×4 matrix $\mathbf{g}$ given by Equation (2.60) produces the 16×16 exomorphism matrix

$$\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{array}\right]. \tag{2.61}$$

The matrix $\mathbf{G}$ is called the *metric exomorphism*, but we will often shorten that to simply "the metric" even in contexts where multivectors are involved.

In Equation (2.61), the columns are put in the same order as shown in Figure 2.11, and the entries in each column also correspond to the basis elements in the same order. This exomorphism of the metric transforms a 16-component multivector $\mathbf{u}$ of coefficients, again listed in the same order, into another multivector of coefficients given by $\mathbf{Gu}$. The blocks shown along the diagonal correspond to the submatrices of $\mathbf{G}$ that transform the parts of $\mathbf{u}$ having different grades. The top-left entry transforms the scalar component, the 4×4 submatrix following it along the diagonal is the original metric $\mathbf{g}$ that transforms the grade-one vector components, the 6×6 submatrix in the center is the second compound matrix $C_2(\mathbf{g})$ that transforms the bivector components, the 4×4 submatrix that comes next is the third compound matrix $C_3(\mathbf{g})$ that transforms the trivector components, and the bottom-right entry transforms the antiscalar component.

Whereas the original metric $\mathbf{g}$ given by Equation (2.60) eliminates the single $\mathbf{e}_4$ component of a grade-one vector $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4$, the exomorphism $\mathbf{G}$ given by Equation (2.61) eliminates fully half the components of a complete 16-component multivector

$$\mathbf{u} = s\mathbf{1} + p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4 + l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12} + g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321} + t\mathbb{1} \tag{2.62}$$

because that's how many components contain a factor of $\mathbf{e}_4$. This corresponds to the fact that there are eight zeros along the diagonal of $\mathbf{G}$.

2.8.2 The Antimetric

We can regard the transformation $\mathbf{Gu}$ as a unary operation that does something to the generic multivector $\mathbf{u}$. That being the case, we can construct the corresponding anti-operation by first taking the complement of $\mathbf{u}$, then multiplying by the matrix $\mathbf{G}$, and finally taking the inverse complement of the result. This produces a new unary operation $\mathbb{G}\mathbf{u}$ defined by

Antimetric

$$\boxed{\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\overline{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}},} \tag{2.63}$$

where the matrix $\mathbb{G}$, which we write in a blackboard bold typeface, is another $2^n \times 2^n$ matrix called the *metric antiexomorphism*. This term will usually be shortened to simply "the antimetric". It should come as no surprise that the defining property of an antiexomorphism $\mathbb{G}$ is the requirement that $\mathbb{G}(\mathbf{a} \vee \mathbf{b}) = (\mathbb{G}\mathbf{a}) \vee (\mathbb{G}\mathbf{b})$ for any multivectors $\mathbf{a}$ and $\mathbf{b}$, where the wedge product has been replaced by the antiwedge product. The 16×16 antimetric $\mathbb{G}$ corresponding to the metric $\mathbf{G}$ shown in Equation (2.61) is given by

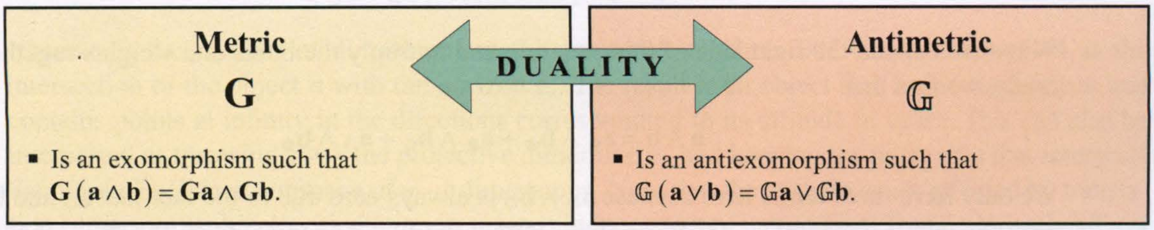
$$\mathbb{G} = \left[\begin{array}{cccccccccccccccc}
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1
\end{array}\right]. \tag{2.64}$$

The entries of the antimetric $\mathbb{G}$ are nothing more than a rearrangement of the entries of the metric $\mathbf{G}$. If we define a function $f(k)$ that maps the index k of a basis element to the index of its complement, ignoring sign, then the (i, j) entry of $\mathbb{G}$ is equal to the $(f(i), f(j))$ entry of $\mathbf{G}$. When we list the basis elements in the order shown in Figure 2.11, the difference between $\mathbf{G}$ and $\mathbb{G}$ is that the order of the submatrices along the diagonal is reversed, and the entries of the center submatrix (shaded blue) are mirrored across the diagonal running from the lower-left corner to the upper-right corner. This relationship is visible when comparing Equations (2.61) and (2.64).

The part of the metric $\mathbf{G}$ that transforms antivectors is the adjugate transpose of the part that transforms vectors. Since these parts are exchanged in the antimetric, it must be the case that

$$\boxed{\mathbf{G}\mathbb{G} = \det(\mathbf{g})\,\mathbf{I}} \tag{2.65}$$

because the adjugate transpose of a matrix $\mathbf{m}$ differs from the inverse of $\mathbf{m}$ by a factor of the determinant. In the case that $\mathbf{g}$ is degenerate, like it is in the projective exterior algebra, we have $\mathbf{G}\mathbb{G} = 0$. Otherwise, when all of the basis vectors square under the dot product to ± 1, we have $\mathbf{G}\mathbb{G} = \pm\mathbf{I}$, where the sign is determined by whether an even or odd number of basis vectors square to -1.



2.8.3 Bulk and Weight

Notice that the locations where ones appear along the diagonal of the matrix $\mathbb{G}$ are exactly the locations where zeros appear along the diagonal of the matrix $\mathbf{G}$. This is, of course, necessary because complements invert the condition of whether a basis element contains a factor of $\mathbf{e}_4$. The practical effect is that the operations $\mathbf{G}\mathbf{u}$ and $\mathbb{G}\mathbf{u}$ divide the components of $\mathbf{u}$ into two disjoint parts, those that do not have a factor that squares to zero and those that do have a factor that squares to zero. We call these two parts the *bulk* of $\mathbf{u}$ and the *weight* of $\mathbf{u}$.

The bulk of an object $\mathbf{u}$ is denoted by $\mathbf{u}_\bullet$ with a solid black circle written as a subscript, and it is defined as

Bulk

$$\boxed{\mathbf{u}_\bullet = \mathbf{G}\mathbf{u}.} \tag{2.66}$$

The weight of an object $\mathbf{u}$ is denoted by $\mathbf{u}_\circ$ with an empty white circle written as a subscript, and it is defined as

Weight

$$\boxed{\mathbf{u}_\circ = \mathbb{G}\mathbf{u}.} \tag{2.67}$$

In the 4D projective space representing geometries of 3D Euclidean space, the bulk of an object consists of all the components that do not have a factor of $\mathbf{e}_4$, and the weight of an object consists of all the components that do have a factor of $\mathbf{e}_4$. This divides each of our representations of points, lines, and planes into the two parts shown in Table 2.8.

When either the bulk or weight of an object is zero, it can be interpreted in a special way. An object with zero bulk always contains the origin. The origin itself has only an $\mathbf{e}_4$ component, a line

Type	Bulk	Weight
Point $\mathbf{p}$	$\mathbf{p}_\bullet = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$	$\mathbf{p}_\circ = p_w\mathbf{e}_4$
Line $\boldsymbol{l}$	$\boldsymbol{l}_\bullet = l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$	$\boldsymbol{l}_\circ = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}$
Plane $\mathbf{g}$	$\mathbf{g}_\bullet = g_w\mathbf{e}_{321}$	$\mathbf{g}_\circ = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}$

Table 2.8. These are the bulks and weights of geometric objects in the 4D projective space.

passing through the origin has a zero moment, and a plane containing the origin has no $\mathbf{e}_{321}$ component. Symmetrically, an object with zero weight is always contained by the horizon. A point with no $\mathbf{e}_4$ component is a point at infinity, a line with a zero direction also lies at infinity, and a plane with a zero normal is the horizon itself.

Since the bulk and weight are disjoint under a degenerate metric, any object $\mathbf{u}$ can be written as a sum of its bulk and weight parts as $\mathbf{u} = \mathbf{u}_\bullet + \mathbf{u}_\circ$. When we multiply two objects $\mathbf{a}$ and $\mathbf{b}$ together with the wedge product, we can write

$$\mathbf{a} \wedge \mathbf{b} = (\mathbf{a}_\bullet + \mathbf{a}_\circ) \wedge (\mathbf{b}_\bullet + \mathbf{b}_\circ). \tag{2.68}$$

When we expand the right side of this equation and multiply the bulks and weights together separately, we find that

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a}_\bullet \wedge \mathbf{b}_\bullet + \mathbf{a}_\bullet \wedge \mathbf{b}_\circ + \mathbf{a}_\circ \wedge \mathbf{b}_\bullet. \tag{2.69}$$

We only have three terms here because $\mathbf{a}_\circ \wedge \mathbf{b}_\circ$ is always zero due to the fact that $\mathbf{a}_\circ$ and $\mathbf{b}_\circ$ must both contain a factor of $\mathbf{e}_4$. Of these three terms, the first one constitutes the bulk of the entire product because it's the only term for which neither factor contains a factor of $\mathbf{e}_4$. The remaining two terms constitute the weight of the entire product, and we can thus express the bulk and weight of $\mathbf{a} \wedge \mathbf{b}$ as

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b})_\bullet &= \mathbf{a}_\bullet \wedge \mathbf{b}_\bullet \\ (\mathbf{a} \wedge \mathbf{b})_\circ &= \mathbf{a}_\bullet \wedge \mathbf{b}_\circ + \mathbf{a}_\circ \wedge \mathbf{b}_\bullet. \end{aligned} \tag{2.70}$$

The antiwedge product can be decomposed in a similar way. When we multiply the bulks and weights of two objects $\mathbf{a}$ and $\mathbf{b}$ together with the antiwedge product, we have

$$\begin{aligned} \mathbf{a} \vee \mathbf{b} &= (\mathbf{a}_\bullet + \mathbf{a}_\circ) \vee (\mathbf{b}_\bullet + \mathbf{b}_\circ) \\ &= \mathbf{a}_\bullet \vee \mathbf{b}_\circ + \mathbf{a}_\circ \vee \mathbf{b}_\bullet + \mathbf{a}_\circ \vee \mathbf{b}_\circ. \end{aligned} \tag{2.71}$$

This time, the product of the bulks $\mathbf{a}_\bullet \wedge \mathbf{b}_\bullet$ is always zero because both factors $\mathbf{a}_\bullet$ and $\mathbf{b}_\bullet$ are missing a factor of $\mathbf{e}_4$. The first two terms constitute the bulk of the entire product, and the last term constitutes the weight, so we can express the bulk and weight of $\mathbf{a} \vee \mathbf{b}$ as

$$\begin{aligned} (\mathbf{a} \vee \mathbf{b})_\bullet &= \mathbf{a}_\bullet \vee \mathbf{b}_\circ + \mathbf{a}_\circ \vee \mathbf{b}_\bullet \\ (\mathbf{a} \vee \mathbf{b})_\circ &= \mathbf{a}_\circ \vee \mathbf{b}_\circ. \end{aligned} \tag{2.72}$$

Bulk $\mathbf{u}_\bullet$	DUALITY	Weight $\mathbf{u}_\circ$
▪ Selects components of **u** without a factor of $\mathbf{e}_n$.		▪ Selects components of **u** with a factor of $\mathbf{e}_n$.
▪ Contains positional information about **u**.		▪ Contains directional information about **u**.
▪ Zero bulk means **u** contains the origin.		▪ Zero weight means **u** is contained in the horizon.

2.8.4 Attitude

The partitioning of an object's components into bulk and weight has a geometric meaning that appears all the time in projective algebras. The bulk of an object contains information about the position of the object relative to the origin, and the weight contains information about the attitude and orientation of the object independently of its position. For example, the weight of a line is specifically the 3D direction vector along which the line runs, and the weight of a plane is specifically its 3D normal bivector, but each of these is multiplied by the projective basis vector $\mathbf{e}_4$. In order to extract this information as a purely directional nonprojective quantity, we define a function that throws away the bulk of an object and removes the factor of $\mathbf{e}_4$ from its weight. The *attitude* of an object $\mathbf{u}$ is denoted by $\operatorname{att}(\mathbf{u})$ and defined as

Attitude

$$\boxed{\operatorname{att}(\mathbf{u}) = \mathbf{u} \vee \overline{\mathbf{e}}_n,} \tag{2.73}$$

where n is the number of dimensions in the projective algebra. This can be interpreted as the intersection of the object $\mathbf{u}$ with the horizon $\overline{\mathbf{e}}_n$. The result is an object that lies in the horizon and contains points at infinity in the directions corresponding to its attitude in space. This can also be interpreted as throwing away the projective dimension $\mathbf{e}_n$ and keeping the piece of $\mathbf{u}$ that intersects the purely Euclidean subspace of $n-1$ dimensions. Note that the attitude is not affected by translation because any object's orientation in space is independent of its position. Translation only affects the bulk of a point, line, or plane.

In the 4D projective algebra, we extract the attitude of an object by taking the antiwedge product with $\overline{\mathbf{e}}_4 = \mathbf{e}_{321}$. The attitudes of points, lines, and planes are listed in Table 2.9. The attitude of a point is nothing more than its weight as a scalar, so it doesn't have much geometric meaning. But the attitude of a line is a vector representing the point at infinity in the direction parallel to the line, and the attitude of a plane is a bivector representing a line at infinity containing points in all directions parallel to the plane.

Type	Attitude
Point $\mathbf{p}$	$\operatorname{att}(\mathbf{p}) = p_w\mathbf{1}$
Line $\boldsymbol{l}$	$\operatorname{att}(\boldsymbol{l}) = l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2 + l_{vz}\mathbf{e}_3$
Plane $\mathbf{g}$	$\operatorname{att}(\mathbf{g}) = g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}$

Table 2.9. These are the attitudes of geometric objects in three dimensions.

2.9 Inner Products

In the previous section, we defined the metric $\mathbf{g}$ as an $n \times n$ matrix whose entries were given by all possible dot products between grade-one basis vectors in n-dimensional space. We were able to choose how each basis vector squared, but then the remainder of the metric exomorphism $\mathbf{G}$ was fully determined by the requirement that $\mathbf{G}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{Ga}) \wedge (\mathbf{Gb})$. We now use this larger $2^n \times 2^n$ matrix to extend the definition of the dot product to basis elements of all grades within the exterior algebra and thus to all multivector quantities.

The extended *dot product* $\mathbf{a} \bullet \mathbf{b}$, denoted by a fat solid black dot, is given by

Dot product

$$\boxed{\mathbf{a} \bullet \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b}\right)\mathbf{1},} \tag{2.74}$$

where $\mathbf{a}$ and $\mathbf{b}$ are treated as 2^n-dimensional vectors of coefficients corresponding to the 2^n basis elements in the n-dimensional exterior algebra. The right side of the definition is the product of a

1×2^n row vector (because $\mathbf{a}$ is transposed), a $2^n \times 2^n$ matrix, and a $2^n \times 1$ column vector. The final multiplication by $\mathbf{1}$ simply maps the scalar result onto the embedding of scalars as grade-zero elements of the exterior algebra.

The definition given by Equation (2.74) provides an inner product that extends linearly to the whole exterior algebra. It's easy to demonstrate that $(\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c}$ for any multivectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, that $(s\mathbf{a}) \bullet \mathbf{b} = \mathbf{a} \bullet (s\mathbf{b}) = s(\mathbf{a} \bullet \mathbf{b})$ for any scalar s, and that $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$. Due to the block diagonal form of the exomorphism matrix $\mathbf{G}$, the inner product between two quantities having different grades is always zero. Contributions to the final result of $\mathbf{a} \bullet \mathbf{b}$ are made only by terms of $\mathbf{a}$ and $\mathbf{b}$ that have matching grades. This is further restricted to terms having identical basis elements if $\mathbf{G}$ is a diagonal matrix, which always happens when the $n \times n$ metric $\mathbf{g}$ is a diagonal matrix. This innate simplicity of the dot and antidot products is necessary for the proper functioning of norms. However, there is a group of related but more general products called *interior products*, discussed in Section 2.13 below, and they generate meaningful results when two objects of different grades are multiplied together.

As discussed previously in Section 2.3, a product in exterior algebra always has a corresponding antiproduct, and this is just as true for the dot product as it was for the wedge product. The *antidot product* $\mathbf{a} \circ \mathbf{b}$, denoted by a fat empty white dot, can be defined as

Antidot product

$$\boxed{\mathbf{a} \circ \mathbf{b} = \left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1},} \tag{2.75}$$

which is clearly symmetric to Equation (2.74). The only differences are that the metric $\mathbf{G}$ has been replaced by the antimetric $\mathbb{G}$, and the scalar result is mapped onto the grade-n elements (which are antigrade-zero elements) instead of the grade-zero elements. The result of a dot product is always a scalar, and the result of an antidot product is always an antiscalar.

The usual method for defining an antiproduct through De Morgan's laws is also valid, so an alternative definition of the antidot product is

$$\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}. \tag{2.76}$$

We can show that Equations (2.75) and (2.76) are equivalent by first expanding the definition of the dot product on the right side of Equation (2.76) as

$$\overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}} = \overline{\left(\underline{\mathbf{a}}^{\mathrm{T}} \mathbf{G} \underline{\mathbf{b}}\right) \mathbf{1}}. \tag{2.77}$$

By taking the left complement in the definition of the antimetric $\mathbb{G}$ given in Equation (2.63), we have $\mathbf{G}\underline{\mathbf{b}} = \underline{\mathbb{G}\mathbf{b}}$, so we can now write

$$\overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}} = \overline{\left(\underline{\mathbf{a}}^{\mathrm{T}} \underline{\mathbb{G}\mathbf{b}}\right) \mathbf{1}}. \tag{2.78}$$

In the matrix product $\underline{\mathbf{a}}^{\mathrm{T}} \underline{\mathbb{G}\mathbf{b}}$, the left complement operation reorders the entries of $\mathbf{a}$ and $\mathbb{G}\mathbf{b}$ in the same way, so the matrix product between the two does not change, and we know that $\underline{\mathbf{a}}^{\mathrm{T}} \underline{\mathbb{G}\mathbf{b}} = \mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}$. This lets us write

$$\overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}} = \overline{\left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbf{1}} = \left(\mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b}\right) \mathbb{1}, \tag{2.79}$$

which demonstrates the equivalency of the two definitions of the antidot product.

The dot and antidot products of the 16 basis elements in the 4D projective algebra with themselves are listed in Table 2.10. The products between any two distinct basis elements are all zero because the metric and antimetric are both diagonal matrices. As a consequence of the fact that the metric is degenerate here, the dot product $\mathbf{u} \bullet \mathbf{u}$ ends up being the sum of the squares of all the components of $\mathbf{u}_\bullet$ (the bulk of $\mathbf{u}$) as a scalar, and the antidot product $\mathbf{u} \circ \mathbf{u}$ ends up being the sum of the squares of all the components of $\mathbf{u}_\circ$ (the weight of $\mathbf{u}$) as an antiscalar. The infix symbols used for dot product and antidot product reflect this relationship by being smaller versions of the symbols used as subscripts to denote bulk and weight. It is a combination of the dot product and antidot product that allows us to determine magnitudes in the homogeneous geometric model, and this the main concept in the discussion of norms in the next section.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u} \bullet \mathbf{u}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	$\mathbf{1}$	0
$\mathbf{u} \circ \mathbf{u}$	0	0	0	0	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	0	0	0	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	0	$\mathbb{1}$

Table 2.10. These are the dot and antidot products between each of the 16 basis elements in the 4D projective exterior algebra with themselves.

Due to the way in which compound matrices are constructed from an $n \times n$ matrix that transforms grade-one vectors, the dot product between objects of higher grade can always be reduced to a set of dot products between combinations of the grade-one factors of those objects. This doesn't really provide us with any computational advantages, but it does give us a theoretical tool that we can use to derive properties of other operations. In particular, the relationship that we discuss here will be an important stepping stone when we establish a rule for decomposing interior products in Section 2.13.

First, we consider the dot product between two basis bivectors $\mathbf{e}_{ij}$ and $\mathbf{e}_{kl}$ under an arbitrary metric tensor $\mathfrak{g}$. The value of $\mathbf{e}_{ij} \bullet \mathbf{e}_{kl}$ is given by the (s, t) entry of the second compound matrix $C_2(\mathfrak{g})$, where s and t are the indices of the basis bivectors $\mathbf{e}_{ij}$ and $\mathbf{e}_{kl}$ within the ordered list of all basis bivectors. (For example, in the 4D projective algebra, the index of $\mathbf{e}_{23}$ is 4 because it is the fourth item in the list $\{\mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$.) Since the metric tensor is symmetric, the entries of row s and column s of $C_2(\mathfrak{g})$ are both given by $\mathfrak{g}_{[i]} \wedge \mathfrak{g}_{[j]}$, which is the wedge product between columns i and j of $\mathfrak{g}$. Similarly, the entries of row t and column t of of $C_2(\mathfrak{g})$ are both given by $\mathfrak{g}_{[k]} \wedge \mathfrak{g}_{[l]}$. This means the we can write $[C_2(\mathfrak{g})]_{st}$ in the two different forms

$$[C_2(\mathfrak{g})]_{st} = (\mathfrak{g}_{[i]} \wedge \mathfrak{g}_{[j]})_t = (\mathfrak{g}_{[k]} \wedge \mathfrak{g}_{[l]})_s. \tag{2.80}$$

In both cases, we obtain

$$\mathbf{e}_{ij} \bullet \mathbf{e}_{kl} = [C_2(\mathfrak{g})]_{st} = \mathfrak{g}_{ik}\mathfrak{g}_{jl} - \mathfrak{g}_{il}\mathfrak{g}_{jk}. \tag{2.81}$$

The entries of $\mathfrak{g}$ are defined to be dot products between basis vectors, so we can rewrite this as

$$\begin{aligned} \mathbf{e}_{ij} \bullet \mathbf{e}_{kl} &= (\mathbf{e}_i \bullet \mathbf{e}_k)(\mathbf{e}_j \bullet \mathbf{e}_l) - (\mathbf{e}_i \bullet \mathbf{e}_l)(\mathbf{e}_j \bullet \mathbf{e}_k) \\ &= \begin{vmatrix} \mathbf{e}_i \bullet \mathbf{e}_k & \mathbf{e}_i \bullet \mathbf{e}_l \\ \mathbf{e}_j \bullet \mathbf{e}_k & \mathbf{e}_j \bullet \mathbf{e}_l \end{vmatrix}. \end{aligned} \tag{2.82}$$

That is, the dot product between the bivectors $\mathbf{e}_{ij}$ and $\mathbf{e}_{kl}$ is equal to the determinant of the 2×2 matrix whose entries are the dot products between their vector factors, where the row determines which factor of the first bivector is involved and the column determines which factor of the second bivector is involved. When we generalize this to grade-k basis elements, we find that the dot product between two basis k-vectors is similarly given by the determinant of a $k \times k$ matrix whose entries are dot products between all combinations of their vector factors. Since all of this is linear, we can replace each grade-k basis element with an arbitrary product of k vectors to arrive at the formula

$$(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k) \bullet (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) = \begin{vmatrix} \mathbf{a}_1 \bullet \mathbf{b}_1 & \mathbf{a}_1 \bullet \mathbf{b}_2 & \cdots & \mathbf{a}_1 \bullet \mathbf{b}_k \\ \mathbf{a}_2 \bullet \mathbf{b}_1 & \mathbf{a}_2 \bullet \mathbf{b}_2 & \cdots & \mathbf{a}_2 \bullet \mathbf{b}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_k \bullet \mathbf{b}_1 & \mathbf{a}_k \bullet \mathbf{b}_2 & \cdots & \mathbf{a}_k \bullet \mathbf{b}_k \end{vmatrix}, \tag{2.83}$$

where each of the factors $\mathbf{a}_i$ and $\mathbf{b}_i$ are grade-one vectors.

It would never be practical to calculate a dot product using Equation (2.83) because it's much easier to simply use the definition given by Equation (2.74) with the extended metric $\mathbf{G}$ that will always be known to us. However, Equation (2.83) does provide information that will help us prove important relationships later on. We can turn the dot product of two simple k-vectors into a summation over k terms by expanding the determinant on the right side of Equation (2.83) along its first column to get

$$\begin{vmatrix} \mathbf{a}_1 \bullet \mathbf{b}_1 & \mathbf{a}_1 \bullet \mathbf{b}_2 & \cdots & \mathbf{a}_1 \bullet \mathbf{b}_k \\ \mathbf{a}_2 \bullet \mathbf{b}_1 & \mathbf{a}_2 \bullet \mathbf{b}_2 & \cdots & \mathbf{a}_2 \bullet \mathbf{b}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_k \bullet \mathbf{b}_1 & \mathbf{a}_k \bullet \mathbf{b}_2 & \cdots & \mathbf{a}_k \bullet \mathbf{b}_k \end{vmatrix} = \sum_{i=1}^{k} (-1)^{i-1} (\mathbf{a}_i \bullet \mathbf{b}_1) \begin{vmatrix} \mathbf{a}_1 \bullet \mathbf{b}_2 & \mathbf{a}_1 \bullet \mathbf{b}_3 & \cdots & \mathbf{a}_1 \bullet \mathbf{b}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{i-1} \bullet \mathbf{b}_2 & \mathbf{a}_{i-1} \bullet \mathbf{b}_3 & \cdots & \mathbf{a}_{i-1} \bullet \mathbf{b}_k \\ \mathbf{a}_{i+1} \bullet \mathbf{b}_2 & \mathbf{a}_{i+1} \bullet \mathbf{b}_3 & \cdots & \mathbf{a}_{i+1} \bullet \mathbf{b}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_k \bullet \mathbf{b}_2 & \mathbf{a}_k \bullet \mathbf{b}_3 & \cdots & \mathbf{a}_k \bullet \mathbf{b}_k \end{vmatrix}. \tag{2.84}$$

The determinant of the $(k-1) \times (k-1)$ matrix appearing on the right side of this equation (which is part of the summand) is equivalent to the dot product between two simple $(k-1)$-vectors. The first is $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{i-1} \wedge \mathbf{a}_{i+1} \wedge \cdots \wedge \mathbf{a}_k$, which depends on the index of summation i because it is missing the factor $\mathbf{a}_i$, and the second is $\mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k$. Since the second factor is the same for all terms of the summation, we can pull it out and write

$$\begin{aligned} &(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k) \bullet (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) \\ &= \left[\sum_{i=1}^{k} (-1)^{i-1} (\mathbf{a}_i \bullet \mathbf{b}_1)(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{i-1} \wedge \mathbf{a}_{i+1} \wedge \cdots \mathbf{a}_k) \right] \bullet (\mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k). \end{aligned} \tag{2.85}$$

This result is as far as we go for now. In Section 2.13, we will be able to isolate this summation by dropping the factor $\mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k$ and equating it to another expression in order to establish a fundamental property of interior products.

Dot Product	DUALITY	Antidot Product
$\mathbf{a} \bullet \mathbf{b}$		$\mathbf{a} \circ \mathbf{b}$
▪ Defined as $(\mathbf{a}^{\mathrm{T}}\mathbf{G}\mathbf{b})\mathbf{1}$.		▪ Defined as $(\mathbf{a}^{\mathrm{T}}\mathbb{G}\mathbf{b})\mathbb{1}$.
▪ Inner product between bulks.		▪ Inner product between weights.

2.10 Norms

With a metric and dot product established, we now have the means to quantitatively describe the magnitude of a geometric object. This is accomplished through the definition of a *norm* that maps an element of a vector space to a real number representing some abstract notion of distance from the origin. The conventional way to do this is to define the norm of a vector $\mathbf{v}$, which is denoted by $\|\mathbf{v}\|$, as the square root of the dot product of $\mathbf{v}$ with itself. That is, the dot product induces the norm given by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{2.86}$$

When we attempt to extend this definition to the whole projective space by using the extended dot product $\bullet$ defined in the previous section, it continues to work well for Euclidean vectors, bivectors, and trivectors by producing their lengths, areas, and volumes. But it fails to produce a meaningful measure of magnitude for anything having a nonzero weight part because it involves the projective dimension $\mathbf{e}_4$. We need to account for the homogeneity of the projective space so that nonzero scalar multiples of any particular geometric object all end up having the same distance from the origin when the norm is applied.

We first review what properties a norm must possess in order to be called a norm because it will be necessary to make some generalizations later on. The conventional norm must satisfy the following three conditions:

1. The norm of $\mathbf{v}$ is zero if and only if $\mathbf{v}$ is the zero vector.
2. If a vector $\mathbf{v}$ is multiplied by a scalar s, then its norm is multiplied by $|s|$.
3. The norm of the sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ cannot be greater than the sum of their norms.

The first of these properties guarantees that only the origin has a magnitude of zero. It also ensures that the difference between any two distinct vectors has a nonzero magnitude. The second property says that whenever a vector's components are scaled by a factor s, the vector's distance from the origin must also increase or decrease by the same factor s, ignoring sign. The third property is known as the *triangle inequality*, and it states that adding two vectors $\mathbf{v}$ and $\mathbf{w}$ cannot result in a new vector that is magically farther away from the origin than the sum of the distances of $\mathbf{v}$ and $\mathbf{w}$ themselves to the origin.

If we start replacing the vector $\mathbf{v}$ with geometric objects from the projective algebra, we quickly find that the above properties lose their effectiveness when the extended dot product $\bullet$ is applied in Equation (2.86). Any line that passes through the origin would have a norm of zero despite being a nonzero quantity $l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43}$, so the first property is not satisfied. The second property is technically satisfied, but it demonstrates that the norm of any geometric object could be made to have any arbitrary value without changing the geometry itself, which renders the norm completely meaningless. The triangle inequality is also technically satisfied, but it is equally meaningless for similar reasons.

2.10.1 Bulk and Weight Norms

We have two different dot products, which are defined by Equations (2.74) and (2.75), and it turns out that a proper norm capturing a meaningful distance from the origin requires that both dot products participate. We first define separate norms with each dot product, where we use the term "norm" loosely because these definitions do not satisfy the first property above. Since one norm is based on the dot product and the other norm is based on the antidot product, we can rather generically call them the *norm* and *antinorm*, but in the presence of a degenerate metric, it will be fitting to give them the more distinctive names that follow.

The *bulk norm* of an object $\mathbf{u}$, denoted by $\|\mathbf{u}\|_\bullet$ with the symbol for bulk written as a subscript, is based on the dot product and defined as

Norm / Bulk norm

$$\boxed{\|\mathbf{u}\|_\bullet = \sqrt{\mathbf{u} \bullet \mathbf{u}}.} \tag{2.87}$$

The *weight norm* of an object $\mathbf{u}$, denoted by $\|\mathbf{u}\|_\circ$ with the symbol for weight written as a subscript, is based on the antidot product and defined as

Antinorm / Weight norm

$$\boxed{\|\mathbf{u}\|_\circ = \sqrt{\mathbf{u} \circ \mathbf{u}}.} \tag{2.88}$$

The bulk norms and weight norms of points, lines, and planes are listed in Table 2.11. As should be expected, the bulk norm measures the size of the bulk components of an object, and the weight norm measures the size of the weight components of an object. It's important to recognize that the bulk norm produces a scalar quantity, but the weight norm produces an antiscalar quantity. When the bulk norm of $\mathbf{u}$ is the scalar $\mathbf{1}$, we say that $\mathbf{u}$ is *bulk normalized*. Similarly, when the weight norm of $\mathbf{u}$ is the antiscalar $\mathbb{1}$, we say that $\mathbf{u}$ is *weight normalized*.

Type	Bulk Norm	Weight Norm
Point $\mathbf{p}$	$\|\mathbf{p}\|_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$	$\|\mathbf{p}\|_\circ = \|p_w\|\mathbb{1}$
Line $\boldsymbol{l}$	$\|\boldsymbol{l}\|_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\|\boldsymbol{l}\|_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane $\mathbf{g}$	$\|\mathbf{g}\|_\bullet = \|g_w\|\mathbf{1}$	$\|\mathbf{g}\|_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$

Table 2.11. These are the bulk norms and weight norms of geometric objects in the 4D projective exterior algebra.

2.10.2 Unitization

The weight norm of an object can be regarded as the collective magnitude of the components that extend in the direction of the basis vector $\mathbf{e}_4$. We project the higher-dimensional representations of geometric objects into 3D Euclidean space by scaling so that the weight has unit magnitude. For a homogeneous point, all we need to do is divide by the w coordinate because there is only one component that includes a factor of $\mathbf{e}_4$. Lines and planes, however, each have three components that include a factor of $\mathbf{e}_4$, and we need to divide by their size as a whole. Once the weight has been adjusted to have unit magnitude, we call the object *unitized* as a shorter alternative to calling it weight normalized. In general, unitization is the process of scaling the components of a geometric object so that its weight norm becomes the antiscalar $\mathbb{1}$. An object $\mathbf{u}$ is unitized by calculating

Unitization

$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|_\circ} = \frac{\mathbf{u}}{\sqrt{\mathbf{u} \circ \mathbf{u}}},} \tag{2.89}$$

and we indicate that something has been unitized by writing it with a hat above the variable or expression. Table 2.12 lists the conditions under which points, lines, and planes are considered to be unitized because their weight norms would have unit size.

Type	Definition	Unitization
Point $\mathbf{p}$	$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4$	$p_w^2 = 1$
Line $\boldsymbol{l}$	$\boldsymbol{l} = l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43} + l_{mx} \mathbf{e}_{23} + l_{my} \mathbf{e}_{31} + l_{mz} \mathbf{e}_{12}$	$l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$
Plane $\mathbf{g}$	$\mathbf{g} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} + g_w \mathbf{e}_{321}$	$g_x^2 + g_y^2 + g_z^2 = 1$

Table 2.12. The right column lists the conditions under which geometric objects in three dimensions are considered to be unitized.

2.10.3 The Geometric Norm

The bulk norm and weight norm are combined into a single quantity by simply adding them together. The *geometric norm* of an object $\mathbf{u}$, denoted by $\|\mathbf{u}\|$ with no subscript, is defined as

Geometric norm

$$\boxed{\|\mathbf{u}\| = \|\mathbf{u}\|_\bullet + \|\mathbf{u}\|_\circ = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}.} \tag{2.90}$$

The geometric norm produces a quantity $\|\mathbf{u}\| = s\mathbf{1} + t\mathbb{1}$ consisting of a scalar and antiscalar pair of numbers. It is a *homogeneous magnitude* that has properties consistent with the geometric model of homogeneous points, lines, and planes. The scalar part of a homogeneous magnitude is its bulk, and the antiscalar part is its weight. Like all other homogeneous objects, multiplying the whole quantity by a nonzero scalar value does not change its meaning, and it is projected into 3D space by choosing a scale that makes its weight have unit size.

The geometric norm provides us with a concrete measurement of distance from the origin that remains invariant under projection no matter how an object is homogeneously scaled. The geometric norm of a point $\mathbf{p}$ is

$$\|\mathbf{p}\| = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2} + \mathbb{1}\,|p_w|, \tag{2.91}$$

and rescaling so that the antiscalar part has unit magnitude requires the division by p_w to which we are accustomed for points in homogeneous coordinates. The geometric norm of a line $\boldsymbol{l}$ is

$$\|\boldsymbol{l}\| = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2} + \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}, \tag{2.92}$$

and it is projected into 3D space by requiring that the three components (l_{vx}, l_{vy}, l_{vz}) of the line's direction make a unit vector. When this is the case, the magnitude of the line's moment, with components (l_{mx}, l_{my}, l_{mz}), is equal to the perpendicular distance between the line and the origin. Finally, the geometric norm of a plane $\mathbf{g}$ is

$$\|\mathbf{g}\| = \mathbf{1}\,|g_w| + \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}, \tag{2.93}$$

and it is projected into 3D space by requiring that the three components (g_x, g_y, g_z) of the plane's normal have unit magnitude. When this is the case, the absolute value of g_w is equal to the perpendicular distance between the plane and the origin.

Assuming the weight norm of an object $\mathbf{u}$ is not zero, the unitized version of its homogeneous magnitude $\|\mathbf{u}\| = s\mathbf{1} + t\mathbb{1}$ is given by

$$\widehat{\|\mathbf{u}\|} = \frac{s}{t}\mathbf{1} + \mathbb{1}. \tag{2.94}$$

The antiscalar part no longer has any meaning, and we can simply write the magnitude as a scalar distance value. (This is similar to how the w coordinate of a homogeneous point is typically dropped

Type	Geometric Norm	Interpretation
Point **p**	$\left\lVert\widehat{\mathbf{p}}\right\rVert = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point **p**.
Line ***l***	$\left\lVert\widehat{\boldsymbol{l}}\right\rVert = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line ***l***.
Plane **g**	$\left\lVert\widehat{\mathbf{g}}\right\rVert = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane **g**.

Table 2.13. These are the scalar parts of the geometric norms of objects in three dimensions after unitization. The geometric norm of an object is equal to the distance between the object and the origin.

when projecting into 3D space.) When we unitize the geometric norms of points, lines, and planes, we obtain the values listed in Table 2.13, and they all correspond to the distances between the geometric objects and the origin.

The geometric norm satisfies a generalization of the three properties required for a conventional norm, and the differences are summarized in Table 2.14. We first have the property that the norm of a vector is zero if and only if the vector itself is zero. This is generalized so that the geometric norm of an object in the projective space is zero if and only if the object contains the origin. Since we are measuring the distance from the origin, any lines or planes that pass through the origin, in addition to the point at the origin itself, have a geometric norm of zero. Next, we have the property that scaling a vector causes the vector's norm to scale by the same amount. This does not apply in our projective space because any nonzero scale of an object's components does not change the geometric meaning of the object. Instead, we think about the points that make up an object and consider what happens when all of those points are scaled by the same factor in order to dilate the entire object with respect to the origin. In this case, we expect the geometric norm to be scaled by the same factor as well, and that is indeed what occurs. Finally, we have the triangle inequality for conventional norms. This property generalizes by considering what happens if we translate all of the points that make up an object. When we do this, the distance between the object and the origin cannot increase by more than the distance that the object is translated.

Math Library Notes

- The `SquaredBulkNorm()` and `SquaredWeightNorm()` functions return the squares of the bulk norm and weight norm. Squares are returned so the quotient can be taken before a single square root is applied.
- Points, lines, and planes can be scaled to have unit weight by calling the `Unitize()` function.

2.11 Euclidean Distances

Given a norm on a vector space, the conventional way to measure the distance between two vectors **a** and **b** is to calculate $\lVert \mathbf{a} - \mathbf{b} \rVert$. This is equivalent to translating the coordinate system so that either **a** or **b** coincides with the origin and then taking the norm of the other vector. This is still an effective way to measure distances in the 4D projective algebra if one of the objects happens to be a point **p**. We can just translate the other object by $-\mathbf{p}_{xyz} / p_w$, which can be accomplished for a point, line or plane by using the matrix in Equation (2.53), (2.54), or (2.55), and then take the geometric norm of the result to measure its distance from the origin.

Conventional Norm	Geometric Norm
1. $\lVert \mathbf{v} \rVert = 0$ if and only if $\mathbf{v} = \mathbf{0}$.	**1.** $\lVert \mathbf{u} \rVert = 0$ if and only if $\mathbf{u}$ contains the origin, which is equivalent to stating $\mathbf{u} \wedge \mathbf{e}_{n+1} = 0$ in n dimensions.
2. For any vector $\mathbf{v}$ and scalar s, $\lVert s\mathbf{v} \rVert = \lvert s \rvert \lVert \mathbf{v} \rVert$.	**2.** Let $D(\mathbf{u}, s)$ be the result of applying a dilation by a scale factor s centered at the origin to the object $\mathbf{u}$. Then for any object $\mathbf{u}$ and scalar s, $\lVert D(\mathbf{u}, s) \rVert = \lvert s \rvert \lVert \mathbf{u} \rVert$.
3. For any vectors $\mathbf{v}$ and $\mathbf{w}$, $\lVert \mathbf{v} + \mathbf{w} \rVert \le \lVert \mathbf{v} \rVert + \lVert \mathbf{w} \rVert$.	**3.** Let $T(\mathbf{u}, \mathbf{t})$ be the result of applying a translation by the displacement vector $\mathbf{t}$ to the object $\mathbf{u}$. Then for any object $\mathbf{u}$ and vector $\mathbf{t}$, $\lVert T(\mathbf{u}, \mathbf{t}) \rVert \le \lVert \mathbf{u} \rVert + \lVert \mathbf{t} \rVert$.

Table 2.14. The definition of the geometric norm of an object $\mathbf{u}$ generalizes the conventional definition of norm to account for the geometric interpretation of objects in projective space.

When we talk about the Euclidean distance between two objects $\mathbf{a}$ and $\mathbf{b}$, what we mean is the length of the shortest straight path connecting a point on $\mathbf{a}$ to a point on $\mathbf{b}$. For example, the distance between a point $\mathbf{p}$ and a plane $\mathbf{g}$ is the length of the straight path connecting $\mathbf{p}$ to the point on $\mathbf{g}$ that is closest to $\mathbf{p}$. When one of the objects involved in a distance measurement is a point $\mathbf{p}$, then there is no choice about what point on that object is closest to the other object because we only have $\mathbf{p}$ itself. In these cases, we can translate $\mathbf{p}$ to the origin and take the geometric norm of the other object as described above. If neither object is a point, however, then we don't know where to translate our coordinate system, and the same strategy doesn't work. In three dimensions, this problem appears only when considering the distance between two lines, but it is more common in higher numbers of dimensions.

In the case of two lines $\boldsymbol{l}$ and $\mathbf{k}$, we can figure out what to do by taking a closer look at the wedge product $\boldsymbol{l} \wedge \mathbf{k}$. Suppose that $\boldsymbol{l} = \mathbf{p} \wedge \mathbf{q}$ and $\mathbf{k} = \mathbf{r} \wedge \mathbf{s}$, where the points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$ all have a w coordinate of one. Then the directions of the lines as 3D vectors are $\boldsymbol{l}_{\mathbf{v}} = \mathbf{q} - \mathbf{p}$ and $\mathbf{k}_{\mathbf{v}} = \mathbf{s} - \mathbf{r}$. Although it's somewhat difficult to visualize, the wedge product

$$\boldsymbol{l} \wedge \mathbf{k} = \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r} \wedge \mathbf{s} \tag{2.95}$$

can be interpreted as the signed volume of a four-dimensional parallelotope whose sides are given by the four-dimensional vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$. Since all four sides have a length of one in the direction of the w axis, that direction can be ignored by essentially dividing it out, leaving behind the three-dimensional parallelepiped shown in Figure 2.12. Without loss of generality, we can assume that the points $\mathbf{p}$ and $\mathbf{r}$ correspond to the points of closest approach on the two lines because sliding the points along each line (keeping the distance between $\mathbf{p}$ and $\mathbf{q}$ constant and the distance between $\mathbf{r}$ and $\mathbf{s}$ constant) only has the effect of skewing the parallelepiped, which does not change its volume.

The area of the base of the parallelepiped is given by $\lVert \boldsymbol{l}_{\mathbf{v}} \wedge \mathbf{k}_{\mathbf{v}} \rVert$. If we divide the absolute volume $\lvert \boldsymbol{l} \wedge \mathbf{k} \rvert$ of the parallelepiped by this area and take out the unit length in the w direction, then the only remaining dimension is the distance between the two lines corresponding to the magnitude of $\mathbf{p} - \mathbf{r}$ in the figure. We are expressing this as an antiscalar, however, so we take the complement of $\boldsymbol{l} \wedge \mathbf{k}$ to turn it into a scalar. Thus, the formula for the distance d between two lines is given by

$$d = \frac{\lvert \boldsymbol{l} \vee \mathbf{k} \rvert}{\lVert \boldsymbol{l}_{\mathbf{v}} \wedge \mathbf{k}_{\mathbf{v}} \rVert}, \tag{2.96}$$

where the numerator contains the antiwedge product between two 4D bivectors, and the denominator contains the wedge product between two 3D vectors.

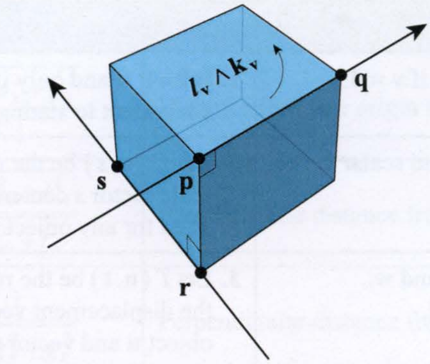


Figure 2.12. The distance between two lines $\boldsymbol{l} = \mathbf{p} \wedge \mathbf{q}$ and $\mathbf{k} = \mathbf{r} \wedge \mathbf{s}$ having 3D direction vectors $\boldsymbol{l}_{\mathrm{v}} = \mathbf{q} - \mathbf{p}$ and $\mathbf{k}_{\mathrm{v}} = \mathbf{s} - \mathbf{r}$ is given by the complement of the volume $\boldsymbol{l} \wedge \mathbf{k}$ divided by the base area $\|\boldsymbol{l}_{\mathrm{v}} \wedge \mathbf{k}_{\mathrm{v}}\|$.

Here, we have recognized that we can construct a four-dimensional volume by multiplying two lines together, remove the projective dimension along the w axis, and divide by the area of a two-dimensional base to obtain a height value that corresponds to the distance between the lines. This idea can be generalized to all types of geometric objects. Suppose that we want to find the distance between two objects **a** and **b** whose representations in the 4D projective algebra have grades $k_{\mathbf{a}}$ and $k_{\mathbf{b}}$. The attitude of their wedge product has $k_{\mathbf{a}} + k_{\mathbf{b}} - 1$ dimensions because it discards the projective dimension, and its volume is directly related to how far apart **a** and **b** are. It makes sense that the bulk of $\mathbf{a} \wedge \mathbf{b}$ is thrown away because the position of $\mathbf{a} \wedge \mathbf{b}$ is not relevant to the distance between **a** and **b**. We can always translate so that $\mathbf{a} \wedge \mathbf{b}$ passes through the origin without affecting the size of $\operatorname{att}(\mathbf{a} \wedge \mathbf{b})$. We just need to divide the volume of this attitude by the area of a base having $k_{\mathbf{a}} + k_{\mathbf{b}} - 2$ dimensions (where we are using the terms volume and area only to mean that they differ in dimensionality by one). This area is given by the wedge product of the separate attitudes of **a** and **b**, so we can express the distance $d(\mathbf{a}, \mathbf{b})$ between objects **a** and **b** as

$$d(\mathbf{a}, \mathbf{b}) = \frac{\|\operatorname{att}(\mathbf{a} \wedge \mathbf{b})\|_{\bullet}}{\|\operatorname{att}(\mathbf{a}) \wedge \operatorname{att}(\mathbf{b})\|_{\bullet}}. \tag{2.97}$$

Since the numerator and denominator each contain the weights of both **a** and **b**, they are divided out, and this formula respects the homogeneous properties of the geometry. We can manipulate the denominator a little bit by replacing $\operatorname{att}(\mathbf{a}) \wedge \operatorname{att}(\mathbf{b})$ with either $\mathbf{a} \wedge \operatorname{att}(\mathbf{b})$ or $\operatorname{att}(\mathbf{a}) \wedge \mathbf{b}$ and taking a weight norm instead of a bulk norm. The result is the same product between the original weights of **a** and **b**, but we can now express the distance as a homogeneous magnitude by writing

$$d(\mathbf{a}, \mathbf{b}) = \|\operatorname{att}(\mathbf{a} \wedge \mathbf{b})\|_{\bullet} + \|\mathbf{a} \wedge \operatorname{att}(\mathbf{b})\|_{\circ}. \tag{2.98}$$

The distance given by Equation (2.98) is an absolute quantity, but we know that it's both possible and very useful to calculate a signed distance between a point and a plane. We have also seen a signed crossing orientation for the relationship between two lines in Equation (2.46). Whenever the sum of the grades of two objects **a** and **b** is the full dimensionality n of the projective space, the wedge product $\mathbf{a} \wedge \mathbf{b}$ is a single numerical value as an antiscalar, and $\mathbf{a} \vee \mathbf{b}$ is the same numerical value as a scalar. In these cases, we can extract some meaning from the sign of the value. In the case of a point **p** and a plane **g**, the sign of $\mathbf{p} \vee \mathbf{g}$ tells us whether **p** falls on the front side or back side of **g**. In the case of two lines $\boldsymbol{l}$ and **k**, the sign of $\boldsymbol{l} \vee \mathbf{k}$ tells us whether the crossing orientation is clockwise or counterclockwise. We just need to be careful about the order of multiplication when the grades of both objects are odd since they don't commute. The sign of $\mathbf{p} \vee \mathbf{g}$ agrees with the conventional 4D dot product $\mathbf{g} \cdot \mathbf{p}$ discussed in Chapter 1, but $\mathbf{g} \vee \mathbf{p}$ is negated. The order in which we multiply two lines does not matter.

After accounting for the possibility that we care about the sign of the result in some cases, the general formula for the distance $d(\mathbf{a}, \mathbf{b})$ between two geometric objects $\mathbf{a}$ and $\mathbf{b}$ as a homogeneous magnitude in an n-dimensional projective space is

Euclidean distance

$$\boxed{d(\mathbf{a},\mathbf{b}) = \begin{cases} \mathbf{a} \vee \mathbf{b} + \left\|\mathbf{a} \wedge \operatorname{att}(\mathbf{b})\right\|_{\circ}, & \text{if } \operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) = n; \\ \left\|\operatorname{att}(\mathbf{a} \wedge \mathbf{b})\right\|_{\bullet} + \left\|\mathbf{a} \wedge \operatorname{att}(\mathbf{b})\right\|_{\circ}, & \text{otherwise.} \end{cases}} \tag{2.99}$$

If we want an absolute distance in the case that $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}) = n$, all we have to do is take the absolute value of $\mathbf{a} \vee \mathbf{b}$. Using Equation (2.99), formulas for the Euclidean distances between the four possible combinations of points, lines, and planes in 3D space are shown in Table 2.15. The distance between two skew lines is explored further in Comparison Chart #2.

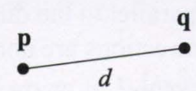
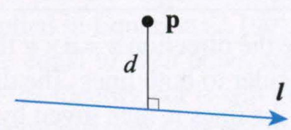
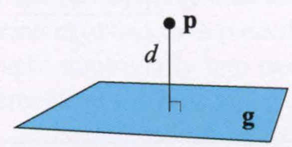
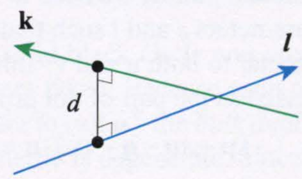
Distance Formula	**Illustration**
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p},\mathbf{q}) = \left\|\mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w\right\| \mathbf{1} + \left\|p_w q_w\right\| \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\boldsymbol{l}$. $d(\mathbf{p},\boldsymbol{l}) = \left\|\boldsymbol{l}_{\mathrm{v}} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_{\mathbf{m}}\right\| \mathbf{1} + \left\|p_w \boldsymbol{l}_{\mathrm{v}}\right\| \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$. $d(\mathbf{p},\mathbf{g}) = (\mathbf{p} \cdot \mathbf{g})\mathbf{1} + \left\|p_w \mathbf{g}_{xyz}\right\| \mathbb{1}$	
Perpendicular distance d between skew lines $\boldsymbol{l}$ and $\mathbf{k}$. $d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_{\mathrm{v}} \cdot \mathbf{k}_{\mathbf{m}} + \boldsymbol{l}_{\mathbf{m}} \cdot \mathbf{k}_{\mathrm{v}})\mathbf{1} + \left\|\boldsymbol{l}_{\mathrm{v}} \times \mathbf{k}_{\mathrm{v}}\right\| \mathbb{1}$	

Table 2.15. These are the Euclidean distances between various combinations of points, lines, and planes in three dimensions, expressed as homogeneous magnitudes.

2.12 Duals

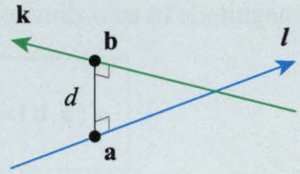
The complement operation exchanges the meanings of full and empty dimensions independently of the metric that has been chosen for an algebra. We can construct another operation that does depend on the metric by simply multiplying by the metric before we take a complement. This produces the *metric dual* of an object $\mathbf{u}$, which is denoted by $\mathbf{u}^{\star}$ with a solid black star written as a superscript. The metric dual of $\mathbf{u}$ is defined as

Dual / Bulk dual

$$\boxed{\mathbf{u}^{\star} = \overline{\mathbf{G}\mathbf{u}},} \tag{2.100}$$

Comparison Chart #2

Closest Points on Skew Lines



Given two skew lines $\boldsymbol{l}$ and $\mathbf{k}$ in 3D space, calculate the point $\mathbf{a}$ lying on $\boldsymbol{l}$ and the point $\mathbf{b}$ lying on $\mathbf{k}$ such that the distance between $\mathbf{a}$ and $\mathbf{b}$ is minimized. Also calculate the shortest distance d between the lines without using the closest points.

<table>
<tr><th>Conventional Methods</th><th>Geometric Algebra</th></tr>
<tr><td>

Let $\boldsymbol{l}(s)=\mathbf{p}+s\mathbf{u}$ and $\mathbf{k}(t)=\mathbf{q}+t\mathbf{v}$ be parametric lines, with the first containing the point $\mathbf{p}$ and running parallel to the direction vector $\mathbf{u}$ and the second containing the point $\mathbf{q}$ and running parallel to the direction vector $\mathbf{v}$. Assume the directions are normalized so that $\lVert\mathbf{u}\rVert=1$ and $\lVert\mathbf{v}\rVert=1$.

</td><td>

Let $\boldsymbol{l}$ and $\mathbf{k}$ be lines as defined in Equation (2.36) with directions $\boldsymbol{l}_{\mathrm{v}}$ and $\mathbf{k}_{\mathrm{v}}$ and moments $\boldsymbol{l}_{\mathrm{m}}$ and $\mathbf{k}_{\mathrm{m}}$. There is no need to assume the lines are unitized.

</td></tr>
<tr><td>

Calculate the direction $\mathbf{z}=\mathbf{u}\times\mathbf{v}$ that is perpendicular to both lines. The distance between the lines is then given by

$$d=\frac{|(\mathbf{p}-\mathbf{q})\cdot\mathbf{z}|}{\lVert\mathbf{z}\rVert}.$$

</td><td>

Using Equation (2.99), calculate the quantities

$$x=|\boldsymbol{l}\vee\mathbf{k}|=|\boldsymbol{l}_{\mathrm{v}}\cdot\mathbf{k}_{\mathrm{m}}+\boldsymbol{l}_{\mathrm{m}}\cdot\mathbf{k}_{\mathrm{v}}|$$

and

$$y=\lVert\boldsymbol{l}\wedge\operatorname{att}(\mathbf{k})\rVert_{\circ}=\lVert\boldsymbol{l}_{\mathrm{v}}\times\mathbf{k}_{\mathrm{v}}\rVert\mathbb{1}.$$

The distance between the lines is $d=x/y$.

</td></tr>
<tr><td>

For the closest points, we need to find the values of the parameters s and t such that $\boldsymbol{l}(s)-\mathbf{k}(t)$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. This condition can be expressed as the pair of dot products

$$(\mathbf{p}+s\mathbf{u}-\mathbf{q}-t\mathbf{v})\cdot\mathbf{u}=0$$
$$(\mathbf{p}+s\mathbf{u}-\mathbf{q}-t\mathbf{v})\cdot\mathbf{v}=0.$$

Write this linear system in matrix form as

$$\begin{bmatrix}1 & -\mathbf{u}\cdot\mathbf{v}\\ \mathbf{u}\cdot\mathbf{v} & -1\end{bmatrix}\begin{bmatrix}s\\ t\end{bmatrix}=\begin{bmatrix}(\mathbf{q}-\mathbf{p})\cdot\mathbf{u}\\ (\mathbf{q}-\mathbf{p})\cdot\mathbf{v}\end{bmatrix}.$$

Solve for s and t by inverting the 2×2 matrix to obtain

$$\begin{bmatrix}s\\ t\end{bmatrix}=\frac{1}{(\mathbf{u}\cdot\mathbf{v})^2-1}\begin{bmatrix}-1 & \mathbf{u}\cdot\mathbf{v}\\ -\mathbf{u}\cdot\mathbf{v} & 1\end{bmatrix}\begin{bmatrix}(\mathbf{q}-\mathbf{p})\cdot\mathbf{u}\\ (\mathbf{q}-\mathbf{p})\cdot\mathbf{v}\end{bmatrix}.$$

The points $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a}=\mathbf{p}+s\mathbf{u}$$
$$\mathbf{b}=\mathbf{q}+t\mathbf{v}.$$

</td><td>

For the closest points, take the direction

$$\mathbf{z}=\boldsymbol{l}_{\mathrm{v}}\times\mathbf{k}_{\mathrm{v}},$$

already calculated above, which is a vector perpendicular to both lines. Now construct the planes $\mathbf{g}=\boldsymbol{l}\wedge\mathbf{z}$ and $\mathbf{h}=\mathbf{k}\wedge\mathbf{z}$. The points $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a}=\mathbf{h}\vee\boldsymbol{l}$$
$$\mathbf{b}=\mathbf{g}\vee\mathbf{k}.$$

</td></tr>
</table>

and we will normally just call this "the dual" of $\mathbf{u}$. The fact that we are using the right complement here instead of the left is an arbitrary choice, and it's made so that our results are consistent with the duals arising from our choice of basis elements and the single complement existing in the five-dimensional algebra discussed in Chapter 4. This choice exists only in even numbers of dimensions, and its effects are limited to changing the sign of some of the objects we work with, which does not change the meaning of those objects in a homogeneous setting.

Naturally, if we can define one dual with the metric $\mathbf{G}$, then we can define another dual with the antimetric $\mathbb{G}$. The *metric antidual* of an object $\mathbf{u}$, denoted by $\mathbf{u}^{\bigwhitestar}$ with an empty white star in the superscript, is defined as

Antidual / Weight dual

$$\boxed{\mathbf{u}^{\bigwhitestar} = \overline{\mathbb{G}\mathbf{u}}.} \tag{2.101}$$

Again, we will normally drop the word "metric" and simply call this "the antidual" of $\mathbf{u}$. We can also arrive at this definition by applying De Morgan's law to the definition of the dual in Equation (2.100). If we take the left complement of $\mathbf{u}$, apply the dual, and then take the right complement of the result, we have

$$\overline{\underline{\mathbf{u}}^{\bigstar}} = \overline{\overline{\mathbf{G}\underline{\mathbf{u}}}} = \overline{\mathbb{G}\mathbf{u}}, \tag{2.102}$$

where the last step makes use of the definition of the antimetric $\mathbb{G}$ given in Equation (2.63). The definition of the antimetric also tells us that $\mathbf{u}^{\bigwhitestar} = \mathbf{G}\overline{\mathbf{u}}$, which is just the application of the right complement and the metric in the opposite order than it was applied in Equation (2.100).

When the metric $\mathbf{G}$ is the identity, the dual and antidual are equal to each other, and they are both the same as the right complement. When the metric is not the identity but is still orthogonal, the duals can negate or permute basis elements in ways that the complement does not. We will see a metric that has these effects in Chapter 4. As long as the metric is invertible, the dual and antidual are either equal or negatives of each other because $\mathbf{G}\mathbb{G} = \pm\mathbf{I}$. However, when we have a degenerate metric as we do in the projective space, the dual and antidual split nontrivially into two separate operations. In this case, the values of $\overline{\mathbf{G}\mathbf{u}}$ and $\overline{\mathbb{G}\mathbf{u}}$ are the complements of the bulk and weight of $\mathbf{u}$, respectively, which we can express as

$$\mathbf{u}^{\bigstar} = \overline{\mathbf{u}_{\bullet}} \quad \text{and} \quad \mathbf{u}^{\bigwhitestar} = \overline{\mathbf{u}_{\circ}}. \tag{2.103}$$

The solid black star and empty white star are paired with the solid black circle and empty white circle corresponding to the components having their complement taken. Because each of the two dual operations keep either the bulk or the weight, we often prefer to call $\mathbf{u}^{\bigstar}$ the *bulk dual* of $\mathbf{u}$, and we prefer to call $\mathbf{u}^{\bigwhitestar}$ the *weight dual* of $\mathbf{u}$ in settings where the metric is degenerate. Since the complement operation has the effect of exchanging which components belong to the bulk and weight, the bulk dual of $\mathbf{u}$ has only weight components, and the weight dual of $\mathbf{u}$ has only bulk components.

For the sake of completeness, we also define duals that are based on the left complement instead of the right complement. It won't be necessary to use these duals except in situations where we can point out the fact that additional operations have left and right variants. When we need to make a distinction, we will call the bulk dual $\mathbf{u}^{\bigstar}$ and the weight dual $\mathbf{u}^{\bigwhitestar}$ that we have already defined the *right* bulk dual and *right* weight dual. We denote the *left* bulk dual of an object $\mathbf{u}$ as $\mathbf{u}_{\bigstar}$ with a solid star in the subscript instead of the superscript to match the position of the left complement notation. The right complement is denoted by an overbar, and a right dual is denoted by a star in the superscript. The left complement is denoted by an underbar, and a left dual is thus denoted by a star in the subscript. The *left* weight dual of an object $\mathbf{u}$ is therefore denoted by $\mathbf{u}_{\bigwhitestar}$. The effects of all four dual operations[5] on the 16 basis elements of the 4D projective algebra are shown in Table 2.16.

[5] The right bulk dual $\mathbf{u}^{\bigstar}$ that we have defined is also known as the *Hodge dual* of $\mathbf{u}$, and it is usually written with the Hodge star operator notation $\bigstar\mathbf{u}$. Our notation provides the greater versatility necessary upon the realization that this particular dual is but one of four related operations.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0
$\mathbf{u}_{\bigstar}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_4$	0
$\mathbf{u}^{\bigwhitestar}$	0	0	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{u}_{\bigwhitestar}$	0	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$

Table 2.16. For each of the 16 basis elements $\mathbf{u}$ in the 4D projective exterior algebra, this table lists the right bulk dual $\mathbf{u}^{\bigstar}$, the left bulk dual $\mathbf{u}_{\bigstar}$, the right weight dual $\mathbf{u}^{\bigwhitestar}$, and the left weight dual $\mathbf{u}_{\bigwhitestar}$.

The bulk duals and weight duals of points, lines, and planes are listed in Table 2.17. Note that because the bulk dual of each object has no bulk components, it must contain the origin. In a symmetric manner, the weight dual of each object has no weight components and therefore must lie in the horizon. In general, the weight dual of an object $\mathbf{u}$ is another object containing all points at infinity in directions perpendicular to $\mathbf{u}$ itself. The weight dual of a plane is the point at infinity in the direction of the plane's normal. The weight dual of a line is the line at infinity containing all points at infinity in directions perpendicular to the original line's direction. The weight dual of a point is the entire horizon. These properties are particularly important for the projection operations discussed in the next section.

The bulk dual and weight dual operations distribute over the wedge product and antiwedge product in a particular way. First, we consider the bulk dual of a wedge product $\mathbf{a} \wedge \mathbf{b}$, which can be expanded as

$$(\mathbf{a} \wedge \mathbf{b})^{\bigstar} = \overline{\mathbf{G}(\mathbf{a} \wedge \mathbf{b})}. \tag{2.104}$$

Because the metric $\mathbf{G}$ is an exomorphism, we can distribute it across the wedge product and write

$$(\mathbf{a} \wedge \mathbf{b})^{\bigstar} = \overline{(\mathbf{Ga}) \wedge (\mathbf{Gb})}. \tag{2.105}$$

Applying De Morgan's law for the wedge product, the right side is transformed into an antiwedge product of complements as

$$(\mathbf{a} \wedge \mathbf{b})^{\bigstar} = \overline{\mathbf{Ga}} \vee \overline{\mathbf{Gb}}. \tag{2.106}$$

Each of the factors on the right side matches the definition of the bulk dual given by Equation (2.100), so we now have the identity

$$\boxed{(\mathbf{a} \wedge \mathbf{b})^{\bigstar} = \mathbf{a}^{\bigstar} \vee \mathbf{b}^{\bigstar}.} \tag{2.107}$$

This is a form of De Morgan's law for the wedge product that makes use of duals instead of complements. If the metric is nondegenerate, then we can apply the inverse dual to both sides so that

Type	**Bulk Dual**	**Weight Dual**
Point $\mathbf{p}$	$\mathbf{p}^{\bigstar} = p_x \mathbf{e}_{423} + p_y \mathbf{e}_{431} + p_z \mathbf{e}_{412}$	$\mathbf{p}^{\bigwhitestar} = p_w \mathbf{e}_{321}$
Line $\boldsymbol{l}$	$\boldsymbol{l}^{\bigstar} = -l_{mx} \mathbf{e}_{41} - l_{my} \mathbf{e}_{42} - l_{mz} \mathbf{e}_{43}$	$\boldsymbol{l}^{\bigwhitestar} = -l_{vx} \mathbf{e}_{23} - l_{vy} \mathbf{e}_{31} - l_{vz} \mathbf{e}_{12}$
Plane $\mathbf{g}$	$\mathbf{g}^{\bigstar} = -g_w \mathbf{e}_4$	$\mathbf{g}^{\bigwhitestar} = -g_x \mathbf{e}_1 - g_y \mathbf{e}_2 - g_z \mathbf{e}_3$

Table 2.17. These are the bulk duals and weight duals of geometric objects in the 4D projective exterior algebra representing 3D Euclidean space.

only $\mathbf{a} \wedge \mathbf{b}$ appears on the left, but this is not possible when the metric is degenerate because the dual cannot be inverted in that case.

We can follow the same procedure for the weight dual of an antiwedge product $\mathbf{a} \vee \mathbf{b}$. Expanding the definition of weight dual, distributing the antimetric, and applying De Morgan's law for the antiwedge product gives us

$$\begin{aligned}(\mathbf{a} \vee \mathbf{b})^{\text{☆}} &= \overline{\mathbb{G}(\mathbf{a} \vee \mathbf{b})} \\ &= \overline{(\mathbb{G}\mathbf{a}) \vee (\mathbb{G}\mathbf{b})} \\ &= \overline{\mathbb{G}\mathbf{a}} \wedge \overline{\mathbb{G}\mathbf{b}}.\end{aligned} \tag{2.108}$$

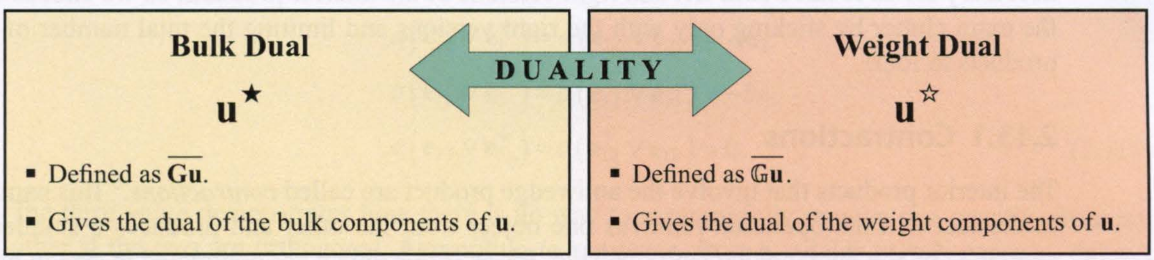
This leads to the identity

$$\boxed{(\mathbf{a} \vee \mathbf{b})^{\text{☆}} = \mathbf{a}^{\text{☆}} \wedge \mathbf{b}^{\text{☆}},} \tag{2.109}$$

which is a form of De Morgan's law for the antiwedge product that makes use of antiduals instead of complements. Again, it is not possible to move all dual operations to the right side if the metric is degenerate.

Math Library Notes

- The right bulk dual and weight dual operations are implemented by the `BulkDual()` and `WeightDual()` functions.



2.13 Interior Products

As discussed in Section 2.5, the join and meet operations combine geometric objects in two different ways. The join of objects **a** and **b** produces the higher-dimensional geometry containing both objects, and its grade is the sum of $\operatorname{gr}(\mathbf{a})$ and $\operatorname{gr}(\mathbf{b})$. The meet of objects **a** and **b** produces the lower-dimensional geometry contained by both objects, and its antigrade is the sum of $\operatorname{ag}(\mathbf{a})$ and $\operatorname{ag}(\mathbf{b})$. There are additional ways to combine two geometric objects by applying a dual operation to one of them first, and this causes grades or antigrades to be subtracted. With the wedge product, antiwedge product, and two dual operations at our disposal, we can combine objects **a** and **b** in four distinct ways to produce $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$, $\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$, $\mathbf{a} \vee \mathbf{b}^{\bigstar}$, and $\mathbf{a} \vee \mathbf{b}^{\text{☆}}$. These are generally known as *interior products*, but we also give each of them the more specific name shown in Table 2.18. All four operations have geometric meanings that we discuss in this section.

In our interior products, the dual operation always appears on the right operand because we defined our duals using the right complement. We can be more specific with the terminology by calling them *right* interior products. It is possible to define four more interior products for which the dual operation appears on the left operand, but that requires using dual operations based on the

Interior Product	Definition
Bulk expansion	$\mathbf{a} \wedge \mathbf{b}^{\bigstar}$
Weight expansion	$\mathbf{a} \wedge \mathbf{b}^{\medwhitestar}$
Bulk contraction	$\mathbf{a} \vee \mathbf{b}^{\bigstar}$
Weight contraction	$\mathbf{a} \vee \mathbf{b}^{\medwhitestar}$

Table 2.18. There are four right interior products that combine a wedge product or antiwedge product with a right bulk dual or weight dual operation.

left complement for proper symmetry. Doing so gives us the additional *left* interior products $\mathbf{b}_{\bigstar} \wedge \mathbf{a}$, $\mathbf{b}_{\medwhitestar} \wedge \mathbf{a}$, $\mathbf{b}_{\bigstar} \vee \mathbf{a}$, and $\mathbf{b}_{\medwhitestar} \vee \mathbf{a}$, where we have intentionally written $\mathbf{b}$ on the left to keep the same name for the dualized operand. The only difference between right and left interior products is a possible change in sign that depends on the grades of the operands. Specifically, the left and right interior products are related by

$$\mathbf{b}_{*} \wedge \mathbf{a} = (-1)^{\operatorname{ag}(\mathbf{b})[\operatorname{ag}(\mathbf{a})+\operatorname{ag}(\mathbf{b})]}\, \mathbf{a} \wedge \mathbf{b}^{*} \tag{2.110}$$

and

$$\mathbf{b}_{*} \vee \mathbf{a} = (-1)^{\operatorname{gr}(\mathbf{b})[\operatorname{gr}(\mathbf{a})+\operatorname{gr}(\mathbf{b})]}\, \mathbf{a} \vee \mathbf{b}^{*}, \tag{2.111}$$

where the asterisk is a placeholder for either the bulk dual or weight dual operation. It won't be necessary for us to have both left and right versions of the interior products, so we choose to avoid the extra clutter by sticking only with the right versions and limiting the total number of interior products to four.

2.13.1 Contractions

The interior products that involve the antiwedge product are called *contractions*.[6] This name is due to the fact that the operation removes one object from the other and produces a smaller object contained by the operand that isn't dualized. It's as if this object has contracted in size to become a lower-dimensional subspace of itself. Depending on which dual operation is involved, there are two different contraction operations. The *bulk contraction* of $\mathbf{a}$ with $\mathbf{b}$ is defined as

Bulk contraction

$$\boxed{\operatorname{bulk\ contraction}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \vee \mathbf{b}^{\bigstar},} \tag{2.112}$$

and it applies the bulk dual to the second operand. When we apply the weight dual to the second operand instead, we get the *weight contraction*, which is defined as

Weight contraction

$$\boxed{\operatorname{weight\ contraction}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \vee \mathbf{b}^{\medwhitestar}.} \tag{2.113}$$

When the metric is the identity, these two contractions are identical, and we simply call them "the contraction". When the metric is degenerate, they produce different results that have the specific applications discussed below.

[6] A notation in which $\mathbf{a} \rfloor \mathbf{b}$ means left contraction and $\mathbf{a} \lfloor \mathbf{b}$ means right contraction can be found in many places throughout the literature. Aside from its deplorable typographical qualities, this notation is not adequate for distinguishing between bulk and weight contractions, it unnecessarily obfuscates the basic underlying operations, and it lacks an analogous notation for expansions.

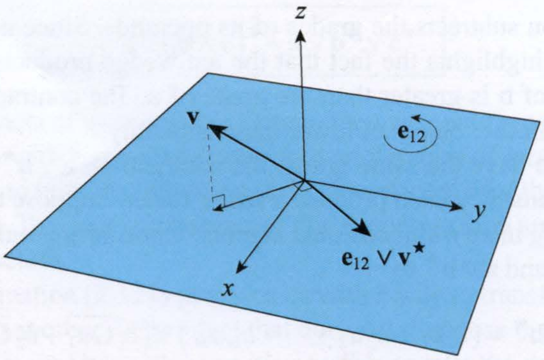


Figure 2.13. The contraction $\mathbf{e}_{12} \vee \mathbf{v}^{\bigstar}$ removes the vector $\mathbf{v}$ from the bivector $\mathbf{e}_{12}$. This produces a new vector that is contained in $\mathbf{e}_{12}$ and is orthogonal to $\mathbf{v}$.

First, to get a feeling for what the contraction actually does, we examine its properties in 3D Euclidean space where $\mathbb{1} = \mathbf{e}_{123}$, the metric is the 3×3 identity matrix, and both dual operations are equivalent to the complement operation. Let us consider the contraction of the bivector $\mathbf{e}_{12}$ with the arbitrary vector $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, which is illustrated in Figure 2.13. Distributing over the vector's components, we can write this contraction as

$$\mathbf{e}_{12} \vee \mathbf{v}^{\bigstar} = a\left(\mathbf{e}_{12} \vee \mathbf{e}_1^{\bigstar}\right) + b\left(\mathbf{e}_{12} \vee \mathbf{e}_2^{\bigstar}\right) + c\left(\mathbf{e}_{12} \vee \mathbf{e}_3^{\bigstar}\right) \tag{2.114}$$

and calculate each term independently. This gives us the three values

$$\begin{aligned} a\left(\mathbf{e}_{12} \vee \mathbf{e}_1^{\bigstar}\right) &= a\left(\mathbf{e}_{12} \vee \mathbf{e}_{23}\right) = a\mathbf{e}_2 \\ b\left(\mathbf{e}_{12} \vee \mathbf{e}_2^{\bigstar}\right) &= b\left(\mathbf{e}_{12} \vee \mathbf{e}_{31}\right) = -b\mathbf{e}_1 \\ c\left(\mathbf{e}_{12} \vee \mathbf{e}_3^{\bigstar}\right) &= c\left(\mathbf{e}_{12} \vee \mathbf{e}_{12}\right) = 0. \end{aligned} \tag{2.115}$$

The third value demonstrates how there is no way to meaningfully remove one quantity from another if the two are orthogonal. Attempting to remove $\mathbf{e}_3$ from $\mathbf{e}_{12}$ yields zero because $\mathbf{e}_{12}$ doesn't contain any component parallel to $\mathbf{e}_3$ to begin with. In general, a contraction throws away parts that are perpendicular to each other. This leaves the first two values, which arise from the components of $\mathbf{v}$ that actually do lie in $\mathbf{e}_{12}$. When we contract $\mathbf{e}_{12}$ with the $\mathbf{e}_1$ and $\mathbf{e}_2$ components of $\mathbf{v}$, those components are removed, and we are left with the orthogonal subspace of $\mathbf{e}_{12}$, giving us

$$\mathbf{e}_{12} \vee \mathbf{v}^{\bigstar} = -b\mathbf{e}_1 + a\mathbf{e}_2. \tag{2.116}$$

The overall effect is that the vector $\mathbf{v}$ is projected onto $\mathbf{e}_{12}$, and that projection is then removed from $\mathbf{e}_{12}$ to produce a new vector that lies in $\mathbf{e}_{12}$ but is perpendicular to $\mathbf{v}$. In this case, the projection of $\mathbf{v}$ onto $\mathbf{e}_{12}$ is rotated 90 degrees counterclockwise about the z axis, matching the orientation of $\mathbf{e}_{12}$. Had we contracted with $\mathbf{e}_{21}$ instead, with the opposite orientation, then the projection of $\mathbf{v}$ would have been rotated clockwise.

The antiwedge product has the effect of summing the antigrades of its operands, but the contraction behaves differently because one of the operands is dualized. In the contractions $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ and $\mathbf{a} \vee \mathbf{b}^{\star}$, the antigrades of $\mathbf{b}^{\bigstar}$ and $\mathbf{b}^{\star}$ are both $n - \operatorname{ag}(\mathbf{b})$, where n is the number of dimensions, so the antigrade of the whole expression is $\operatorname{ag}(\mathbf{a}) + n - \operatorname{ag}(\mathbf{b})$. If we subtract this from n to obtain the grade instead and replace each antigrade $\operatorname{ag}(\mathbf{u})$ with $n - \operatorname{gr}(\mathbf{u})$, then we arrive at the relationship

$$\operatorname{gr}\left(\mathbf{a} \vee \mathbf{b}^{\bigstar}\right) = \operatorname{gr}\left(\mathbf{a} \vee \mathbf{b}^{\star}\right) = \operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b}). \tag{2.117}$$

Thus, a contraction subtracts the grades of its operands. Since nothing can have a negative grade, Equation (2.117) highlights the fact that the antiwedge products $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ and $\mathbf{a} \vee \mathbf{b}^{\star}$ are identically zero if the grade of $\mathbf{b}$ is greater than the grade of $\mathbf{a}$. The contraction does not allow us to remove one object from another object of lower dimensionality.

When $\mathbf{a}$ and $\mathbf{b}$ have the same grade, the contractions $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ and $\mathbf{a} \vee \mathbf{b}^{\star}$ must each have grade zero, and that means they each produce a scalar result. Suppose that $\mathbf{a}$ and $\mathbf{b}$ both have grade k and that we're working in an n-dimensional algebra. Considering only the bulk contraction for the moment, we can expand $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ to

$$\mathbf{a} \vee \mathbf{b}^{\bigstar} = (a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_m \mathbf{u}_m) \vee \left(b_1 \overline{\mathbf{G}\mathbf{u}_1} + b_2 \overline{\mathbf{G}\mathbf{u}_2} + \cdots + b_m \overline{\mathbf{G}\mathbf{u}_m} \right), \tag{2.118}$$

where $m = \binom{n}{k}$, and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are the m basis elements of grade k. Each term of the expanded antiwedge product on the right side has the form $a_i b_j \left(\mathbf{u}_i \vee \overline{\mathbf{G}\mathbf{u}_j} \right)$, which is nonzero only when $\mathbf{G}\mathbf{u}_j$ itself contains a $\mathbf{u}_i$ component that causes the antiwedge product to be $\mathbf{1}$. What this means is that the components of $\mathbf{a}$ and $\mathbf{b}$ are paired and summed in exactly the same way as they are in the definition of the dot product given by Equation (2.74), and we thus have the identity

$$\boxed{\mathbf{a} \vee \mathbf{b}^{\bigstar} = \mathbf{a} \bullet \mathbf{b}, \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}).} \tag{2.119}$$

The bulk contraction reduces to the dot product when its operands have the same grade. Something similar involving the antidot product happens for the weight contraction, but it must still produce a scalar quantity. If we expand $\mathbf{a} \vee \mathbf{b}^{\star}$ in the same way that we expanded the bulk contraction, then the metric $\mathbf{G}$ appearing in Equation (2.118) is replaced by the antimetric $\mathbb{G}$. The components of $\mathbf{a}$ and $\mathbf{b}$ are paired and summed in exactly the same way as they are in the definition of the antidot product given by Equation (2.75), but we get a scalar result instead of an antiscalar, so it's not exactly the antidot product to which the weight contraction reduces when its operands have the same grade. Nevertheless, we can write the identity

$$\boxed{\mathbf{a} \vee \mathbf{b}^{\star} = (\mathbf{a} \circ \mathbf{b}) \vee \mathbf{1}, \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}).} \tag{2.120}$$

It may seem like this presents a small asymmetry that indicates a possible problem with the duality of the interior products, but the bulk and weight contractions are not actually duals of each other. We have exchanged the bulk dual and weight dual operations on the second operand, but to find the correct dual of the whole expression, we also need to replace the antiwedge product with the wedge product. The interior product that reduces exactly to the antidot product as an antiscalar for operands of equal grade is the weight expansion, which is discussed below in Section 2.13.5.

All values produced by the bulk and weight contractions between basis elements in the 4D projective algebra are shown in Table 2.19. The bulk contraction $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ is identically zero whenever $\mathbf{b}$ has a factor of $\mathbf{e}_4$, and the weight contraction $\mathbf{a} \vee \mathbf{b}^{\star}$ is identically zero whenever $\mathbf{b}$ does not have a factor of $\mathbf{e}_4$. This means that only one of the two contractions can be nonzero for any specific value of $\mathbf{b}$. The nonzero values of the bulk contraction are highlighted in green in the table, and the nonzero values of the weight contraction are highlighted in purple. Entries for which the bulk contraction coincides with the dot product or the weight contraction coincides with the scalar version of the antidot product are highlighted in a darker shade. The metric shown in Equation (2.61) is visible in the dark green cells along the diagonal.

There are a few interesting properties that we can derive about the bulk contraction. First, it's very easy to show that the bulk contraction with a value that can be expressed as a wedge product of multiple factors can be transformed into repeated contractions with each of the factors. It follows directly from Equation (2.107) that we can write

$$\boxed{\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c})^{\star} = \mathbf{a} \vee \mathbf{b}^{\star} \vee \mathbf{c}^{\star}.} \tag{2.121}$$

If $\mathbf{b}$ and $\mathbf{c}$ are wedge products of even smaller factors, then we could repeat the process until we've transformed a contraction with a simple k-vector into an iteration of k contractions with its grade-one factors. It's important to note that Equation (2.121) works only for the bulk contraction and not for the weight contraction. The dual counterpart of this decomposition applies to the weight expansion, which is discussed below.

The relationship in Equation (2.121) provides us with a way to transform a dot product involving a bulk contraction into another dot product that does not. Suppose that we have three quantities $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b}) = \operatorname{gr}(\mathbf{c})$ so it's possible to take the dot product $\left(\mathbf{a} \vee \mathbf{b}^{\star}\right) \bullet \mathbf{c}$ and get a nonzero result. Since the dot product is equivalent to a bulk contraction, we can rewrite it and apply Equation (2.121) to obtain

$$\begin{aligned}\left(\mathbf{a} \vee \mathbf{b}^{\star}\right) \bullet \mathbf{c} &= \mathbf{a} \vee \mathbf{b}^{\star} \vee \mathbf{c}^{\star} \\ &= \mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c})^{\star}.\end{aligned} \tag{2.122}$$

The quantities $\mathbf{a}$ and $\mathbf{b} \wedge \mathbf{c}$ have the same grade, so we can turn the bulk contraction on the right side back into a dot product and write

$$\boxed{\left(\mathbf{a} \vee \mathbf{b}^{\star}\right) \bullet \mathbf{c} = \mathbf{a} \bullet (\mathbf{b} \wedge \mathbf{c}), \qquad \text{when } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}) + \operatorname{gr}(\mathbf{c}).} \tag{2.123}$$

Bulk and Weight Contraction — $\mathbf{a} \vee \mathbf{b}^{\star}$ / $\mathbf{a} \bullet \mathbf{b}$ / $\mathbf{a} \vee \mathbf{b}^{\text{☆}}$ / $(\mathbf{a} \circ \mathbf{b}) \vee \mathbb{1}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_2$	$\mathbf{e}_2$	0	$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_3$	$\mathbf{e}_3$	0	0	$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_4$	$\mathbf{e}_4$	0	0	0	$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	0	$\mathbf{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	0	0	$\mathbf{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	0	0	0	$\mathbf{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	0	0	0	0	$\mathbf{1}$	0	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	0	0	0	0	0	$\mathbf{1}$	0	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_4$	0	0	$\mathbf{1}$	0	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_4$	0	0	$\mathbf{1}$	0	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	0	$\mathbf{e}_4$	0	0	$\mathbf{1}$	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	0	0	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	0	0	$\mathbf{1}$	0
$\mathbb{1}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$

Table 2.19. These are the values of the bulk contraction $\mathbf{a} \vee \mathbf{b}^{\star}$ and weight contraction $\mathbf{a} \vee \mathbf{b}^{\text{☆}}$ between all pairs of basis elements $\mathbf{a}$ and $\mathbf{b}$ in the 4D projective exterior algebra. The values of the bulk contraction are highlighted in green, and the values of the weight contraction are highlighted in purple. The darker green cells along the diagonal correspond to the nonzero entries of the metric $\mathbf{G}$.

We now turn to the more difficult case of $(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{v}^{\star}$ in which the left operand of the bulk contraction is a wedge product of two factors. The right operand is limited to a vector quantity, and we use the letter $\mathbf{v}$ to emphasize that it has grade one. The following derivation only works for vector quantities on the right, but as we already know, any contraction can be decomposed into a sequence of contractions with grade-one vectors. The result we obtain below can be repeated for any additional factors.

Suppose $\mathbf{a} \wedge \mathbf{b}$ is a simple k-vector, and let $\mathbf{a} \wedge \mathbf{b} = \mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k$, where each $\mathbf{d}_i$ is a grade-one vector. Because $\mathbf{v}$ has grade one, the result of the bulk contraction $(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^{\star}$ has grade $k-1$. We will employ a small trick and take the dot product with an arbitrary quantity $\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}$ that also has grade $k-1$. Using the identity given by Equation (2.123), we can rearrange this dot product as

$$\left[(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^{\star}\right] \bullet (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}) = (\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \bullet (\mathbf{v} \wedge \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}). \tag{2.124}$$

We now make use of Equation (2.85) to expand the dot product of the two k-vectors on the right side and write this as

$$\begin{aligned}&\left[(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^{\star}\right] \bullet (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}) \\ &\quad = \left[\sum_{i=1}^{k} (-1)^{i-1} (\mathbf{d}_i \bullet \mathbf{v})(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_{i-1} \wedge \mathbf{d}_{i+1} \wedge \cdots \mathbf{d}_k)\right] \bullet (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}).\end{aligned} \tag{2.125}$$

Here, both sides of the equation are dot products between the $(k-1)$-vector that we care about and the arbitrary $(k-1)$-vector $\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}$. The equation must hold for any value of $\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1}$, and in particular, it must hold for the grade-k basis elements. This means that for any basis element that is not annihilated by the metric $\mathbf{G}$, the corresponding component of the left operands of the two dot products must be equal to each other. Altogether, those components make up the bulks of the left operands, so we can infer the equivalence

$$\left[(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^{\star}\right]_{\bullet} = \left[\sum_{i=1}^{k} (-1)^{i-1} (\mathbf{d}_i \bullet \mathbf{v})(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_{i-1} \wedge \mathbf{d}_{i+1} \wedge \cdots \mathbf{d}_k)\right]_{\bullet}. \tag{2.126}$$

If the metric is nondegenerate, then the bulk includes all the components, and the solid circle subscripts are not necessary. To keep things tidy, we drop the bulk notation for the rest of this derivation with the understanding that we are working with only the bulks in each equation. It will be added back to the final result.

Continuing with Equation (2.126), we can divide the summation on the right side into two parts, one consisting of the first j terms and one consisting of all the terms that remain. Every term of the first part has a trailing factor of $\mathbf{d}_{j+1} \wedge \cdots \wedge \mathbf{d}_k$, and every term of the second part has a leading factor of $\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_j$. When we write the summations separately with those factors pulled out on the appropriate sides, we have

$$\begin{aligned}&(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^{\star} \\ &\quad = \left[\sum_{i=1}^{j} (-1)^{i-1} (\mathbf{d}_i \bullet \mathbf{v})(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_{i-1} \wedge \mathbf{d}_{i+1} \wedge \cdots \mathbf{d}_j)\right] \wedge (\mathbf{d}_{j+1} \wedge \cdots \wedge \mathbf{d}_k) \\ &\quad + (\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_j) \wedge \left[\sum_{i=j+1}^{k} (-1)^{i-1} (\mathbf{d}_i \bullet \mathbf{v})(\mathbf{d}_{j+1} \wedge \cdots \wedge \mathbf{d}_{i-1} \wedge \mathbf{d}_{i+1} \wedge \cdots \mathbf{d}_k)\right].\end{aligned} \tag{2.127}$$

Each of the summations inside the brackets has the same form as the right side of Equation (2.126), so we can replace them with bulk contractions to obtain

$$(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^\star = \left[(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_j) \vee \mathbf{v}^\star\right] \wedge (\mathbf{d}_{j+1} \wedge \cdots \wedge \mathbf{d}_k)$$
$$+(-1)^j (\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_j) \wedge \left[(\mathbf{d}_{j+1} \wedge \cdots \wedge \mathbf{d}_k) \vee \mathbf{v}^\star\right]. \tag{2.128}$$

The factor of $(-1)^j$ appearing in this equation accounts for whether the first term of the second summation was positive or negative. When we set $\mathbf{a} = \mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_j$ and $\mathbf{b} = \mathbf{d}_{j+1} \wedge \cdots \wedge \mathbf{d}_k$, we arrive at the result

$$\boxed{\left[(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{v}^\star\right]_\bullet = \left[(\mathbf{a} \vee \mathbf{v}^\star) \wedge \mathbf{b} + (-1)^{\operatorname{gr}(\mathbf{a})}\, \mathbf{a} \wedge (\mathbf{b} \vee \mathbf{v}^\star)\right]_\bullet .} \tag{2.129}$$

Remember that $\mathbf{v}$ is a grade-one vector in this identity, but $\mathbf{a}$ and $\mathbf{b}$ can have any grade.

In the case that $\mathbf{a}$ and $\mathbf{b}$ are both grade-one vectors in a Euclidean space where the metric is the identity, Equation (2.129) simplifies to

$$(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{v}^\star = (\mathbf{a} \bullet \mathbf{v})\,\mathbf{b} - (\mathbf{b} \bullet \mathbf{v})\,\mathbf{a}. \tag{2.130}$$

If this identity seems familiar, it's due to it being a generalization of the conventional vector triple product $\mathbf{v} \times (\mathbf{a} \times \mathbf{b})$ in a 3D vector space. The wedge product $\mathbf{a} \wedge \mathbf{b}$ creates a bivector, and the antiwedge product with $\mathbf{v}^\star$ knocks that back down to a vector. In three dimensions, the per-coordinate calculations for both of those products look like cross products. Now that we have expressed Equation (2.130) in terms of the operations of an exterior algebra, it applies more generally to vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{v}$ having any dimensionality.

2.13.2 Projection and Rejection

Suppose now that the metric is invertible (but not necessarily the identity), $\mathbf{a}$ is a grade-one vector, the vectors $\mathbf{v}$ and $\mathbf{a}$ are the same, and $\mathbf{b}$ has any grade. Then Equation (2.129) becomes

$$(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{a}^\star = (\mathbf{a} \bullet \mathbf{a})\,\mathbf{b} - \mathbf{a} \wedge (\mathbf{b} \vee \mathbf{a}^\star). \tag{2.131}$$

Assuming $\mathbf{a} \bullet \mathbf{a} \neq 0$, this can be rearranged to

$$\mathbf{b} = \frac{1}{\mathbf{a} \bullet \mathbf{a}}\left[\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{a}^\star) + (\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{a}^\star\right]. \tag{2.132}$$

This equation decomposes $\mathbf{b}$ into two separate components with respect to the vector $\mathbf{a}$. Notice that both components have the same factors and the same multiplication operations, but the order in which the products are evaluated is different. The first component must contain the vector $\mathbf{a}$, and it must contain the result of the contraction $\mathbf{b} \vee \mathbf{a}^\star$ that removes $\mathbf{a}$ from $\mathbf{b}$. This component is the *projection* of $\mathbf{b}$ onto $\mathbf{a}$, which we denote by $\mathbf{b}_{\parallel \mathbf{a}}$ with parallel bars in the subscript to indicate that it is the part of $\mathbf{b}$ that is parallel to $\mathbf{a}$. We take this to be the definition of projection when the metric is invertible and write

Projection onto vector

$$\boxed{\mathbf{b}_{\parallel \mathbf{a}} = \frac{\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{a}^\star)}{\mathbf{a} \bullet \mathbf{a}}.} \tag{2.133}$$

The second component in Equation (2.132) contains everything that is not part of the projection of $\mathbf{b}$ onto $\mathbf{a}$. This component cannot contain anything parallel to the vector $\mathbf{a}$ because that particular direction is removed by the contraction operation. Thus, the second component represents all the parts of $\mathbf{b}$ that are perpendicular to $\mathbf{a}$. We call this component the *rejection* of $\mathbf{b}$ from $\mathbf{a}$ and denote it by $\mathbf{b}_{\perp \mathbf{a}}$ with a perpendicular symbol in the subscript, following the notation used for projection. As before, we take this to be the definition of rejection and write

Rejection from vector

$$\boxed{\mathbf{b}_{\perp\mathbf{a}} = \frac{(\mathbf{a} \wedge \mathbf{b}) \vee \mathbf{a}^{\star}}{\mathbf{a} \bullet \mathbf{a}}.} \tag{2.134}$$

When the metric is the identity, the bulk dual operation reduces to the right complement. In this case, Equation (2.132) becomes a way of separating $\mathbf{b}$ into components that contain a factor of $\mathbf{a}$ and those that do not. In particular, if $\mathbf{a}$ is a unit vector, then we can write

$$\mathbf{b} = \mathbf{a} \wedge (\mathbf{b} \vee \overline{\mathbf{a}}) + (\mathbf{a} \wedge \mathbf{b}) \vee \overline{\mathbf{a}}. \tag{2.135}$$

The first term on the right side of this equation includes all components of $\mathbf{b}$ that have a factor of $\mathbf{a}$, and the second term includes all components of $\mathbf{b}$ that do not have a factor of $\mathbf{a}$. If we set $\mathbf{a} = \mathbf{e}_n$ in an n-dimensional projective space, then this gives us alternative expressions for the bulk and weight of $\mathbf{b}$ that we can write as

$$\mathbf{b}_{\bullet} = (\mathbf{e}_n \wedge \mathbf{b}) \vee \overline{\mathbf{e}}_n \quad \text{and} \quad \mathbf{b}_{\circ} = \mathbf{e}_n \wedge (\mathbf{b} \vee \overline{\mathbf{e}}_n). \tag{2.136}$$

These are not definitions, but they are identities that sometimes come in handy. We will use one of them to make a transformation below.

2.13.3 Euclidean Angles

When a dot product has been defined, the cosine of the Euclidean angle ϕ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is canonically given by

$$\cos\phi = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}, \tag{2.137}$$

where $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ are each the induced norm defined by Equation (2.86). If we insert the extended dot product $\bullet$ defined earlier by Equation (2.74), then this formula works well for Euclidean vectors, bivectors, trivectors, and so on, producing the correct angle between any such objects of the same grade. However, it does not allow us to measure the angle between two objects $\mathbf{a}$ and $\mathbf{b}$ of different grades because $\mathbf{a} \bullet \mathbf{b}$ is always zero in that case. To calculate the angle between two objects like a vector and a bivector, we need to do some generalizing.

The bulk contraction $\mathbf{a} \vee \mathbf{b}^{\star}$ is equivalent to $\mathbf{a} \bullet \mathbf{b}$ when the grades of $\mathbf{a}$ and $\mathbf{b}$ are the same but also produces meaningful results when the grades are different. Since the contraction projects its dualized operand onto its other operand, as shown earlier in Figure 2.13, it introduces exactly the cosine factor that we need. But now, that factor is incorporated into the magnitude of a nonscalar result, so we are forced to take a norm in order to calculate an angle. That gives us the formula

$$\cos\phi = \frac{\|\mathbf{a} \vee \mathbf{b}^{\star}\|_{\bullet}}{\|\mathbf{a}\|_{\bullet}\|\mathbf{b}\|_{\bullet}} \tag{2.138}$$

for the cosine of the angle ϕ made between two objects $\mathbf{a}$ and $\mathbf{b}$ having different grades. Here, $\mathbf{a}$ must be the object of higher grade or else the contraction is identically zero. The magnitudes in the denominator are both scalar values multiplying the basis element $\mathbf{1}$, so they are implicitly multiplied together with the wedge product.

Since we are taking a norm that always produces a positive number in the numerator, Equation (2.138) can never give us an angle ϕ larger than 90 degrees. While it is possible to think of two objects having the same grade as pointing in roughly the same direction, with an angle less than 90 degrees between them, or in roughly opposite directions, with an angle greater than 90 degrees between them, the same notion does not apply to objects having different grades. A vector can never make an angle larger than 90 degrees with a bivector, for instance. This is reflected in the fact that the contraction between two objects produces a scalar result that can be positive or negative only when the grades of those objects are the same.

When it comes to measuring the angle between any pairing of homogeneous lines and planes in a projective space, all we have to do it plug their attitudes into Equation (2.138). The attitude of a line or plane is a purely Euclidean expression of its directional orientation in space, so we can calculate

$$\cos\phi = \frac{\left\|\operatorname{att}(\mathbf{a}) \vee \operatorname{att}(\mathbf{b})^{\bigstar}\right\|_{\bullet}}{\left\|\operatorname{att}(\mathbf{a})\right\|_{\bullet}\left\|\operatorname{att}(\mathbf{b})\right\|_{\bullet}} \tag{2.139}$$

to obtain the cosine of the angle between two geometric objects **a** and **b**. If the two objects have the same grade, then we don't take the norm in the numerator, and that gives us a signed scalar value.

Equation (2.139) can be simplified in a couple ways, and we can transform the whole expression of $\cos\phi$ into a homogeneous magnitude. First, we rewrite the contraction in the numerator using the definitions of attitude and bulk dual to get

$$\operatorname{att}(\mathbf{a}) \vee \operatorname{att}(\mathbf{b})^{\bigstar} = (\mathbf{a} \vee \overline{\mathbf{e}}_n) \vee \overline{(\mathbf{b} \vee \overline{\mathbf{e}}_n)_{\bullet}}. \tag{2.140}$$

Since the quantity $\mathbf{b} \vee \overline{\mathbf{e}}_n$ cannot contain a factor of $\mathbf{e}_n$, applying the bulk operation to it has no effect, so we can simply drop it. Reassociating the antiwedge products and applying De Morgan's law to the right complement of $\mathbf{b} \vee \overline{\mathbf{e}}_n$ gives us

$$\operatorname{att}(\mathbf{a}) \vee \operatorname{att}(\mathbf{b})^{\bigstar} = \mathbf{a} \vee \left[\overline{\mathbf{e}}_n \vee \left(\overline{\mathbf{b}} \wedge \overline{\overline{\mathbf{e}}}_n\right)\right]. \tag{2.141}$$

According to Equation (2.23), applying the double right complement to $\mathbf{e}_n$ introduces a factor of $(-1)^{n-1}$. If we also reverse the terms of the antiwedge product $\overline{\mathbf{e}}_n \vee \left(\overline{\mathbf{b}} \wedge \mathbf{e}_n\right)$, then it introduces another factor of $(-1)^{\operatorname{gr}(\mathbf{b})-1}$ according to Equation (2.32). Reversing the terms of the wedge product $\overline{\mathbf{b}} \wedge \mathbf{e}_n$ introduces yet another factor of $(-1)^{n-\operatorname{gr}(\mathbf{b})}$ according to Equation (2.18). These three factors accumulate to $(-1)^{2n-2}$, so there is no overall sign change when we rewrite Equation (2.141) as

$$\operatorname{att}(\mathbf{a}) \vee \operatorname{att}(\mathbf{b})^{\bigstar} = \mathbf{a} \vee \left[\left(\mathbf{e}_n \wedge \overline{\mathbf{b}}\right) \vee \overline{\mathbf{e}}_n\right]. \tag{2.142}$$

The part in brackets is now precisely the expression for the bulk of $\overline{\mathbf{b}}$ as given by Equation (2.136). The bulk of the complement is equal to the complement of the weight, which is equal to the weight dual, so we can now write

$$\operatorname{att}(\mathbf{a}) \vee \operatorname{att}(\mathbf{b})^{\bigstar} = \mathbf{a} \vee \overline{\mathbf{b}}_{\bullet} = \mathbf{a} \vee \overline{\mathbf{b}_{\circ}} = \mathbf{a} \vee \mathbf{b}^{\star}. \tag{2.143}$$

That takes care of the numerator in Equation (2.139). In the denominator, we simply recognize that the bulk norm of the attitude of an object **u** is really the weight norm of **u** expressed as a scalar instead of an antiscalar. We can write this relationship as

$$\left\|\mathbf{u}\right\|_{\circ} = \left\|\operatorname{att}(\mathbf{u})\right\|_{\bullet} \wedge \mathbb{1}. \tag{2.144}$$

The fact that Equation (2.139) is a ratio in which the numerator is a scalar and the denominator can now be expressed as the product of two antiscalars leads us to the expression

$$\cos\phi = \left\|\mathbf{a} \vee \mathbf{b}^{\star}\right\|_{\bullet} + \left\|\mathbf{a}\right\|_{\circ}\left\|\mathbf{b}\right\|_{\circ} \tag{2.145}$$

as a homogeneous magnitude representing the Euclidean angle between two projective geometries **a** and **b**. (There is an implicit antiwedge product between the two weight norms.) When **a** and **b** have the same grade, we still want a signed value in the scalar part, so we drop the bulk norm in that case to arrive at

Euclidean angle

$$\cos\phi(\mathbf{a},\mathbf{b}) = \begin{cases} \mathbf{a} \vee \mathbf{b}^{\star} + \|\mathbf{a}\|_{\circ}\|\mathbf{b}\|_{\circ}, & \text{if } \operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}); \\ \left\|\mathbf{a} \vee \mathbf{b}^{\star}\right\|_{\bullet} + \|\mathbf{a}\|_{\circ}\|\mathbf{b}\|_{\circ}, & \text{otherwise.} \end{cases} \tag{2.146}$$

Formulas for the cosines of the Euclidean angles made between the three possible combinations of planes and lines in 3D space are shown in Table 2.20. In the case of two planes, the scalar part of the homogeneous magnitude is simply given by the ordinary dot product between the planes' normal vectors, in conventional terms. In the case of two lines, the situation is similar, and the scalar part is given by the dot product between the lines' direction vectors. The case of a plane and a line is slightly more involved. By taking the magnitude of the cross product between the plane's normal vector and the line's direction vector, we are calculating the sine of the angle between those vectors, which is the cosine of the angle made between the line and the planar surface. That angle can never be larger than 90 degrees, so it makes sense that we have to evaluate a norm that can never be negative.

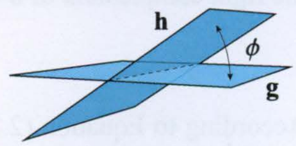
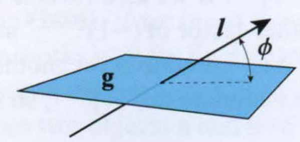
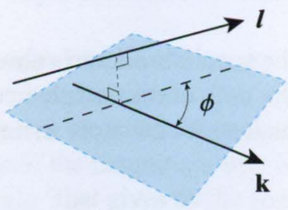
Angle Formula	Illustration
Cosine of angle ϕ between planes $\mathbf{g}$ and $\mathbf{h}$. $\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz} \cdot \mathbf{h}_{xyz})\mathbf{1} + \|\mathbf{g}\|_{\circ}\|\mathbf{h}\|_{\circ}$	
Cosine of angle ϕ between plane $\mathbf{g}$ and line $\boldsymbol{l}$. $\cos\phi(\mathbf{g},\boldsymbol{l}) = \|\mathbf{g}_{xyz} \times \boldsymbol{l}_{\mathbf{v}}\|\mathbf{1} + \|\mathbf{g}\|_{\circ}\|\boldsymbol{l}\|_{\circ}$	
Cosine of angle ϕ between lines $\boldsymbol{l}$ and $\mathbf{k}$. $\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{v}})\mathbf{1} + \|\boldsymbol{l}\|_{\circ}\|\mathbf{k}\|_{\circ}$	

Table 2.20. These are the cosines of the Euclidean angles between lines and planes in three dimensions, expressed as homogeneous magnitudes.

2.13.4 Parametric Forms

The bulk contraction provides us with a way to express each type of geometric object containing more than one point using a parametric form. In the 4D projective algebra, the set of such objects is limited to only lines and planes, but the same formulation will be applicable to a larger variety of objects in Chapter 4.

Let $\mathbf{u}$ represent a geometric object of grade two or higher, and let $\mathbf{p}_0$ be a unitized point that's known to be contained by $\mathbf{u}$. Directional information about $\mathbf{u}$ is stored in its attitude, so it's reasonable to expect that other points contained by $\mathbf{u}$ differ from $\mathbf{p}_0$ by some multiple of $\operatorname{att}(\mathbf{u})$. For a line

$\boldsymbol{l}$, this is clearly the case because $\operatorname{att}(\boldsymbol{l}) = l_{vx}\,\mathbf{e}_1 + l_{vy}\,\mathbf{e}_2 + l_{vz}\,\mathbf{e}_3$ is a vector pointing along the direction of the line. We can just add scalar multiples of this attitude to the known point $\mathbf{p}_0$ to generate every other point on the line in exactly the same way that we did conventionally in Equation (1.15).

For a plane $\mathbf{g}$, the situation is different for a couple of reasons. First, the attitude of a plane is a bivector $\operatorname{att}(\mathbf{g}) = g_x\,\mathbf{e}_{23} + g_y\,\mathbf{e}_{31} + g_z\,\mathbf{e}_{12}$, and second, a single scalar parameter is not sufficient to generate a two-dimensional surface. Anything that we add to the known point $\mathbf{p}_0$ to generate other points in the plane must be a vector, so whatever operation we apply to $\operatorname{att}(\mathbf{g})$ must turn a bivector quantity into a vector quantity. This happens when we make the parameter a vector instead of a scalar and use the contraction to remove that vector parameter from the attitude to leave behind an orthogonal vector that is parallel to the plane. This is exactly what was happening conventionally in Equation (1.18).

The parametric forms for lines and planes differed from each other significantly in Chapter 1, but we are now able to unify them into one form that applies to both lines and planes here and to three more types of objects later in Section 4.6. In general, the parametric form of a geometric object $\mathbf{u}$ having grade k is given by

$$\boxed{\mathbf{p}(\alpha) = \mathbf{p}_0 + \operatorname{att}(\mathbf{u}) \vee \alpha^{\bigstar},} \tag{2.147}$$

where $\mathbf{p}_0$ is a point known ahead of time to be contained in $\mathbf{u}$, and α is an arbitrary parameter having grade $k-2$. (If no point $\mathbf{p}_0$ is known, one can always be found be projecting the origin onto $\mathbf{u}$ as discussed in Section 2.13.6 below.) The contraction always produces a vector quantity because the attitude of $\mathbf{u}$ has grade $k-1$, and the contraction operation subtracts the grade of α from it. A line has grade two, so its parametric form requires that α be a scalar with grade zero. A plane has grade three, so its parametric form requires that α be a vector with grade one. If we were working in a higher number of dimensions, this could be continued to parameters of higher grade. In all cases, α is a purely Euclidean quantity and has no weight components, though if it did, they would be stripped away by the bulk dual operation anyway.

2.13.5 Expansions

The mathematical and conceptual dual to the contraction is the *expansion*, which is an interior product that involves the wedge product instead of the antiwedge product. Contractions and expansions are mutual antiproducts, and we can construct one from the other in the usual manner by applying the right complement to the operands and applying the left complement to the result. When we do this for the bulk contraction $\mathbf{a} \vee \mathbf{b}^{\bigstar}$ defined by Equation (2.112), we get

$$\begin{aligned}
\underline{\overline{\mathbf{a}} \vee \overline{\mathbf{b}}^{\bigstar}} &= \underline{\overline{\mathbf{a}} \vee \overline{\mathbf{G}\overline{\mathbf{b}}}} \\
&= \mathbf{a} \wedge \mathbf{G}\overline{\mathbf{b}} \\
&= \mathbf{a} \wedge \overline{\mathbb{G}\mathbf{b}} \\
&= \mathbf{a} \wedge \mathbf{b}^{\star}.
\end{aligned} \tag{2.148}$$

The outcome of this short derivation is that the antiwedge product has been replaced by the wedge product, and the bulk dual has been replaced by the weight dual. This new operation is called the *weight expansion*, which we explicitly define as

Weight expansion

$$\boxed{\text{weight expansion}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \wedge \mathbf{b}^{\star}.} \tag{2.149}$$

When we apply the same procedure to the weight contraction defined by Equation (2.113), we obtain the final combination of wedge product and bulk dual from Table 2.18. This operation is called the *bulk expansion*, defined as

Bulk expansion

$$\boxed{\text{bulk expansion}\left(\mathbf{a},\mathbf{b}\right)=\mathbf{a}\wedge\mathbf{b}^{\bigstar}.} \tag{2.150}$$

As with contractions, these two expansions are identical when the metric is the identity. In that case, we simply call them "the expansion". When the metric is degenerate, they produce different results due to the way that the duals select different components of the second operand.

Recall that a contraction transforms an object **a** into a lower-grade quantity that it contains by removing any parts of another object **b** that are parallel to it. The object **a** contracts into a smaller part of itself. In the opposite sense, an expansion transforms an object **a** into a higher-grade quantity that contains it by adding any parts of another object **b** that are perpendicular to it. This time, the object **a** expands into something larger. In both the cases of contraction and expansion, the result is an object that is orthogonal to the operand that is dualized.

As we did for the contraction, we can examine the properties of the expansion in 3D Euclidean space where $\mathbb{1}=\mathbf{e}_{123}$, the metric is the 3×3 identity matrix, and both dual operations are equivalent to the complement operation. We'll consider the expansion of an arbitrary vector $\mathbf{v}=a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_3$ onto the bivector $\mathbf{e}_{12}$ shown in Figure 2.14. Distributing over the vector's components, we can write this expansion as

$$\mathbf{v}\wedge\mathbf{e}_{12}^{\bigstar}=a\left(\mathbf{e}_1\wedge\mathbf{e}_{12}^{\bigstar}\right)+b\left(\mathbf{e}_2\wedge\mathbf{e}_{12}^{\bigstar}\right)+c\left(\mathbf{e}_3\wedge\mathbf{e}_{12}^{\bigstar}\right) \tag{2.151}$$

and calculate each term independently. The dual of $\mathbf{e}_{12}$ is $\mathbf{e}_3$, so we have the three values

$$\begin{aligned} a\left(\mathbf{e}_1\wedge\mathbf{e}_{12}^{\bigstar}\right)&=-a\mathbf{e}_{31}\\ b\left(\mathbf{e}_2\wedge\mathbf{e}_{12}^{\bigstar}\right)&=b\mathbf{e}_{23}\\ c\left(\mathbf{e}_3\wedge\mathbf{e}_{12}^{\bigstar}\right)&=0. \end{aligned} \tag{2.152}$$

The third value demonstrates that we can't add more dimensions to one quantity that are perpendicular to another quantity if the two are already orthogonal. In general, an expansion takes the parts of one object that are parallel to another object and combines them with the space that is perpendicular to that other object. When we expand the $\mathbf{e}_3$ component of $\mathbf{v}$ onto $\mathbf{e}_{12}$, there is nothing parallel to $\mathbf{e}_{12}$ to combine with $\mathbf{e}_{12}^{\bigstar}$, so that component is thrown away. We are left with

$$\mathbf{v}\wedge\mathbf{e}_{12}^{\bigstar}=b\mathbf{e}_{23}-a\mathbf{e}_{31}, \tag{2.153}$$

which is a bivector that contains $\mathbf{v}$ and is orthogonal to $\mathbf{e}_{12}$ as shown in the figure.

For every property of contractions that we previously discussed, there is a corresponding dual property that applies to expansions. In each case, we find that the expansion property is similar to the contraction property but with grade replaced by antigrade, bulk replaced by weight, and all

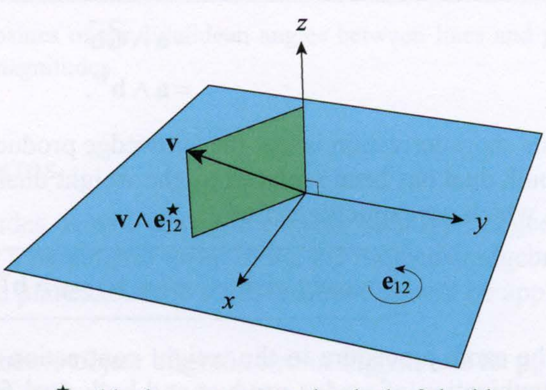


Figure 2.14. The expansion $\mathbf{v}\wedge\mathbf{e}_{12}^{\bigstar}$ adds the vector $\mathbf{v}$ to the dual of the bivector $\mathbf{e}_{12}$ (blue). This produces a new bivector (green) that contains $\mathbf{v}$ and is orthogonal to $\mathbf{e}_{12}$.

products replaced by their antiproducts. To begin with, the contraction subtracts the grades of its operands, but the expansion subtracts the antigrades. In the expansions $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ and $\mathbf{a} \wedge \mathbf{b}^{\star}$, the wedge product adds $\operatorname{gr}(\mathbf{a})$ and $n - \operatorname{gr}(\mathbf{b})$, where n is the number of dimensions. If we subtract this sum from n to get an antigrade and replace each grade $\operatorname{gr}(\mathbf{u})$ with $n - \operatorname{ag}(\mathbf{u})$, then we have

$$\operatorname{ag}\left(\mathbf{a} \wedge \mathbf{b}^{\bigstar}\right) = \operatorname{ag}\left(\mathbf{a} \wedge \mathbf{b}^{\star}\right) = \operatorname{ag}(\mathbf{a}) - \operatorname{ag}(\mathbf{b}), \tag{2.154}$$

which is the analog of Equation (2.117). For the antigrade of the result to be nonnegative, $\operatorname{ag}(\mathbf{b})$ must be no greater than $\operatorname{ag}(\mathbf{a})$, which means that we must have $\operatorname{gr}(\mathbf{b}) \geq \operatorname{gr}(\mathbf{a})$ for expansions.

When **a** and **b** have the same grade, which is equivalent to saying they have the same antigrade, the expansions $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ and $\mathbf{a} \wedge \mathbf{b}^{\star}$ must produce an antiscalar because the antigrade of the result is zero. The weight expansion, being the dual of the bulk contraction, reduces to the antidot product when the antigrades of its operands are the same, which we can write as

$$\boxed{\mathbf{a} \wedge \mathbf{b}^{\star} = \mathbf{a} \circ \mathbf{b}, \quad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b}).} \tag{2.155}$$

The bulk expansion similarly reduces to the value of the dot product transformed into an antiscalar, which we express as

$$\boxed{\mathbf{a} \wedge \mathbf{b}^{\bigstar} = (\mathbf{a} \bullet \mathbf{b}) \wedge \mathbb{1}, \quad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b}).} \tag{2.156}$$

These two identities are the analogs of Equations (2.119) and (2.120).

All values produced by the bulk and weight expansions between basis elements in the 4D projective algebra are shown in Table 2.21. The bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ is identically zero whenever **b** has a factor of $\mathbf{e}_4$, and the weight expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ is identically zero whenever **b** does not have a factor of $\mathbf{e}_4$. As with contractions, this means that only one of the two expansions can be nonzero for any specific value of **b**. The nonzero values of the bulk expansion are highlighted in green in the table, and the nonzero values of the weight expansion are highlighted in purple. Entries for which the weight expansion coincides with the antidot product or the bulk expansion coincides with the antiscalar version of the dot product are highlighted in a darker shade. The antimetric shown in Equation (2.64) is visible in the dark purple cells along the diagonal.

The decomposition properties of expansions are similar to those of contractions. It follows directly from Equation (2.109) that we can write

$$\boxed{\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})^{\star} = \mathbf{a} \wedge \mathbf{b}^{\star} \wedge \mathbf{c}^{\star},} \tag{2.157}$$

which is the analog of Equation (2.121). Because the weight expansion is equivalent to the antidot product when the operands have the same grade, this leads to the identity

$$\boxed{\left(\mathbf{a} \wedge \mathbf{b}^{\star}\right) \circ \mathbf{c} = \mathbf{a} \circ (\mathbf{b} \vee \mathbf{c}), \quad \text{when } \operatorname{ag}(\mathbf{a}) = \operatorname{ag}(\mathbf{b}) + \operatorname{ag}(\mathbf{c}),} \tag{2.158}$$

which is the analog of Equation (2.123). The final identity, which is the analog of Equation (2.129), could be derived using a process similar to the one used for the contraction, but it's much easier to simply take the complement of both sides of Equation (2.129) and simplify. Doing so gives us

$$\boxed{\left[(\mathbf{a} \vee \mathbf{b}) \wedge \mathbf{v}^{\star}\right]_{\circ} = \left[\left(\mathbf{a} \wedge \mathbf{v}^{\star}\right) \vee \mathbf{b} + (-1)^{\operatorname{ag}(\mathbf{a})}\, \mathbf{a} \vee \left(\mathbf{b} \wedge \mathbf{v}^{\star}\right)\right]_{\circ},} \tag{2.159}$$

where **v** is now an antivector, not a vector, but **a** and **b** can have any grade.

Bulk and Weight Expansion — $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ — $(\mathbf{a} \bullet \mathbf{b}) \wedge \mathbb{1}$ — $\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$ — $\mathbf{a} \circ \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{1}$
$\mathbf{e}_1$	0	$\mathbb{1}$	0	0	0	$\mathbf{e}_{321}$	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{41}$	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	$\mathbb{1}$	0	0	0	$\mathbf{e}_{321}$	0	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	$\mathbf{e}_{12}$	0	$-\mathbf{e}_{23}$	$\mathbf{e}_{42}$	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	$\mathbb{1}$	0	0	0	$\mathbf{e}_{321}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	0	$\mathbf{e}_{43}$	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	0	$\mathbf{e}_4$
$\mathbf{e}_{41}$	0	0	0	0	0	$\mathbb{1}$	0	0	0	0	0	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	$\mathbb{1}$	0	0	0	0	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	$\mathbb{1}$	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	0	0	0	0	$\mathbb{1}$	0	0	$\mathbf{e}_{321}$	0	0	$-\mathbf{e}_{423}$	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	0	0	0	0	$\mathbb{1}$	0	0	$\mathbf{e}_{321}$	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	0	0	0	$\mathbb{1}$	0	0	$\mathbf{e}_{321}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{1}$	0	0	0	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{1}$	0	0	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{1}$	0	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{1}$	$\mathbf{e}_{321}$
$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{1}$

Table 2.21. These are the values of the bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ and weight expansion $\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$ between all pairs of basis elements $\mathbf{a}$ and $\mathbf{b}$ in the 4D projective exterior algebra. The values of the bulk expansion are highlighted in green, and the values of the weight expansion are highlighted in purple. The darker purple cells along the diagonal correspond to the nonzero entries of the antimetric $\mathbb{G}$.

The primary application of the expansion operation arises when we apply the weight expansion to pairs of projective geometries. Given two geometric objects $\mathbf{a}$ and $\mathbf{b}$ for which $\mathbf{a}$ has a lower grade, the weight expansion $\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$ produces a new geometry that contains $\mathbf{a}$ and is orthogonal to $\mathbf{b}$. In the 4D projective algebra, the condition that $\operatorname{gr}(\mathbf{a}) < \operatorname{gr}(\mathbf{b})$ leaves three possible combinations of points, lines, and planes for which the weight expansion operation is meaningful, and these are listed in Table 2.22 along with illustrations.

As long as the higher-dimensional object $\mathbf{b}$ does not lie in the horizon and thus have a zero weight, it is always possible to construct a new geometry that is perpendicular to it and also contains the object $\mathbf{a}$ no matter what the geometric relationship is between $\mathbf{a}$ and $\mathbf{b}$. For example, the weight expansion of a point onto a plane produces a line that is orthogonal to the plane and passes through the point even if the point lies in the plane. The weight expansion operation does not produce null results in special cases like the join and meet operations sometimes do.

The weight expansion $\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$ works the way that it does because the dual operation applied to the operand $\mathbf{b}$ produces a relative position lying in the horizon, specifically one of those listed in the rightmost column of Table 2.17. When one of these relative positions is joined with another geometry $\mathbf{a}$, the result must contain $\mathbf{a}$, but the original position of $\mathbf{b}$ is ignored. The attitude of the result must be a combination of the weight of $\mathbf{a}$ and the entire value of $\mathbf{b}^{\text{☆}}$, and so it is parallel to the attitude of $\mathbf{a}$ but perpendicular to the attitude of $\mathbf{b}$. Importantly, the new geometry contains the shortest path from any point on $\mathbf{a}$ to the closest point on $\mathbf{b}$, thus establishing the most direct connection between the two.

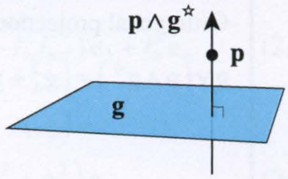
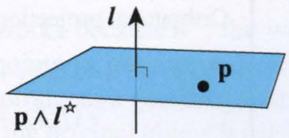
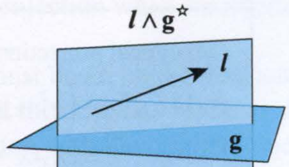
Expansion Operation	**Illustration**
Line containing point **p** and orthogonal to plane **g**. $\begin{aligned}\mathbf{p}\wedge\mathbf{g}^{\star} = &-p_w g_x \mathbf{e}_{41} + (p_z g_y - p_y g_z)\mathbf{e}_{23}\\ &- p_w g_y \mathbf{e}_{42} + (p_x g_z - p_z g_x)\mathbf{e}_{31}\\ &- p_w g_z \mathbf{e}_{43} + (p_y g_x - p_x g_y)\mathbf{e}_{12}\end{aligned}$	
Plane containing point **p** and orthogonal to line ***l***. $\begin{aligned}\mathbf{p}\wedge\boldsymbol{l}^{\star} = &-p_w l_{vx}\mathbf{e}_{423} - p_w l_{vy}\mathbf{e}_{431} - p_w l_{vz}\mathbf{e}_{412}\\ &+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\mathbf{e}_{321}\end{aligned}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\begin{aligned}\boldsymbol{l}\wedge\mathbf{g}^{\star} = &(l_{vy}g_z - l_{vz}g_y)\mathbf{e}_{423}\\ &+ (l_{vz}g_x - l_{vx}g_z)\mathbf{e}_{431}\\ &+ (l_{vx}g_y - l_{vy}g_x)\mathbf{e}_{412}\\ &- (l_{mx}g_x + l_{my}g_y + l_{mz}g_z)\mathbf{e}_{321}\end{aligned}$	

Table 2.22. These are the weight expansions $\mathbf{a}\wedge\mathbf{b}^{\star}$ between points, lines, and planes in three dimensions. The result is an object that contains **a** and is orthogonal to **b**.

Contraction	**DUALITY**	**Expansion**
$\mathbf{a}\vee\mathbf{b}^{\bigstar}$		$\mathbf{a}\wedge\mathbf{b}^{\star}$
▪ Decreases the grade of **a** by removing the subspace spanned by **b**.		▪ Increases the grade of **a** by adding the subspace not spanned by **b**.
▪ The result is a lower-dimensional object that is contained by **a** and is orthogonal to **b**.		▪ The result is a higher-dimensional object that contains **a** and is orthogonal to **b**.
▪ The grade of the result is $\operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b})$.		▪ The antigrade of the result is $\operatorname{ag}(\mathbf{a}) - \operatorname{ag}(\mathbf{b})$.

2.13.6 Geometric Projection

The weight expansion has an immediate application that should now be very conspicuous. If the operation $\mathbf{a}\wedge\mathbf{b}^{\star}$ produces a new object **c** that contains **a** and is perpendicular to **b**, then a simple intersection of **c** with the original object **b** has the effect of moving **a** along **c** until it lies in **b**. Since **c** is perpendicular to **b**, this performs an *orthogonal projection* of **a** onto **b**. The intersection of **c** and **b** is accomplished with the meet operation, so we can express the entire projection of **a** onto **b** with the formula

Orthogonal projection

$$\text{orthogonal projection}(\mathbf{a},\mathbf{b}) = \mathbf{b}\vee(\mathbf{a}\wedge\mathbf{b}^{\star}). \tag{2.160}$$

In three dimensions, this formula can be used to project a point onto a plane, to project a point onto a line, or to project a line onto a plane. The exact calculations for these specific cases are listed in expanded form in Table 2.23.

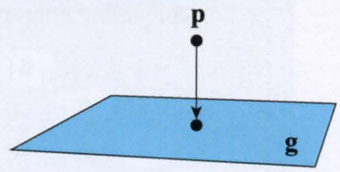
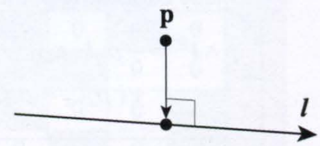
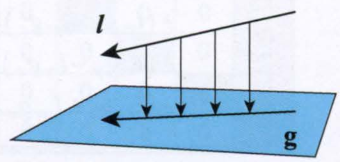
Projection Operation	Illustration
Orthogonal projection of point **p** onto plane **g**. $\mathbf{g}\vee(\mathbf{p}\wedge\mathbf{g}^{\star})=(g_x^2+g_y^2+g_z^2)(p_x\mathbf{e}_1+p_y\mathbf{e}_2+p_z\mathbf{e}_3+p_w\mathbf{e}_4)$ $-(g_xp_x+g_yp_y+g_zp_z+g_wp_w)(g_x\mathbf{e}_1+g_y\mathbf{e}_2+g_z\mathbf{e}_3)$	
Orthogonal projection of point **p** onto line ***l***. $\boldsymbol{l}\vee(\mathbf{p}\wedge\boldsymbol{l}^{\star})=(l_{vx}p_x+l_{vy}p_y+l_{vz}p_z)(l_{vx}\mathbf{e}_1+l_{vy}\mathbf{e}_2+l_{vz}\mathbf{e}_3)$ $+(l_{vy}l_{mz}-l_{vz}l_{my})p_w\mathbf{e}_1+(l_{vz}l_{mx}-l_{vx}l_{mz})p_w\mathbf{e}_2$ $+(l_{vx}l_{my}-l_{vy}l_{mx})p_w\mathbf{e}_3+(l_{vx}^2+l_{vy}^2+l_{vz}^2)p_w\mathbf{e}_4$	
Orthogonal projection of line ***l*** onto plane **g**. $\mathbf{g}\vee(\boldsymbol{l}\wedge\mathbf{g}^{\star})=(g_x^2+g_y^2+g_z^2)(l_{vx}\mathbf{e}_{41}+l_{vy}\mathbf{e}_{42}+l_{vz}\mathbf{e}_{43})$ $-(g_xl_{vx}+g_yl_{vy}+g_zl_{vz})(g_x\mathbf{e}_{41}+g_y\mathbf{e}_{42}+g_z\mathbf{e}_{43})$ $+(g_xl_{mx}+g_yl_{my}+g_zl_{mz})(g_x\mathbf{e}_{23}+g_y\mathbf{e}_{31}+g_z\mathbf{e}_{12})$ $+(g_zl_{vy}-g_yl_{vz})g_w\mathbf{e}_{23}+(g_xl_{vz}-g_zl_{vx})g_w\mathbf{e}_{31}$ $+(g_yl_{vx}-g_xl_{vy})g_w\mathbf{e}_{12}$	

Table 2.23. These are all the possible orthogonal projections between any two different kinds of flat geometry in three dimensions.

The projection performed by Equation (2.160) preserves the original orientation of the object being projected as much as possible. The weight of a projected point keeps its original sign, and this is easily verified in the formulas shown in Table 2.23. The direction of a projected line makes a nonnegative dot product with its original direction. This can be seen by extracting the attitude from the formula for $\mathbf{g}\vee(\boldsymbol{l}\wedge\mathbf{g}^{\star})$ in the table and taking a dot product with the attitude of the original line $\boldsymbol{l}$ to get

$$\operatorname{att}(\mathbf{g}\vee(\boldsymbol{l}\wedge\mathbf{g}^{\star}))\cdot\operatorname{att}(\boldsymbol{l})=\mathbf{g}_{xyz}^2\boldsymbol{l}_{\mathbf{v}}^2-(\mathbf{g}_{xyz}\cdot\boldsymbol{l}_{\mathbf{v}})^2=\mathbf{g}_{xyz}^2\boldsymbol{l}_{\mathbf{v}}^2(1-\cos^2\theta)\geq 0, \tag{2.161}$$

where θ is the angle made between the line direction $\boldsymbol{l}_{\mathbf{v}}$ and the plane normal $\mathbf{g}_{xyz}$.

Because the object **b** onto which something is being projected appears twice in Equation (2.160), the weight of the result is multiplied by squared weight of **b**. To preserve the original weight of the projected object **a**, we need to divide by $\|\mathbf{b}\|_{\circ}^2=\mathbf{b}\circ\mathbf{b}$ unless it is already known that **b** is unitized. Of course, this does not change the geometric meaning of the projected result, but doing this division may be convenient or necessary to make sure that weights don't get too large and start creating numerical issues.

When a point **p** is projected onto a line or plane, the result is the point closest to **p** on the line or plane. When **p** is the origin, we give the projected point a special name, the *support*. The support of an object **u**, denoted by $\operatorname{sup}(\mathbf{u})$, is the point contained in **u** that is closest to the origin. By plugging the origin $\mathbf{e}_n$ into Equation (2.160), we can write this definition as

Support

$$\boxed{\operatorname{sup}(\mathbf{u})=\mathbf{u}\vee(\mathbf{e}_n\wedge\mathbf{u}^{\star}).} \tag{2.162}$$

For a point $\mathbf{p}$ in the 4D projective algebra, the weight expansion part of this formula is $\mathbf{e}_4 \wedge \mathbf{p}^{\star} = p_w \mathbb{1}$, and $\sup(\mathbf{p})$ is simply the same point $\mathbf{p}$ scaled by its own weight. For a line $\boldsymbol{l}$, we have

$$\sup(\boldsymbol{l}) = (l_{vy} l_{mz} - l_{vz} l_{my})\,\mathbf{e}_1 + (l_{vz} l_{mx} - l_{vx} l_{mz})\,\mathbf{e}_2 + (l_{vx} l_{my} - l_{vy} l_{mx})\,\mathbf{e}_3 + \boldsymbol{l}_{\mathbf{v}}^2\,\mathbf{e}_4. \tag{2.163}$$

And for a plane $\mathbf{g}$, we have

$$\sup(\mathbf{g}) = -g_x g_w \mathbf{e}_1 - g_y g_w \mathbf{e}_2 - g_w g_z \mathbf{e}_3 + \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_4. \tag{2.164}$$

The orthogonal projection formula given by Equation (2.160) works because $\mathbf{b}^{\star}$, the weight dual of $\mathbf{b}$, is an object contained in the horizon consisting of all points at infinity in directions perpendicular to the attitude of $\mathbf{b}$. Thus, the weight expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ produces a geometry that contains $\mathbf{a}$ and is perpendicular to $\mathbf{b}$, and intersecting that geometry with the original object $\mathbf{b}$ performs the projection. If we replace the weight expansion with a bulk expansion, then the intermediate object $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ is something different, and we get a different kind of projection when we intersect it with the original object $\mathbf{b}$.

The bulk dual of $\mathbf{b}$ has only weight components and therefore must be an object that contains the origin. This means the bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ contains both $\mathbf{a}$ and the origin as well as a direct path connecting the two. When we intersect the result of the expansion with the original object $\mathbf{b}$, it has the effect of moving $\mathbf{a}$ onto $\mathbf{b}$ along directions between $\mathbf{a}$ and the origin. This performs a *central projection* of $\mathbf{a}$ onto $\mathbf{b}$ with respect to the origin, which we define as

Central projection

$$\boxed{\text{central projection}(\mathbf{a}, \mathbf{b}) = \mathbf{b} \vee \left(\mathbf{a} \wedge \mathbf{b}^{\bigstar}\right).} \tag{2.165}$$

As with the orthogonal projection, this formula can be used to project a point onto a plane, a point onto a line, or a line onto a plane in three dimensions. The exact calculations for these specific cases are listed in expanded form in Table 2.24.

In the case that we are centrally projecting onto a plane $\mathbf{g}$, the bulk dual of $\mathbf{g}$ is simply $g_w \mathbf{e}_4$, which is the origin weighted by g_w. The bulk expansion $\mathbf{p} \wedge \mathbf{g}^{\bigstar}$ involving a point $\mathbf{p}$ is then the line connecting $\mathbf{p}$ and the origin, and the bulk expansion $\boldsymbol{l} \wedge \mathbf{g}^{\bigstar}$ is the plane containing both the line and the origin. Intersecting either of these with the plane $\mathbf{g}$ moves them onto the plane along the direction toward or away from the origin. The case of centrally projecting a point $\mathbf{p}$ onto a line $\boldsymbol{l}$ is somewhat different because there may not be a point on the line that is directly in between $\mathbf{p}$ and the origin. The bulk dual of $\boldsymbol{l}$ is $-l_{mx}\,\mathbf{e}_{41} - l_{my}\,\mathbf{e}_{42} - l_{mz}\,\mathbf{e}_{43}$, which is a line through the origin perpendicular to the moment bivector of $\boldsymbol{l}$. The bulk expansion $\mathbf{p} \wedge \boldsymbol{l}^{\bigstar}$ is the plane containing this line and the point $\mathbf{p}$, and the intersection of that plane and the line $\boldsymbol{l}$ gives us a point on the line representing the central projection of $\mathbf{p}$. The plane $\mathbf{p} \wedge \boldsymbol{l}^{\bigstar}$ is not the plane shown in Table 2.24, however. Instead, the table illustrates a more intuitive equivalent in which the point $\mathbf{p}$ is first projected orthogonally onto the plane containing $\boldsymbol{l}$ and the origin before being centrally projected onto $\boldsymbol{l}$ within that plane.

If we try to centrally project an object $\mathbf{a}$ that contains the origin onto something else, then we run into trouble. For the point in $\mathbf{a}$ at the origin, there is no way to define the direction toward or away from the origin, so a projected image of that point onto another object doesn't make sense. In this case, the bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$ is always zero because both $\mathbf{a}$ and $\mathbf{b}^{\bigstar}$ have no bulk components. We can also see within the formulas listed in Table 2.24 that the weight of $\mathbf{a}$ does not appear anywhere in the projected geometries, so if $\mathbf{a}$ has no bulk due to it containing the origin, then the result must be zero. The opposite case in which the object $\mathbf{b}$ that we are projecting onto contains the origin also produces zero. Every part of the formulas in Table 2.24 contains a factor of the bulk of $\mathbf{b}$, so if $\mathbf{b}$ has no bulk, then we get a null result. The math is telling us that we just can't do a central projection unless both objects involved are located somewhere away from the origin.

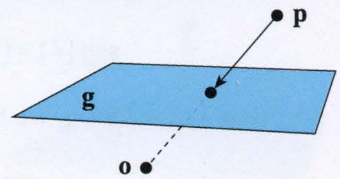
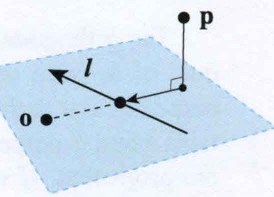
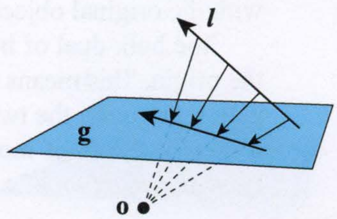
Projection Operation	Illustration
Central projection of point $\mathbf{p}$ onto plane $\mathbf{g}$. $$\mathbf{g}\vee(\mathbf{p}\wedge\mathbf{g}^{\star}) = g_w^2(p_x\mathbf{e}_1+p_y\mathbf{e}_2+p_z\mathbf{e}_3) - (g_xp_x+g_yp_y+g_zp_z)g_w\mathbf{e}_4$$	
Central projection of point $\mathbf{p}$ onto line $\boldsymbol{l}$. $$\boldsymbol{l}\vee(\mathbf{p}\wedge\boldsymbol{l}^{\star}) = (l_{mx}^2+l_{my}^2+l_{mz}^2)(p_x\mathbf{e}_1+p_y\mathbf{e}_2+p_z\mathbf{e}_3) - (l_{mx}p_x+l_{my}p_y+l_{mz}p_z)(l_{mx}\mathbf{e}_1+l_{my}\mathbf{e}_2+l_{mz}\mathbf{e}_3) + (l_{mx}(l_{vz}p_y-l_{vy}p_z)+l_{my}(l_{vx}p_z-l_{vz}p_x)+l_{mz}(l_{vy}p_x-l_{vx}p_y))\mathbf{e}_4$$	
Central projection of line $\boldsymbol{l}$ onto plane $\mathbf{g}$. $$\mathbf{g}\vee(\boldsymbol{l}\wedge\mathbf{g}^{\star}) = (g_yl_{mz}-g_zl_{my})g_w\mathbf{e}_{41}+g_w^2l_{mx}\mathbf{e}_{23} + (g_zl_{mx}-g_xl_{mz})g_w\mathbf{e}_{42}+g_w^2l_{my}\mathbf{e}_{31} + (g_xl_{my}-g_yl_{mx})g_w\mathbf{e}_{43}+g_w^2l_{mz}\mathbf{e}_{12}$$	

Table 2.24. These are all the possible central projections (with respect to the origin $\mathbf{o}$) between any two different kinds of flat geometry in three dimensions.

We now take a look at the anti-operations associated with the orthogonal and central projections by taking right complements of the inputs and the left complement of the result. Applying this procedure first to the central projection formula given by Equation (2.165), we have

$$\begin{aligned}\underline{\overline{\mathbf{b}}\vee(\overline{\mathbf{a}}\wedge\overline{\mathbf{b}}^{\star})} &= \underline{\overline{\mathbf{b}}\vee(\overline{\mathbf{a}}\wedge\overline{\mathbf{b}^{\text{☆}}})} \\ &= \underline{\overline{\mathbf{b}}\vee\overline{(\mathbf{a}\vee\mathbf{b}^{\text{☆}})}} \\ &= \mathbf{b}\wedge(\mathbf{a}\vee\mathbf{b}^{\text{☆}}).\end{aligned} \tag{2.166}$$

This new operation calculates the weight contraction of $\mathbf{a}$ with $\mathbf{b}$ and joins the result with the original object $\mathbf{b}$. Since we're taking a contraction now instead of an expansion, the object $\mathbf{a}$ must be the one having higher grade, suggesting that we are somehow projecting in a manner opposite to the way in which projections containing an expansion work. Indeed, when we calculate the weight contraction $\mathbf{a}\vee\mathbf{b}^{\text{☆}}$, we are throwing away the bulk of $\mathbf{a}$, leaving only its directional information behind. Taking the wedge product with $\mathbf{b}$ then moves $\mathbf{a}$ into a position that contains $\mathbf{b}$ without changing its original attitude. We call this operation the *orthogonal antiprojection* of $\mathbf{a}$ onto $\mathbf{b}$ and define it as

Orthogonal antiprojection

$$\boxed{\text{orthogonal antiprojection}(\mathbf{a},\mathbf{b}) = \mathbf{b}\wedge(\mathbf{a}\vee\mathbf{b}^{\text{☆}}).} \tag{2.167}$$

In the opposite sense that Equation (2.160) performs a projection, this formula can be used to antiproject a plane onto a point, to antiproject a line onto a point, or to antiproject a plane onto a line in three dimensions, as shown in Table 2.25.

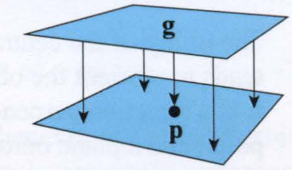
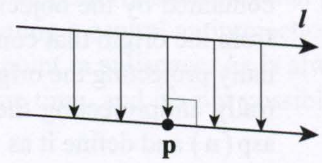
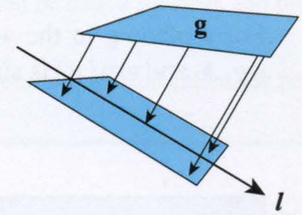
Projection Operation	**Illustration**
Orthogonal antiprojection of plane $\mathbf{g}$ onto point $\mathbf{p}$. $\mathbf{p} \wedge (\mathbf{g} \vee \mathbf{p}^{\star}) = p_w^2\,(g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412})$ $\quad - (p_x g_x + p_y g_y + p_z g_z)\, p_w \mathbf{e}_{321}$	
Orthogonal antiprojection of line $\boldsymbol{l}$ onto point $\mathbf{p}$. $\mathbf{p} \wedge (\boldsymbol{l} \vee \mathbf{p}^{\star}) = p_w^2\,(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43})$ $\quad + (p_y l_{vz} - p_z l_{vy})\, p_w \mathbf{e}_{23}$ $\quad + (p_z l_{vx} - p_x l_{vz})\, p_w \mathbf{e}_{31}$ $\quad + (p_x l_{vy} - p_y l_{vx})\, p_w \mathbf{e}_{12}$	
Orthogonal antiprojection of plane $\mathbf{g}$ onto line $\boldsymbol{l}$. $\boldsymbol{l} \wedge (\mathbf{g} \vee \boldsymbol{l}^{\star}) = (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)(g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412})$ $\quad - (l_{vx} g_x + l_{vy} g_y + l_{vz} g_z)(l_{vx} \mathbf{e}_{423} + l_{vy} \mathbf{e}_{431} + l_{vz} \mathbf{e}_{412})$ $\quad + (l_{vz} l_{my} - l_{vy} l_{mz})\, g_x \mathbf{e}_{321}$ $\quad + (l_{vx} l_{mz} - l_{vz} l_{mx})\, g_y \mathbf{e}_{321}$ $\quad + (l_{vy} l_{mx} - l_{vx} l_{my})\, g_z \mathbf{e}_{321}$	

Table 2.25. These are all the possible orthogonal antiprojections between any two different kinds of flat geometry in three dimensions.

Projection	DUALITY	**Antiprojection**
$\mathbf{b} \vee (\mathbf{a} \wedge \mathbf{b}^{\star})$		$\mathbf{b} \wedge (\mathbf{a} \vee \mathbf{b}^{\bigstar})$
▪ Projects lower-grade object **a** onto higher-grade object **b**.		▪ Projects higher-grade object **a** onto lower-grade object **b**.

The difference between a projection and an antiprojection is that a projection moves the smaller, lower-dimensional object onto the larger, higher-dimensional object, and an antiprojection works in reverse by moving the larger object onto the smaller object. In the case that we are antiprojecting onto a point, using Equation (2.167) is overkill because we can easily recalculate the position of an object without affecting its attitude. To antiproject a plane $\mathbf{g}$ onto a unitized point $\mathbf{p}$, all we have to do is replace the g_w coordinate of the plane, its bulk, with the dot product $-\mathbf{g}_{xyz} \cdot \mathbf{p}_{xyz}$. To antiproject a line $\boldsymbol{l}$ onto a unitized point $\mathbf{p}$, we just replace the moment of the line with the cross product $\mathbf{p}_{xyz} \times \boldsymbol{l}_{\mathbf{v}}$. To handle the possibility that the point is not unitized in either of these cases, the weight of the plane or line simply needs to be multiplied by p_w after the new bulk has been calculated.

The anti-operation associated with the orthogonal projection is similar to Equation (2.167), but the weight contraction is replaced by a bulk contraction. Because it involves a contraction $\mathbf{a} \vee \mathbf{b}^{\bigstar}$, the object $\mathbf{a}$ must again be the one having higher grade. This is a fourth kind of projection that we call the *central antiprojection* of $\mathbf{a}$ onto $\mathbf{b}$ and define as

Central antiprojection

$$\boxed{\operatorname{central\ antiprojection}(\mathbf{a},\mathbf{b}) = \mathbf{b} \wedge (\mathbf{a} \vee \mathbf{b}^{\bigstar}).} \tag{2.168}$$

The utility of the central antiprojection is questionable, but it is included here for completeness. It tends to reorient the object **a** being antiprojected so that it contains the object **b** instead of moving it in a direction perpendicular to **b**. The central antiprojections of a plane onto a point, a line onto a point, and a plane onto a line in three dimensions are shown in Table 2.26.

One thing the central antiprojection is good for is calculating the geometry having the dual meaning of the support. Whereas the support of an object is the point closest to the origin that's contained by the object, its dual counterpart, which we call the *antisupport*, is the plane farthest from the origin that contains the object. The support is calculated in Equation (2.162) by orthogonally projecting the origin $\mathbf{e}_n$ onto an object. In the dual sense, the antisupport is calculated by centrally antiprojecting the horizon $\overline{\mathbf{e}}_n$ onto an object. We denote the antisupport of an object $\mathbf{u}$ by $\operatorname{asp}(\mathbf{u})$ and define it as

Antisupport

$$\boxed{\operatorname{asp}(\mathbf{u}) = \mathbf{u} \wedge (\overline{\mathbf{e}}_n \vee \mathbf{u}^{\bigstar}).} \tag{2.169}$$

For a plane $\mathbf{g}$ in the 4D projective algebra, the bulk expansion part of this formula is $\overline{\mathbf{e}}_4 \vee \mathbf{g}^{\bigstar} = -g_w\mathbf{1}$, and $\operatorname{asp}(\mathbf{g})$ is simply the flipped plane $\mathbf{g}$ scaled by its own bulk. For a line $\boldsymbol{l}$, we have

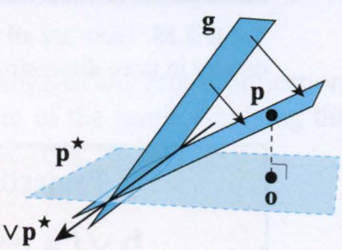
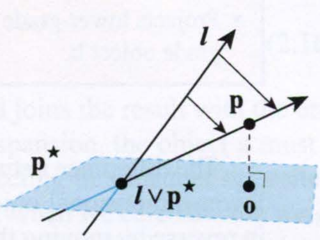
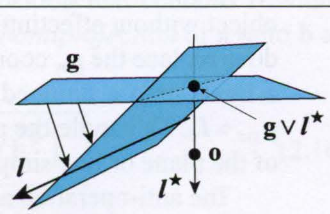
Projection Operation	**Illustration**
Central antiprojection of plane $\mathbf{g}$ onto point $\mathbf{p}$. $\begin{aligned}\mathbf{p} \wedge (\mathbf{g} \vee \mathbf{p}^{\bigstar}) &= \left[(p_y^2 + p_z^2)g_x - (p_y g_y + p_z g_z + p_w g_w)p_x\right]\mathbf{e}_{423} \\ &+ \left[(p_z^2 + p_x^2)g_y - (p_x g_x + p_z g_z + p_w g_w)p_y\right]\mathbf{e}_{431} \\ &+ \left[(p_x^2 + p_y^2)g_z - (p_x g_x + p_y g_y + p_w g_w)p_z\right]\mathbf{e}_{412} \\ &+ (p_x^2 + p_y^2 + p_z^2)g_w\mathbf{e}_{321}\end{aligned}$	
Central antiprojection of line $\boldsymbol{l}$ onto point $\mathbf{p}$. $\begin{aligned}\mathbf{p} \wedge (\boldsymbol{l} \vee \mathbf{p}^{\bigstar}) &= (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})(p_x\mathbf{e}_{41} + p_y\mathbf{e}_{42} + p_z\mathbf{e}_{43}) \\ &+ (p_y^2 + p_z^2)l_{mx}\mathbf{e}_{23} + (p_z^2 + p_x^2)l_{my}\mathbf{e}_{31} + (p_x^2 + p_y^2)l_{mz}\mathbf{e}_{12} \\ &+ (p_z l_{my} - p_y l_{mz})p_w\mathbf{e}_{41} - (p_y l_{my} + p_z l_{mz})p_x\mathbf{e}_{23} \\ &+ (p_x l_{mz} - p_z l_{mx})p_w\mathbf{e}_{42} - (p_z l_{mz} + p_x l_{mx})p_y\mathbf{e}_{31} \\ &+ (p_y l_{mx} - p_x l_{my})p_w\mathbf{e}_{43} - (p_x l_{mx} + p_y l_{my})p_z\mathbf{e}_{12}\end{aligned}$	
Central antiprojection of plane $\mathbf{g}$ onto line $\boldsymbol{l}$. $\begin{aligned}\boldsymbol{l} \wedge (\mathbf{g} \vee \boldsymbol{l}^{\bigstar}) &= (l_{mx}g_x + l_{my}g_y + l_{mz}g_z)(l_{mx}\mathbf{e}_{423} + l_{my}\mathbf{e}_{431} + l_{mz}\mathbf{e}_{412}) \\ &+ (l_{my}l_{vz} - l_{mz}l_{vy})g_w\mathbf{e}_{423} + (l_{mz}l_{vx} - l_{mx}l_{vz})g_w\mathbf{e}_{431} \\ &+ (l_{mx}l_{vy} - l_{my}l_{vx})g_w\mathbf{e}_{412} + (l_{mx}^2 + l_{my}^2 + l_{mz}^2)g_w\mathbf{e}_{321}\end{aligned}$	

Table 2.26. These are all the possible central antiprojections (with respect to the origin **o**) between any two different kinds of flat geometry in three dimensions.

$$\operatorname{asp}(\boldsymbol{l}) = (l_{vz}l_{my} - l_{vy}l_{mz})\,\mathbf{e}_{423} + (l_{vx}l_{mz} - l_{vz}l_{mx})\,\mathbf{e}_{431} + (l_{vy}l_{mx} - l_{vx}l_{my})\,\mathbf{e}_{412} + \boldsymbol{l}_{\mathbf{m}}^{2}\,\mathbf{e}_{321}. \quad (2.170)$$

And for a point $\mathbf{p}$, we have

$$\operatorname{asp}(\mathbf{p}) = -p_x p_w \mathbf{e}_{423} - p_y p_w \mathbf{e}_{431} - p_z p_w \mathbf{e}_{412} + \left(p_x^2 + p_y^2 + p_z^2\right)\mathbf{e}_{321}. \quad (2.171)$$

Since projections and antiprojections are dual to each other, whenever one kind of projection happens in regular space, its dual operation must be happening simultaneously in antispace. Any one of the four kinds of projection we have discussed is always one part of pair of operations involving one projection and one antiprojection. For example, when we perform an orthogonal projection of a point onto a plane in regular space, we are also performing a central antiprojection of the point's complementary plane onto the plane's complementary point in antispace. As is always true in our projective algebra, two things are happening at the same time, and it's not possible to separate them.

Math Library Notes

- The support operation is implemented by the `Support()` function. It can be called for lines and planes in three dimensions.
- The antisupport operation is implemented by the `Antisupport()` function. It can be called for points and lines in three dimensions.

Support — DUALITY — **Antisupport**

$$\mathbf{u} \vee \left(\mathbf{e}_n \wedge \mathbf{u}^{\star}\right)$$

- Performs an orthogonal projection of the origin $\mathbf{e}_n$ onto $\mathbf{u}$.
- Gives the point contained in $\mathbf{u}$ that is closest to the origin.

$$\mathbf{u} \wedge \left(\overline{\mathbf{e}}_n \vee \mathbf{u}^{\bigstar}\right)$$

- Performs a central antiprojection of the horizon $\overline{\mathbf{e}}_n$ onto $\mathbf{u}$.
- Gives the plane containing $\mathbf{u}$ that is farthest from the origin.

2.14 2D Flat Geometry

All of the mathematical concepts developed in this chapter are just as applicable to any number of dimensions as they are to 3D space. We don't consider any higher dimensions in this book, but we do take a quick look at what happens in 2D space because it has a large amount of practical utility. In the 2D case, homogeneous geometries exist in a 3D projective algebra with extents in the x, y, and z directions. A geometric object is now projected into 2D space by scaling it so that the collective magnitude of its components extending into the z direction is one.

The 3D projective exterior algebra representing geometries in 2D space has eight basis elements over four possible grades, and they are listed in Table 2.27. There is one scalar basis element **1**, and there are three vector basis elements $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. The basis vector $\mathbf{e}_3$ corresponds to the projective dimension, and it plays the role that $\mathbf{e}_4$ fills in the 4D algebra. For the bivector basis elements, we have a choice when it comes to the order of the factors. Bivectors represent lines, so it makes sense that two components of a bivector would correspond to the direction of a line, and that happens if we choose $\mathbf{e}_{31}$, $\mathbf{e}_{32}$, and $\mathbf{e}_{12}$ as our basis elements. However, there are many instances in which lines in 2D behave more likes planes in 3D, so it also makes sense to choose $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, and $\mathbf{e}_{12}$, in which

Type	Grade / Antigrade	Values
Scalar	0 / 3 □□□	$\mathbf{1}$
Vectors	1 / 2 ■□□ □■□ □□■	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$
Bivectors / Antivectors	2 / 1 □■■ ■□■ ■■□	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$
Trivector / Antiscalar	3 / 0 ■■■	$\mathbb{1} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$

Table 2.27. These are the 8 basis elements of the 3D projective exterior algebra.

case two of the components correspond to the normal of a line. Both choices would produce the same results geometrically, but they would have different conventions for the order of components and their signs. The first option has a small defect in that $\mathbf{e}_{31}$ and $\mathbf{e}_{32}$ are not consistently the complements of $\mathbf{e}_2$ and $\mathbf{e}_1$, respectively, making it less attractive. In this book, we choose to use the second option, which also has the advantage that the basis bivectors in the 3D algebra are exactly the same as the subset basis bivectors in the 4D algebra that don't have a factor of $\mathbf{e}_4$. Finally, we must also choose a single trivector that acts as the volume element $\mathbb{1}$. For reasons that won't become fully apparent until later in Section 3.8, we choose $\mathbb{1} = \mathbf{e}_{321}$ for the volume element, which happens to also match one of the trivector basis elements in the 4D algebra. Note that this volume element has the opposite orientation of the volume element $\mathbf{e}_{123}$ that we've previously used in the 3D non-projective algebra, so the signs of some complements and antiwedge products are different here as a result.

The complements of all eight basis elements are shown in Table 2.28. As always, the scalar unit $\mathbf{1}$ and the volume element $\mathbb{1}$ are complements of each other. In the 3D algebra, bivectors have grade one less than the total dimension of the algebra, so they are the antivectors. Thus, vectors and bivectors are complements of each other, but they flip signs when complemented due to the orientation of the volume element. In three dimensions, left complements and right complements are the same, so each basis element has only one complement, and we don't bother specifying left or right.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{1}$

Table 2.28. For each of the 8 basis elements $\mathbf{u}$ in the 3D projective exterior algebra, this table lists the complement $\overline{\mathbf{u}}$ with respect to the volume element $\mathbb{1} = \mathbf{e}_{321}$. (Right and left complements are equivalent in odd numbers of dimensions.)

The full multiplication tables for the wedge product and antiwedge product with the eight basis elements of the 3D projective exterior algebra are shown in Table 2.29. There is nothing special happening here, and the antiwedge product continues to be defined by

$$\mathbf{a} \vee \mathbf{b} = \overline{\overline{\mathbf{a}} \wedge \overline{\mathbf{b}}}, \tag{2.172}$$

where we are using only the overbar notation because there is only one complement operation.

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbb{1}$	0	0	0
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	0	$-\mathbb{1}$	0	0
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	0	0	$-\mathbb{1}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbb{1}$	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	0	0	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	$-\mathbf{1}$	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	$-\mathbf{1}$	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_3$
$\mathbf{e}_{23}$	0	$-\mathbf{1}$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	$-\mathbf{1}$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	$-\mathbf{1}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$\mathbf{e}_{12}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$

Table 2.29. These are the multiplication tables for the wedge product and antiwedge product between the 8 basis elements in the 3D projective exterior algebra representing 2D Euclidean space.

A general 2D homogeneous point $\mathbf{p}$ is represented by the vector

Point (2D)

$$\mathbf{p} = \underbrace{p_x\mathbf{e}_1 + p_y\mathbf{e}_2}_{\text{Position}} + \underbrace{p_z\mathbf{e}_3}_{\text{Weight}} \tag{2.173}$$

with the three coordinates p_x, p_y, and p_z assigned to the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. The basis vector $\mathbf{e}_3$ by itself corresponds to the origin of the 2D plane. If the p_z coordinate is zero, then the point $\mathbf{p}$ is a point at infinity in the direction given by p_x and p_y. A general 2D homogeneous line $\mathbf{g}$ is represented by the bivector

Line (2D)

$$\mathbf{g} = \underbrace{g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31}}_{\text{Normal}} + \underbrace{g_z\mathbf{e}_{12}}_{\text{Position}}, \tag{2.174}$$

where the coordinates g_x and g_y make up a normal vector that points in the direction perpendicular to the line, and the coordinate g_z corresponds to the distance from the origin. If the g_z coordinate is zero, then the line passes through the origin. The basis bivector $\mathbf{e}_{12}$ by itself corresponds to the horizon of the 2D plane, containing all points at infinity.

The join of two points $\mathbf{p}$ and $\mathbf{q}$ is given by

$$\mathbf{p} \wedge \mathbf{q} = (p_yq_z - p_zq_y)\mathbf{e}_{23} + (p_zq_x - p_xq_z)\mathbf{e}_{31} + (p_xq_y - p_yq_x)\mathbf{e}_{12}, \tag{2.175}$$

and it represents the line containing both $\mathbf{p}$ and $\mathbf{q}$. The meet of two lines $\mathbf{g}$ and $\mathbf{h}$ is given by

$$\mathbf{g} \vee \mathbf{h} = (g_zh_y - g_yh_z)\mathbf{e}_1 + (g_xh_z - g_zh_x)\mathbf{e}_2 + (g_yh_x - g_xh_y)\mathbf{e}_3, \tag{2.176}$$

and it represents the homogeneous point where the lines $\mathbf{g}$ and $\mathbf{h}$ intersect. These operations are illustrated in Table 2.30.

The metric $\mathbf{g}$ for the 3D projective algebra is given by

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.177}$$

which means that the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ both square to one under the dot product, and the basis vector $\mathbf{e}_3$ is now the one without physical measure that squares to zero. The metric exomorphism $\mathbf{G}$ and the metric antiexomorphism $\mathbb{G}$ based on the conventional metric $\mathbf{g}$ are 8×8 matrices, and their exact forms are shown in Figure 2.15. These matrices induce the dot products and antidot products shown in Table 2.31 for a basis element $\mathbf{u}$ multiplied by itself.

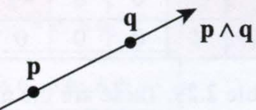
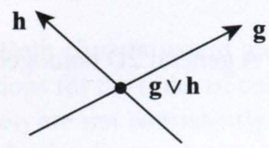
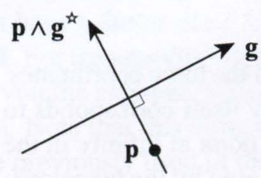
Join / Meet / Expansion Operation	**Illustration**
Line containing points $\mathbf{p}$ and $\mathbf{q}$. $\mathbf{p} \wedge \mathbf{q} = (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$	
Point where lines $\mathbf{g}$ and $\mathbf{h}$ intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_1 + (g_x h_z - g_z h_x)\,\mathbf{e}_2 + (g_y h_x - g_x h_y)\,\mathbf{e}_3$	
Line containing point $\mathbf{p}$ and orthogonal to line $\mathbf{g}$. $\mathbf{p} \wedge \mathbf{g}^{\star} = p_z g_y \mathbf{e}_{23} - p_z g_x \mathbf{e}_{31} + (p_y g_x - p_x g_y)\,\mathbf{e}_{12}$	

Table 2.30. These are the join, meet, and weight expansion operations in two dimensions.

$$\mathbf{G} = \begin{bmatrix} 1 & \square & \square & \square & \square & \square & \square & \square \\ \square & 1 & 0 & 0 & \square & \square & \square & \square \\ \square & 0 & 1 & 0 & \square & \square & \square & \square \\ \square & 0 & 0 & 0 & \square & \square & \square & \square \\ \square & \square & \square & \square & 0 & 0 & 0 & \square \\ \square & \square & \square & \square & 0 & 0 & 0 & \square \\ \square & \square & \square & \square & 0 & 0 & 1 & \square \\ \square & \square & \square & \square & \square & \square & \square & 0 \end{bmatrix} \qquad \mathbb{G} = \begin{bmatrix} 0 & \square & \square & \square & \square & \square & \square & \square \\ \square & 0 & 0 & 0 & \square & \square & \square & \square \\ \square & 0 & 0 & 0 & \square & \square & \square & \square \\ \square & 0 & 0 & 1 & \square & \square & \square & \square \\ \square & \square & \square & \square & 1 & 0 & 0 & \square \\ \square & \square & \square & \square & 0 & 1 & 0 & \square \\ \square & \square & \square & \square & 0 & 0 & 0 & \square \\ \square & \square & \square & \square & \square & \square & \square & 1 \end{bmatrix}$$

Figure 2.15. These 8×8 matrices are the metric exomorphism $\mathbf{G}$ and metric antiexomorphism $\mathbb{G}$ in the 3D projective exterior algebra representing 2D Euclidean space.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{u} \bullet \mathbf{u}$	1	1	1	0	0	0	1	0
$\mathbf{u} \circ \mathbf{u}$	0	0	0	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	0	$\mathbb{1}$

Table 2.31. These are the dot and antidot products between each of the 8 basis elements in the 3D projective exterior algebra with themselves.

The bulk of an object $\mathbf{u}$ is composed of the components of $\mathbf{u}$ that do not contain a factor of $\mathbf{e}_3$, and these can be derived from the metric through the product $\mathbf{Gu}$. The weight is composed of the components that do contain a factor of $\mathbf{e}_3$, and these can be derived from the antimetric through the product $\mathbb{G}\mathbf{u}$. The bulks and weights of points and lines are listed in Table 2.32.

Type	Bulk	Weight
Point $\mathbf{p}$	$\mathbf{p}_\bullet = p_x\mathbf{e}_1 + p_y\mathbf{e}_2$	$\mathbf{p}_\circ = p_z\mathbf{e}_3$
Line $\mathbf{g}$	$\mathbf{g}_\bullet = g_z\mathbf{e}_{12}$	$\mathbf{g}_\circ = g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31}$

Table 2.32. These are the bulks and weights of geometric objects in two dimensions.

The attitude of an object $\mathbf{u}$ is calculated by intersecting $\mathbf{u}$ with the complement of the origin, which is defined in the 3D projective algebra as

Attitude (2D)

$$\boxed{\operatorname{att}(\mathbf{u}) = \mathbf{u} \vee \overline{\mathbf{e}}_3.} \tag{2.178}$$

The attitudes of points and lines are listed in Table 2.33. As in all dimensions, the attitude of a point is just a scalar weight that's not terribly interesting. The attitude of a line $\mathbf{g}$ is a vector that points along the line's direction, and it's a 90-degree clockwise turn of the normal given by the g_x and g_y coordinates. As a side note, had we decided to use basis bivectors $\mathbf{e}_{31}$ and $\mathbf{e}_{32}$ so that a line $\boldsymbol{l}$ could be written as $\boldsymbol{l} = l_{vx}\mathbf{e}_{31} + l_{vy}\mathbf{e}_{32} + l_m\mathbf{e}_{12}$ with (l_{vx}, l_{vy}) corresponding to the line's direction, then the attitude of such a line would be given by $l_{vx}\mathbf{e}_1 + l_{vy}\mathbf{e}_2$. The result is the same either way.

Type	Attitude
Point $\mathbf{p}$	$\operatorname{att}(\mathbf{p}) = p_z\mathbf{1}$
Line $\mathbf{g}$	$\operatorname{att}(\mathbf{g}) = g_y\mathbf{e}_1 - g_x\mathbf{e}_2$

Table 2.33. These are the attitudes of geometric objects in two dimensions.

The bulk norms and weight norms of geometric objects are calculated with the dot and antidot products exactly as defined previously in Section 2.10. Formulas for these values in two dimensions are listed in Table 2.34. Geometric objects are unitized by making the weight norm equal to the unit antiscalar. The conditions under which this is true are listed in Table 2.35. The geometric norms for objects in two dimensions are listed in Table 2.36, and these correspond to the distances between the objects and the origin.

Type	Bulk Norm	Weight Norm
Point $\mathbf{p}$	$\left\Vert\mathbf{p}\right\Vert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2}$	$\left\Vert\mathbf{p}\right\Vert_\circ = \lvert p_z\rvert\,\mathbb{1}$
Line $\mathbf{g}$	$\left\Vert\mathbf{g}\right\Vert_\bullet = \lvert g_z\rvert\,\mathbf{1}$	$\left\Vert\mathbf{g}\right\Vert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2}$

Table 2.34. These are the bulk norms and weight norms of geometric objects in two dimensions.

Type	Definition	Unitization
Point **p**	$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$	$p_z^2 = 1$
Line **g**	$\mathbf{g} = g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12}$	$g_x^2 + g_y^2 = 1$

Table 2.35. The right column lists the conditions under which geometric objects in two dimensions are considered to be unitized.

Type	Geometric Norm	Interpretation
Point **p**	$\widehat{\lVert\mathbf{p}\rVert} = \dfrac{\sqrt{p_x^2 + p_y^2}}{\lvert p_z \rvert}$	Distance from the origin to the point **p**.
Line **g**	$\widehat{\lVert\mathbf{g}\rVert} = \dfrac{\lvert g_z \rvert}{\sqrt{g_x^2 + g_y^2}}$	Perpendicular distance from the origin to the line **g**.

Table 2.36. These are the scalar parts of the geometric norms of objects in two dimensions after unitization.

The distance between two geometric objects is given in general by Equation (2.99), and the specific cases arising in two dimensions are listed in Table 2.37. In the case of two points, the distance formula in two dimensions is identical to the formula for two points in three dimensions except for the removal of one coordinate. In the case of a point and a line, the distance formula in two dimensions is similar to the formula for a point and a plane in three dimensions, not a point and a line. This distance is signed because it's possible to classify a point **p** as being on the front side or the back side of a line **g**. However, the sign of the antiwedge product $\mathbf{p} \vee \mathbf{g}$ is flipped compared to the sign we get when classifying a point with respect to a plane in three dimensions. This product is commutative in two dimensions, but the comparable product between a point and a plane in three dimensions is anticommutative. Thus, if we were to calculate $\mathbf{g} \vee \mathbf{p}$ in both cases, then we would always get a negative value for a position lying on the side of a plane toward which the normal points. If we always wanted a positive value instead for this case, we could just negate the underlying terms of $\mathbf{g} \vee \mathbf{p}$, and we would find that it's the same as calculating an ordinary dot product $\mathbf{p} \cdot \mathbf{g}$ as if **p** and **g** were just treated as ordinary vectors having three components in the two-dimensional case and four components in the three-dimensional case.

The bulk dual of an object **u** is the complement of its bulk, and it is given by $\mathbf{u}^\star = \overline{\mathbf{G}\mathbf{u}}$. The weight dual is the complement of the weight, and it is given by $\mathbf{u}^{\smallstar} = \overline{\mathbb{G}\mathbf{u}}$. There is no difference between right and left duals in the 3D exterior algebra because there is no difference between right and left complements. The bulk duals and weight duals of the eight basis elements are shown in Table 2.38, and the bulk duals and weight duals of points and lines are listed in Table 2.39.

The four different types of interior products are constructed by taking the wedge product or antiwedge product of one object with the bulk dual or weight dual of another object. The interior product that is most useful when is comes to geometric manipulation is the weight expansion of a point **p** onto a line **g**, which is given by

$$\mathbf{p} \wedge \mathbf{g}^{\smallstar} = -p_z g_y \mathbf{e}_{23} + p_z g_x \mathbf{e}_{31} + \left(p_x g_y - p_y g_x\right) \mathbf{e}_{12}. \tag{2.179}$$

This produces a new line that contains the point **p** and is orthogonal to the line **g** as illustrated at the bottom of Table 2.30. Intersecting this new line with the original line **g** using the antiwedge product projects the point **p** onto **g**.

The weight contraction $\mathbf{g} \vee \mathbf{h}^{\star}$ produces the angle between two lines $\mathbf{g}$ and $\mathbf{h}$ using the formula given by Equation (2.146). The simplified formula in two dimensions is listed at the bottom of Table 2.37, and it is very similar to the formula for the angle between two planes in three dimensions. In both cases, the cosine of the angle is allowed to be positive or negative because the angle can be anything between zero and 180 degrees.

The four separate interior products allow us to calculate the same four types of projections in two dimensions that we had in three dimensions, but the only nontrivial pairing of geometries here is a point and a line. Formulas for orthogonal and central projections and antiprojections involving a point and a line are listed in Table 2.40 along with illustrations. The support of a line and the antisupport of a point are specific cases of these projections. The support of a line $\mathbf{g}$ is the point given by the orthogonal projection of the origin $\mathbf{e}_3$ onto $\mathbf{g}$, and the explicit formula is

$$\operatorname{sup}(\mathbf{g}) = -g_x g_z \mathbf{e}_1 - g_y g_z \mathbf{e}_2 + \left(g_x^2 + g_y^2\right)\mathbf{e}_3. \tag{2.180}$$

The antisupport of a point $\mathbf{p}$ is the line given by the central antiprojection of the complement of the origin $\overline{\mathbf{e}}_3$ onto $\mathbf{p}$, and the explicit formula is

$$\operatorname{asp}(\mathbf{p}) = p_x p_z \mathbf{e}_{23} + p_y p_z \mathbf{e}_{31} - \left(p_x^2 + p_y^2\right)\mathbf{e}_{12}. \tag{2.181}$$

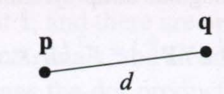
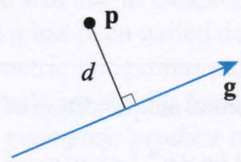
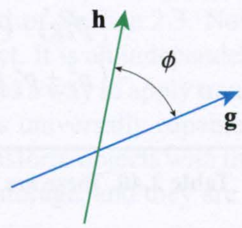
Distance / Angle Formula	**Illustration**
Distance d between points $\mathbf{p}$ and $\mathbf{q}$. $d(\mathbf{p},\mathbf{q}) = \lVert \mathbf{q}_{xy} p_z - \mathbf{p}_{xy} q_z \rVert \mathbf{1} + \lvert p_z q_z \rvert \mathbb{1}$	
Perpendicular distance d between point $\mathbf{p}$ and line $\mathbf{g}$. $d(\mathbf{p},\mathbf{g}) = -(\mathbf{p}\cdot\mathbf{g})\mathbf{1} + \lVert \mathbf{g}_{xy} p_z \rVert \mathbb{1}$	
Cosine of angle ϕ between lines $\mathbf{g}$ and $\mathbf{h}$. $\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xy}\cdot\mathbf{h}_{xy})\mathbf{1} + \lVert\mathbf{g}\rVert_{\circ}\lVert\mathbf{h}\rVert_{\circ}$	

Table 2.37. These are the Euclidean distances and cosines of the Euclidean angles involving lines and planes in two dimensions, expressed as homogeneous magnitudes.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{u}^{\star}$	$\mathbb{1}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	0	0	0	$-\mathbf{e}_3$	0
$\mathbf{u}^{\star}$	0	0	0	$-\mathbf{e}_{12}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	0	$\mathbf{1}$

Table 2.38. For each of the 8 basis elements $\mathbf{u}$ in the 3D projective exterior algebra, this table lists the bulk dual $\mathbf{u}^{\star}$ and the weight dual $\mathbf{u}^{\star}$. (Right and left duals are equivalent in odd numbers of dimensions.)

Type	Bulk Dual	Weight Dual
Point $\mathbf{p}$	$\mathbf{p}^{\bigstar} = -p_x\mathbf{e}_{23} - p_y\mathbf{e}_{31}$	$\mathbf{p}^{\text{☆}} = -p_z\mathbf{e}_{12}$
Line $\mathbf{g}$	$\mathbf{g}^{\bigstar} = -g_z\mathbf{e}_3$	$\mathbf{g}^{\text{☆}} = -g_x\mathbf{e}_1 - g_y\mathbf{e}_2$

Table 2.39. These are the bulk duals and weight duals of geometric objects in two dimensions.

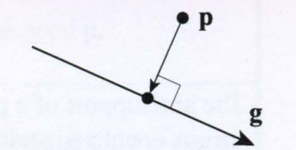
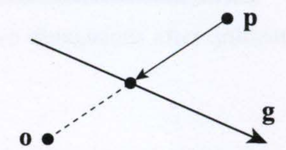
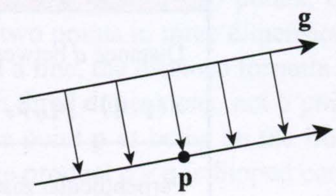
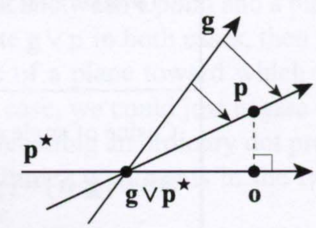
Projection Operation	Illustration
Orthogonal projection of point $\mathbf{p}$ onto line $\mathbf{g}$. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\text{☆}}) = (g_x^2 + g_y^2)(p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3) - (g_xp_x + g_yp_y + g_zp_z)(g_x\mathbf{e}_1 + g_y\mathbf{e}_2)$	
Central projection of point $\mathbf{p}$ onto line $\mathbf{g}$. $\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\bigstar}) = g_z^2(p_x\mathbf{e}_1 + p_y\mathbf{e}_2) - (g_xp_x + g_yp_y)g_z\mathbf{e}_3$	
Orthogonal antiprojection of line $\mathbf{g}$ onto point $\mathbf{p}$. $\mathbf{p} \wedge (\mathbf{g} \vee \mathbf{p}^{\text{☆}}) = p_z^2(g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31}) - (p_xg_x + p_yg_y)p_z\mathbf{e}_{12}$	
Central antiprojection of line $\mathbf{g}$ onto point $\mathbf{p}$. $\mathbf{p} \wedge (\mathbf{g} \vee \mathbf{p}^{\bigstar}) = [p_y^2g_x - (p_yg_y + p_zg_z)p_x]\mathbf{e}_{23} + [p_x^2g_y - (p_xg_x + p_zg_z)p_y]\mathbf{e}_{31} + (p_x^2 + p_y^2)g_z\mathbf{e}_{12}$	

Table 2.40. These are the four types of projection involving a point and a line in two dimensions.

Math Library Notes

- The `Vector2D` and `Point2D` classes both store only the x and y coordinates of a homogeneous 3D vector. The z coordinate is implicitly zero for a `Vector2D` object, and it is implicitly one for a `Point2D` object. There is also a `FlatPoint2D` class that stores all three coordinates explicitly.
- The `Line2D` class stores the three coordinates of a 3D bivector representing a flat line, and they are named `x`, `y`, and `z`.
- The join of two points and the meet of two lines are implemented by the `Wedge()` and `Antiwedge()` functions. These can both also be calculated by using the ^ symbol as an infix operator.

2.15 Dependencies

Despite our focus on projective geometry, this chapter has established the foundations for a wide variety of geometric algebras that are based on the wedge product and a metric. Given a dimensionality n, we start with n basis vectors and generate the entire exterior algebra with the wedge product. The only choice we have at this point is the convention for ordering the factors of basis elements having grades higher than one. The order that we choose for the volume element $\mathbb{1}$ is particularly important because it affects the signs of complements, which in turn affect the signs of values produced by anti-operations. These choices don't impact the geometric interpretation in any significant way other than to change the orientation of some things here and there. The choice that does matter substantially is that of the metric tensor. The manner in which basis vectors multiply under the dot product determines the overall structure of the entire algebra. In this chapter, we have focused on a 4D algebra in which three basis vectors square to one, and the remaining basis vector squares to zero. The elements of this algebra are interpreted as flat geometries that arise in 3D space. In Chapters 4 and 5, we will encounter another 4D algebra whose elements are interpreted as round geometries that arise in 2D space, one dimension less. Both algebras use the same exact set of basis elements, and they have the same structure under the wedge and antiwedge products, but a different metric leads to different dot products, different duals, and a different geometric meaning.

Figure 2.16 illustrates the dependencies among all of the mathematical constructs within geometric algebra that we have discussed in this chapter plus a couple more that will be introduced at the beginning of the next chapter. There are two configurable pieces where we are able to make some choices, and they are highlighted in bold green boxes. Once those have been established, the rest of the algebraic structure is strictly determined by the definitions of the operations. First, we have a choice to make about the orientation of the volume element $\mathbb{1}$, and there are only two possibilities. The orientation that we select ultimately affects complements, duals, and all anti-operations. Second, we choose the form of the $n \times n$ metric tensor $\mathbf{g}$ that defines the dot products between all pairs of basis vectors. This is usually a diagonal matrix, but as we will see in Chapter 4, it can be convenient to transform it into a matrix that is not diagonal. Once $\mathbf{g}$ has been nailed down, the full metric exomorphism $\mathbf{G}$ can be built in only one way. Then the metric antiexomorphism, the dot product, the antidot product, the geometric norm, duals, and antiduals are rigidly fixed.

There are two new operations appearing in Figure 2.16, the *geometric product* between two objects $\mathbf{a}$ and $\mathbf{b}$ and the unary *reverse* operation. Each of these also has a corresponding anti-operation that is derived using the same method discussed at the end of Section 2.3. Nothing we've covered in this chapter has a dependency on the geometric product. It is an independent operation that we add to the exterior algebra in the next chapter, and it provides a way to apply transformations to geometric objects using other geometric objects. Though not as universally capable as the exomorphism matrices discussed in Section 2.7, the operators that transform objects with the geometric product have a greater connection to geometry, they require less storage, and they are more easily parameterized.

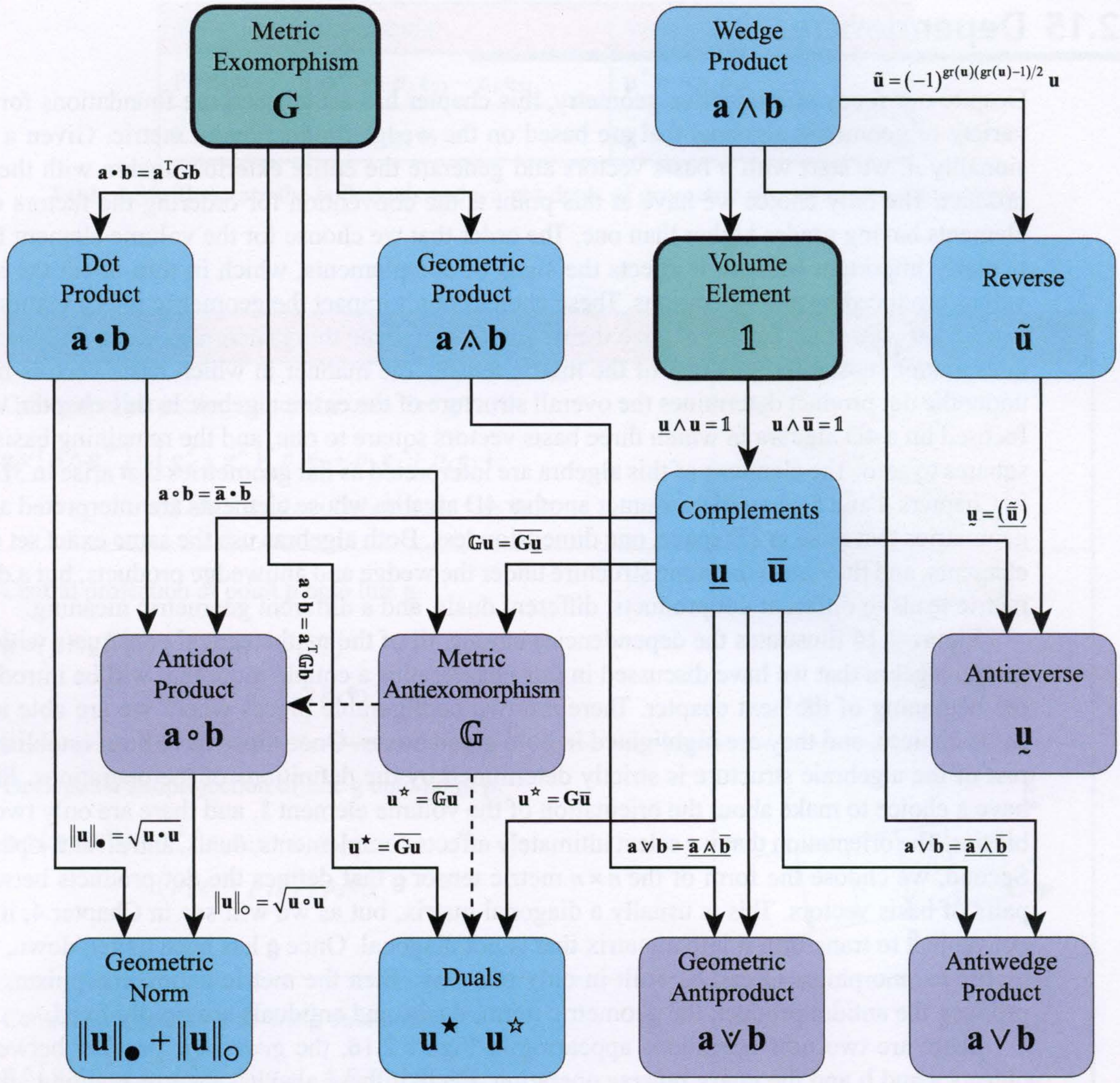


Figure 2.16. The complete structure of any geometric algebra depends on two configurable pieces, the metric exomorphism $\mathbf{G}$ and the orientation of the volume element $\mathbb{1}$, shown in bold green boxes. Everything else is then strictly determined by the rigid definitions of operations shown in the blue boxes. Each purple box contains an anti-operation defined by its complementary relationship with the corresponding regular operation. The dashed lines demonstrate that it is possible to define the antidot product and antidual operations through two different dependency paths.

Historical Remarks

Hermann Grassmann
(1809–1877)

What is today known as exterior algebra was first developed by German mathematician Hermann Grassmann in the mid 1800s, and it is often called Grassmann algebra in his honor. He published a book entitled *Die Lineale Ausdehnungslehre*, (which translates to *Linear Extension Theory*) in 1844, and it contained many of the foundational ideas of linear algebra that would not be formalized in mainstream mathematics until decades later. Due to a writing style from which it was difficult to extract concrete meaning, Grassmann's work was not well understood or appreciated for the ground it broke at the time. His *Linear Extension Theory* was largely ignored even after a thorough revision [Gras1862] was published in 1862. Grassmann algebra became better known once it was incorporated into geometric algebra by Clifford in 1878 (see historical remarks in Chapter 3), but it was overshadowed by the rise of vector analysis around the same time. Only in the 20th century was Grassmann's work properly recognized for being well ahead of its time.

Grassmann did not introduce the upward and downward wedge symbols in use today, nor did he make the same distinction between the exterior product and its antiproduct. His work described a product called the *combinatorial product* that was written as $[EF]$ and operated on values E and F called *extensive magnitudes*. In n-dimensional space, if the grades of E and F satisfied $\operatorname{gr}(E) + \operatorname{gr}(F) \leq n$, then the combinatorial product calculated what we define as $E \wedge F$, and it was called a *progressive product*. Otherwise, if $\operatorname{gr}(E) + \operatorname{gr}(F) > n$, then the combinatorial product calculated what we define as $E \vee F$, and it was called a *regressive product*. When the grades summed to n, the modern wedge product would produce an antiscalar value with a type distinct from a scalar value, but all single-component quantities were considered to be plain old numbers in Grassmann's work. It is now known that scalars and antiscalars must be treated as separate types of values in a complete and correct theory.

The concept of a complement appeared in Grassmann's work, and the complement of a quantity E was denoted by $|E$. Though this was defined as the equivalent of a right complement such that $[E|E] = 1$ for a basis element E, no symmetric left complement was defined to go with it. The regressive product was expressed in terms of complements using a form of the De Morgan relationship that Grassmann wrote as $|[EF] = [|E|F]$. The ambiguity in the combinatorial product meant that this expression served to define $[EF]$ when $\operatorname{gr}(E) + \operatorname{gr}(F) > n$. In the case of a progressive product, where $\operatorname{gr}(E) + \operatorname{gr}(F) \leq n$, Grassmann also used the name *äusseren multiplikation*, which translates to *external* or *exterior multiplication*, for the operation $[EF]$. He explained that the name of this product derived from the fact that $[EF] = 0$ unless one of the factors geometrically lies outside the other, in its exterior. This name was also intended to distinguish exterior multiplication from the operation $[E|F]$ that he called the *inneren multiplikation*, which translates to *internal* or *interior multiplication*. (Grassmann's interior product could be equivalent to a contraction or expansion depending on the grades of E and F.) The notion of a metric had not yet been invented at the time Grassmann did most of his work, so the more general dual that we use in our interior products was never anything more than the complement operation in his interior product, corresponding to the identity metric.

Chapter 3

Rigid Transformations

In the previous chapter, we developed the exterior algebra and used it to model flat homogeneous geometries. With that knowledge, we are able to calculate joins, meets, contractions, and expansions to combine geometric objects in various ways. We can perform projections, and we can measure distances and angles. Using exomorphism matrices, we can extend any linear transformation from points to all types of geometries represented in the algebra. Everything we have done so far subsumes the conventional models of points, lines, and planes discussed in Chapter 1, but we have not yet discussed the place of quaternions and dual quaternions in geometric algebra. In this chapter, we focus on transformations that are performed with operators built directly from the elements of the projective exterior algebra and applied by using a new operation called the *geometric product*. The transformations that we study now are all rigid isometries, meaning that they preserve lengths and angles. Basically, the size and shape of an object does not change when a rigid transformation is applied. In Chapter 5, we will expand our set of operations to include conformal transformations, which preserve angles but not necessarily lengths, so sizes and shapes will be able to change in certain ways.

3.1 The Geometric Product

The geometric product operates on the same elements of the exterior algebra as the wedge product does. We write the geometric product between **a** and **b** using the notation $\mathbf{a} \mathbin{\dot\wedge} \mathbf{b}$, with an upward pointing wedge symbol containing a dot,[1] and we read this as "**a** wedge-dot **b**". The geometric product actually includes the wedge product in its result, but it also produces additional components that make it behave differently from the wedge product. The infix symbol $\dot\wedge$ is meant to reflect the fact that the geometric product is the wedge product plus something more. Whereas the defining characteristic of the wedge product given by Equation (2.1) states that any vector multiplied by itself must be zero, the geometric product is built from the requirement that

$$\boxed{\mathbf{v} \mathbin{\dot\wedge} \mathbf{v} = \mathbf{v} \bullet \mathbf{v}} \tag{3.1}$$

for any vector **v**. That is, the geometric product of any vector with itself must be equal to the dot product of that vector with itself. This means the geometric product depends on the metric **g**, unlike the wedge product, because the dot product of **v** with itself is defined as $\mathbf{v} \bullet \mathbf{v} = \mathbf{v}^{\mathrm{T}} \mathbf{g} \mathbf{v}$.

[1] The geometric product has historically been written as juxtaposition without any multiplication symbol so that the geometric product between two quantities **a** and **b** is simply denoted by **ab**. However, in a complete picture of geometric algebra based on a full understanding of duality, the geometric product appears as a pair of complementary operations just like everything else. The notation **ab** is not adequate for distinguishing between the two products, and juxtaposition is used in this book only for matrix products and multiplication by scalars.

We can define an antiproduct for the geometric product just as we have for all other products defined over the basis elements of the exterior algebra. The *geometric antiproduct* between **a** and **b** is denoted by $\mathbf{a} ⟇ \mathbf{b}$, with a downward pointing wedge symbol containing a dot, and it is read as "**a** antiwedge-dot **b**". The values produced by the geometric antiproduct are defined with the usual De Morgan laws such that

$$\mathbf{a} ⟇ \mathbf{b} = \overline{\underline{\mathbf{a}} ⟑ \underline{\mathbf{b}}} \quad \text{and} \quad \mathbf{a} ⟇ \mathbf{b} = \underline{\overline{\mathbf{a}} ⟑ \overline{\mathbf{b}}}. \tag{3.2}$$

The geometric antiproduct satisfies a property dual to Equation (3.1) that applies to antivectors instead of vectors. For any antivector **u** (which has grade $n-1$ in an n-dimensional algebra), we have the relationship

$$\mathbf{u} ⟇ \mathbf{u} = \mathbf{u} \circ \mathbf{u}. \tag{3.3}$$

In this section and the next, we focus on the geometric product, but the antiproduct will be important for the greater part of this chapter when we talk about rigid transformations of points, lines, and planes in a projective space.

As summarized in Table 3.1, the geometric product possesses the same associative and distributive properties as the wedge product, and it behaves the same way under scalar multiplication. Because the geometric product contains the wedge product, it must exhibit some anticommutative properties when we start multiplying vectors and higher-grade elements together, but it gets more complicated due to the additional terms it produces. If we consider the geometric product of a sum of two vectors **a** and **b**, as we did for the wedge product in Section 2.1.1, then we have the equality

$$(\mathbf{a}+\mathbf{b}) ⟑ (\mathbf{a}+\mathbf{b}) = (\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}+\mathbf{b}). \tag{3.4}$$

Expanding both sides of this equation independently gives us

$$\mathbf{a} ⟑ \mathbf{a} + \mathbf{a} ⟑ \mathbf{b} + \mathbf{b} ⟑ \mathbf{a} + \mathbf{b} ⟑ \mathbf{b} = \mathbf{a} \bullet \mathbf{a} + 2\,\mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{b}, \tag{3.5}$$

where we have used the fact that the dot product is commutative. The products $\mathbf{a} ⟑ \mathbf{a}$ and $\mathbf{b} ⟑ \mathbf{b}$ on the left side cancel the products $\mathbf{a} \bullet \mathbf{a}$ and $\mathbf{b} \bullet \mathbf{b}$ on the right side, and we are left with

$$\mathbf{a} ⟑ \mathbf{b} + \mathbf{b} ⟑ \mathbf{a} = 2\,\mathbf{a} \bullet \mathbf{b} \tag{3.6}$$

as a property of the geometric product that holds for all vectors. If **a** and **b** are orthogonal such that $\mathbf{a} \bullet \mathbf{b} = 0$, then we recover the anticommutative relationship $\mathbf{a} ⟑ \mathbf{b} = -\mathbf{b} ⟑ \mathbf{a}$ from the wedge product.

We can now form a complete picture of what the geometric product does when two vectors are multiplied together. As an example case, we consider arbitrary 3D vectors $\mathbf{a} = a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3$ and $\mathbf{b} = b_x\mathbf{e}_1 + b_y\mathbf{e}_2 + b_z\mathbf{e}_3$. The geometric product $\mathbf{a} ⟑ \mathbf{b}$ is given by

$$\begin{aligned}(a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3) ⟑ (b_x\mathbf{e}_1 + b_y\mathbf{e}_2 + b_z\mathbf{e}_3) &= a_xb_x\,\mathbf{e}_1 ⟑ \mathbf{e}_1 + a_yb_y\,\mathbf{e}_2 ⟑ \mathbf{e}_2 + a_zb_z\,\mathbf{e}_3 ⟑ \mathbf{e}_3 \\ &+ (a_yb_z - a_zb_y)\,\mathbf{e}_2 ⟑ \mathbf{e}_3 + (a_zb_x - a_xb_z)\,\mathbf{e}_3 ⟑ \mathbf{e}_1 + (a_xb_y - a_yb_x)\,\mathbf{e}_1 ⟑ \mathbf{e}_2.\end{aligned} \tag{3.7}$$

Each product of a basis vector with itself can be replaced by $\mathbf{e}_i \bullet \mathbf{e}_i$ through the defining property given by Equation (3.1). The terms containing the product of two distinct basis vectors have exactly the same coefficients as those given by the wedge product. We come to the conclusion that the geometric product between two vectors can be written as

Geometric product of vectors

$$\boxed{\mathbf{a} ⟑ \mathbf{b} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b},} \tag{3.8}$$

which is a multivector containing both a scalar part having grade zero and a bivector part having grade two.

Property	Description
$(\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b}) \mathbin{\wedge\!\!\!\!\wedge} \mathbf{c} = \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} (\mathbf{b} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{c})$	Associative law for the geometric product.
$\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} (\mathbf{b}+\mathbf{c}) = \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b} + \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{c}$ $(\mathbf{a}+\mathbf{b}) \mathbin{\wedge\!\!\!\!\wedge} \mathbf{c} = \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{c} + \mathbf{b} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{c}$	Distributive laws for the geometric product.
$(s \mathbin{\wedge\!\!\!\!\wedge} \mathbf{a}) \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b} = \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} (s \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b}) = s(\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b})$	Scalar factorization for the geometric product.
$s \mathbin{\wedge\!\!\!\!\wedge} t = st$	Geometric product between scalars.
$s \mathbin{\wedge\!\!\!\!\wedge} \mathbf{a} = \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} s = s\mathbf{a}$	Geometric product between a scalar and a multivector.

Table 3.1. These are the basic properties of the geometric product. The letters **a**, **b**, and **c** represent arbitrary multivectors, and the letters s and t represent scalar values.

It's important to realize that Equation (3.8) applies only to vectors, and there are generally more parts created through the geometric product $\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b}$ between two quantities **a** and **b** having arbitrary grades. Each of these parts can have a grade k within the limits set by

$$|\operatorname{gr}(\mathbf{a}) - \operatorname{gr}(\mathbf{b})| \le k \le \operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b}), \tag{3.9}$$

but k must differ from either end of this range by an even number. The reason for this is that any piece of the geometric product that yields something of a grade lower than the upper limit does so because a particular basis vector in one factor is paired with the same basis vector in the other factor, and the two eliminate each other when they multiply to produce a scalar. The part of the geometric product $\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b}$ that does have the maximum grade is equal to the wedge product $\mathbf{a} \wedge \mathbf{b}$, but this only exists in the geometric product if $\operatorname{gr}(\mathbf{a}) + \operatorname{gr}(\mathbf{b})$ is not greater than the dimensionality of the algebra.

If we solve Equation (3.6) for $\mathbf{a} \bullet \mathbf{b}$ and substitute it in Equation (3.8), then we obtain the pair of relationships

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \tfrac{1}{2}(\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b} + \mathbf{b} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{a}) \\ \mathbf{a} \wedge \mathbf{b} &= \tfrac{1}{2}(\mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b} - \mathbf{b} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{a}). \end{aligned} \tag{3.10}$$

These equations highlight the fact that the geometric product yields both commutative and anticommutative components, represented by the dot product and the wedge product. The geometric product is completely commutative only when the vectors **a** and **b** are parallel because that's when the wedge product is zero. Otherwise, solving the second equation for $\mathbf{b} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{a}$ tells us that

$$\mathbf{b} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{a} = \mathbf{a} \mathbin{\wedge\!\!\!\!\wedge} \mathbf{b} - 2\,\mathbf{a} \wedge \mathbf{b}. \tag{3.11}$$

The geometric product is completely anticommutative only when the vectors **a** and **b** are perpendicular because that's when the dot product is zero. Otherwise, when **a** and **b** are neither parallel nor perpendicular, their geometric product contains a mixture of commutative and anticommutative parts.

In any geometric algebra, an arbitrary multivector is a sum of scalar multiples of the distinct basis elements. We can multiply any two multivectors together with the geometric product if we know how to multiply any two basis elements together. The distributive and scalar factorization properties listed in Table 3.1 give us everything else we need. We can just multiply component by component and combine any scalar factors in each pairing of the components belonging to the two multivectors. Each basis element is either the scalar **1**, a basis vector $\mathbf{e}_i$, or the wedge product of two or more basis vectors. We can assume that all pairs of basis vectors are orthogonal such that $\mathbf{e}_i \bullet \mathbf{e}_j = 0$ unless $i = j$, so it's always the case that $\mathbf{e}_i \mathbin{⟑} \mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$ and $\mathbf{e}_j \mathbin{⟑} \mathbf{e}_i = -\mathbf{e}_i \wedge \mathbf{e}_j$ if $i \neq j$. (This follows from the fact that it's always possible to choose a basis in which the metric tensor is a diagonal matrix.) Given two basis elements **a** and **b**, we can now follow a few simple rules to calculate the geometric product $\mathbf{a} \mathbin{⟑} \mathbf{b}$. Suppose that the vector factorizations of **a** and **b** are

$$\mathbf{a} = \mathbf{e}_{a_1} \wedge \mathbf{e}_{a_2} \wedge \cdots \quad \text{and} \quad \mathbf{b} = \mathbf{e}_{b_1} \wedge \mathbf{e}_{b_2} \wedge \cdots. \tag{3.12}$$

For any vector factor $\mathbf{e}_k$ common to both **a** and **b**, we move $\mathbf{e}_k$ to the right in the factorization of **a** so it's the last factor, and we move $\mathbf{e}_k$ to the left in the factorization of **b** so it's the first factor. Each time we swap the order of two consecutive factors in either **a** or **b** to accomplish this, we have to negate, so the overall change in sign will be $(-1)^{r+s}$, where r is the number of places that $\mathbf{e}_k$ moved right in **a**, and s is the number of places that $\mathbf{e}_k$ moved left in **b**. After we do this, the product $\mathbf{a} \mathbin{⟑} \mathbf{b}$ looks like

$$\mathbf{a} \mathbin{⟑} \mathbf{b} = (-1)^{r+s} (\mathbf{e}_{a_1} \wedge \mathbf{e}_{a_2} \wedge \cdots \wedge \mathbf{e}_k) \mathbin{⟑} (\mathbf{e}_k \wedge \mathbf{e}_{b_1} \wedge \mathbf{e}_{b_2} \wedge \cdots), \tag{3.13}$$

where the index k has been removed from the lists $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$. Since the two factors of $\mathbf{e}_k$ are now next to each other, we can reassociate and calculate $\mathbf{e}_k \mathbin{⟑} \mathbf{e}_k$ first, which becomes the scalar value given by the dot product $\mathbf{e}_k \bullet \mathbf{e}_k$. (This dot product will often be $+1$, but it could also be -1 or 0 if the metric tensor is not the identity.) The factor $\mathbf{e}_k$ has been eliminated from both **a** and **b** at this point, lowering the grade of each operand by one and the entire product by two. We repeat this process for any other common factors belonging to **a** and **b** until we can go no further. If any basis vectors remain, then they are all distinct, and we permute them into the preferred order that we've chosen for the unique basis element having those vector factors, keeping in mind that we need to negate if the permutation is odd. This basis element, multiplied by all the dot products that were calculated and the powers of -1 that were accumulated, is the final geometric product of **a** and **b**. As an example, consider the basis elements $\mathbf{a} = \mathbf{e}_{12}$ and $\mathbf{b} = \mathbf{e}_{431}$ in the 4D geometric algebra with the metric given by Equation (2.60). These have the vector factor $\mathbf{e}_1$ in common, so we move it right one place in **a**, which incurs a negation, and we move it left two places in **b**, which does not change the sign. The geometric product is then

$$\mathbf{a} \mathbin{⟑} \mathbf{b} = -\mathbf{e}_2 \wedge \mathbf{e}_1 \mathbin{⟑} \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_3. \tag{3.14}$$

Since $\mathbf{e}_1 \bullet \mathbf{e}_1 = 1$, this reduces to $\mathbf{a} \mathbin{⟑} \mathbf{b} = -\mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_3 = -\mathbf{e}_{243}$. The preferred ordering that we have chosen for the vectors in the resulting trivector is 423, so we swap the first two factors with a negation and arrive at $\mathbf{e}_{12} \mathbin{⟑} \mathbf{e}_{431} = \mathbf{e}_{423}$.

The main focus of this chapter is the transformation of flat geometries in the 4D projective algebra where the metric defines the dot products $\mathbf{e}_1 \bullet \mathbf{e}_1 = 1$, $\mathbf{e}_2 \bullet \mathbf{e}_2 = 1$, $\mathbf{e}_3 \bullet \mathbf{e}_3 = 1$, and $\mathbf{e}_4 \bullet \mathbf{e}_4 = 0$. The full multiplication table for the geometric product between all pairs of the 16 basis elements is shown in Table 3.2. The degeneracy of the basis vector $\mathbf{e}_4$ means that the geometric product $\mathbf{a} \mathbin{⟑} \mathbf{b}$ of any two basis elements **a** and **b** both containing a factor of $\mathbf{e}_4$ is zero no matter what the grades of **a** and **b** are. When we move the $\mathbf{e}_4$ factor to the right end of **a** and to the left end of **b**, the product $\mathbf{e}_4 \mathbin{⟑} \mathbf{e}_4$ in the middle annihilates the entire term. For this reason, a full quarter of the multiplication table is filled with zeros. The same is true for the multiplication table for the geometric antiproduct, which is shown in Table 3.3. In this case, a zero appears for the geometric antiproduct $\mathbf{a} \mathbin{⟇} \mathbf{b}$ whenever **a** and **b** are both *missing* a factor of $\mathbf{e}_4$.

Because the geometric product generates quantities having components of multiple grades, it will be convenient to extract components of one particular grade from a multivector **u** from time to time. The angled bracket notation $\langle \mathbf{u} \rangle_k$ is widely used to mean all components of **u** that have grade k, and it is called the *grade selection operator*. Using this notation, $\langle \mathbf{u} \rangle_0$ is the scalar part of **u**, $\langle \mathbf{u} \rangle_1$ extracts all vector components of **u**, $\langle \mathbf{u} \rangle_2$ extracts all bivector components of **u**, and so on up to the

Geometric Product **a ⟑ b**

$\mathbf{a}\backslash\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_4$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	0	0	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{423}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_4$	$\mathbf{e}_{423}$	0	0	0	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{431}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_4$	0	0	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{412}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$-\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{41}$	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	0	$\mathbf{e}_{412}$	$-\mathbf{e}_4$	$-\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{42}$	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_4$	0	0	0	$\mathbf{e}_{43}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{1}$	$\mathbf{e}_4$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_4$	0

Table 3.2. This is the multiplication table for the geometric product in the 4D projective algebra representing 3D Euclidean space.

Geometric Antiproduct **a ⟇ b**

$\mathbf{a}\backslash\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	0	$\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	0	0	$\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	0	0	0	$\mathbf{e}_{12}$	$\mathbf{1}$	$-\mathbf{e}_{23}$	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	$-\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{1}$	0	$\mathbf{e}_3$
$\mathbf{e}_4$	$-\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{1}$	$\mathbf{e}_4$
$\mathbf{e}_{41}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	$\mathbf{e}_{431}$	$-\mathbf{e}_{43}$	$-\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_4$	$\mathbf{e}_{423}$	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	$\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	0	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	0	0	0	$-\mathbf{e}_3$	$-\mathbf{e}_{321}$	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{12}$
$\mathbf{e}_{423}$	$-\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_4$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	$-\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_4$	$\mathbf{e}_{423}$	$\mathbf{e}_3$	$\mathbf{e}_{321}$	$-\mathbf{e}_1$	$\mathbf{e}_{43}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	$-\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{321}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbb{1}$	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$-\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Table 3.3. This is the multiplication table for the geometric antiproduct in the 4D projective algebra representing 3D Euclidean space.

antiscalar part of $\mathbf{u}$, which is denoted by $\langle\mathbf{u}\rangle_n$ in an n-dimensional algebra. Attempting to extract a component that does not exist in the multivector $\mathbf{u}$ produces zero. Any multivector $\mathbf{u}$ can be expressed as a summation over all grades of its own components by writing

$$\mathbf{u} = \langle\mathbf{u}\rangle_0 + \langle\mathbf{u}\rangle_1 + \cdots + \langle\mathbf{u}\rangle_n = \sum_{k=0}^{n} \langle\mathbf{u}\rangle_k . \tag{3.15}$$

In addition to the grade selection operator, we introduce the notation $u_{\mathbf{1}}$ and $u_{\mathbb{1}}$ to mean the coordinate value of the grade-zero scalar part of $\mathbf{u}$ and the antigrade-zero antiscalar part of $\mathbf{u}$, respectively. The use of a bold style $\mathbf{1}$ and blackboard bold style $\mathbb{1}$ in the subscript, corresponding to the basis elements for scalars and antiscalars, is intended to make it obvious which component we are referring to. In this case, the extracted coordinates are ordinary real numbers, so the antiscalar part $u_{\mathbb{1}}$ is not an antiscalar value. This differs from the grade selection operator, which retains the grades of the extracted components. The $u_{\mathbf{1}}$ and $u_{\mathbb{1}}$ notation works like the $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ notation for complex numbers. For example, if $\mathbf{u} = s\mathbf{1} + t\mathbb{1}$, then $\langle\mathbf{u}\rangle_n = t\mathbb{1}$, but $u_{\mathbb{1}} = t$.

One of the main differences between the geometric product and the simpler exterior product is that many elements of the algebra have inverses under the geometric product, whereas inverses do not exist under the exterior product. A vector $\mathbf{v}$ for which $\mathbf{v} \bullet \mathbf{v} \neq 0$ has an inverse given by

Vector inverse

$$\boxed{\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v} \bullet \mathbf{v}},} \tag{3.16}$$

and this is due to the fact that the product $\mathbf{v} \curlywedge \mathbf{v}$ is equal to the scalar quantity $\mathbf{v} \bullet \mathbf{v}$. In the case that $\mathbf{v} \bullet \mathbf{v} = 1$, the inverse of $\mathbf{v}$ is just $\mathbf{v}$ itself. The existence of this inverse allows us to investigate what it means to divide by a vector. For two vectors $\mathbf{a}$ and $\mathbf{b}$, the quotient $\mathbf{a}/\mathbf{b}$, which has the same meaning as $\mathbf{a} \curlywedge \mathbf{b}^{-1}$, must be the quantity $\mathbf{c}$ such that $\mathbf{a} = \mathbf{c} \curlywedge \mathbf{b}$. Thus, we can write the equation

$$\mathbf{a} = \frac{\mathbf{a}}{\mathbf{b}} \curlywedge \mathbf{b} = \frac{\mathbf{a} \curlywedge \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}} \curlywedge \mathbf{b}. \tag{3.17}$$

When we expand the product $\mathbf{a} \curlywedge \mathbf{b}$ with Equation (3.8), we get

$$\mathbf{a} = \frac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}} \mathbf{b} + \frac{\mathbf{a} \wedge \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}} \curlywedge \mathbf{b}. \tag{3.18}$$

The first term is exactly the projection of $\mathbf{a}$ onto $\mathbf{b}$ given by Equation (2.133) when $\mathbf{a}$ and $\mathbf{b}$ have equal grades, and this means that the second term must be the rejection of $\mathbf{a}$ from $\mathbf{b}$. We can therefore formulate the projection and rejection operations for vectors as

$$\begin{aligned} \mathbf{a}_{\parallel\mathbf{b}} &= (\mathbf{a} \bullet \mathbf{b}) \curlywedge \mathbf{b}^{-1} \\ \mathbf{a}_{\perp\mathbf{b}} &= (\mathbf{a} \wedge \mathbf{b}) \curlywedge \mathbf{b}^{-1}. \end{aligned} \tag{3.19}$$

The division of a bivector by a vector in the formula for the rejection is interesting because it illustrates the difference in the information contained in the purely bivector result of the wedge product $\mathbf{a} \wedge \mathbf{b}$ and the mixed scalar and bivector result of the geometric product $\mathbf{a} \curlywedge \mathbf{b}$. We mentioned in Section 2.1.2 that a bivector contains no information about its shape. Given a bivector $\mathbf{m} = \mathbf{a} \wedge \mathbf{b}$, we cannot expect to be able to recover the particular vector $\mathbf{a}$ originally used in the wedge product if we were to compute the quotient $\mathbf{m} \curlywedge \mathbf{b}^{-1}$ because there are infinitely many possibilities. However, in the case that $\mathbf{m} = \mathbf{a} \curlywedge \mathbf{b}$, where the geometric product appears in place of the wedge product, the scalar part of the multivector $\mathbf{m}$ carries additional information about the angle between $\mathbf{a}$ and $\mathbf{b}$ through the cosine associated with the dot product. This turns an amorphous bivector into a parallelogram having a definite restriction on the shapes it can assume. The magnitude and orientation of the vectors composing the sides of the parallelogram are still undetermined by $\mathbf{m}$ alone, but as soon as we actually specify one vector $\mathbf{b}$, the other vector $\mathbf{a}$ can always be recovered. The vector given by $\mathbf{m} ⟑ \mathbf{b}^{-1}$ is the unique vector $\mathbf{a}$ possessing the proper magnitude and forming the necessary angle with $\mathbf{b}$ such that $\mathbf{m} = \mathbf{a} ⟑ \mathbf{b}$.

In the case that the wedge product is used to calculate $\mathbf{m} = \mathbf{a} \wedge \mathbf{b}$, the lack of a scalar term in the result means that the shape of the oriented area corresponding to the bivector $\mathbf{m}$ should be a parallelogram having only right angles. This must be true because the cosine of 90 degrees produces the zero value that would end up in the scalar term of $\mathbf{a} ⟑ \mathbf{b}$. Thus, dividing $\mathbf{m} = \mathbf{a} \wedge \mathbf{b}$ by $\mathbf{b}$ yields a vector $\mathbf{c}$ that is orthogonal to $\mathbf{b}$ such that $\mathbf{m} = \mathbf{c} ⟑ \mathbf{b} = \mathbf{c} \wedge \mathbf{b}$. For $\mathbf{c} \wedge \mathbf{b}$ to produce the same area as $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{c}$ must be the rejection of $\mathbf{a}$ from $\mathbf{b}$, as shown in Figure 3.1.

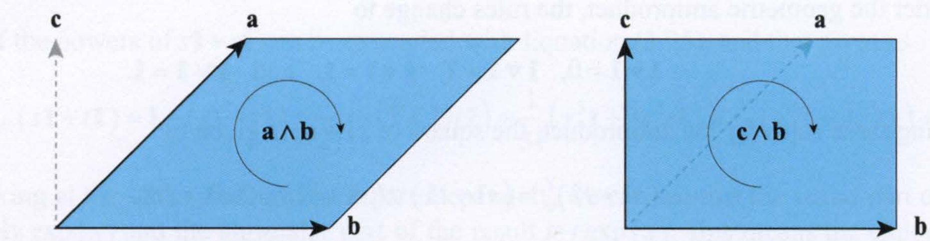


Figure 3.1. The bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c} \wedge \mathbf{b}$ have the same area when $\mathbf{c}$ is the rejection of $\mathbf{a}$ from $\mathbf{b}$. Because its zero scalar part enforces a right angle, the wedge product $\mathbf{a} \wedge \mathbf{b}$ behaves like the geometric product $\mathbf{c} ⟑ \mathbf{b}$, and thus $\mathbf{c}$ is produced by $(\mathbf{a} \wedge \mathbf{b}) ⟑ \mathbf{b}^{-1}$.

3.2 Dual Numbers

Dual numbers were briefly introduced in Section 1.4.4 and defined as two-component quantities $a + b\varepsilon$, where a and b are real numbers, and ε is a special nonzero value that satisfies $\varepsilon^2 = 0$. The use of the word "dual" in this context has no connection to the dual operations $\mathbf{u}^\bigstar$ and $\mathbf{u}^{☆}$ but merely describes that these numbers have two parts. The dual numbers can be found in projective geometric algebra as the set of scalar-antiscalar pairs of the form $s\mathbf{1} + t\mathbb{1}$. Since there are two geometric products, we can perform calculations with dual numbers in two different ways. Under the geometric product, the scalar $\mathbf{1}$ is the identity, and the antiscalar $\mathbb{1}$ has the property $\mathbb{1} ⟑ \mathbb{1} = 0$. In a symmetric manner, the roles of scalar and antiscalar exchange places under the geometric antiproduct such that $\mathbb{1}$ is the identity, and $\mathbf{1}$ has the property $\mathbf{1} ⟇ \mathbf{1} = 0$.

We will need to perform calculations with dual numbers such as finding an inverse, taking a square root, and evaluating trigonometric functions. We derive these operations here and organize the results into the handy reference provided by Table 3.4 so we don't have to worry about any of them later. In order to distinguish between calculations taking place with the geometric product and those taking place with the geometric antiproduct, we insert the product symbol ⟑ or ⟇ next to the multiplicative operation to which it applies. When raising a quantity to a power, the product symbol appears as a subscript so that $x_{⟑}^{2}$ means squared with respect to the geometric product, and $x_{⟇}^{2}$ means squared with respect to the geometric antiproduct. When performing division, we could write $x_{⟑}^{-1}$ for the divisor, but in the case of a stacked fraction, the product symbol is placed at the right end of the fraction bar so that, for example,

$$\frac{1}{x}{}_{⟇} \tag{3.20}$$

means divide by x with the geometric antiproduct. For square roots, the product symbol is placed to the left of the radical as in $\sqrt[⟑]{x}$. Finally, for exponential and trigonometric functions, the product symbol appears as a subscript after the function name so that $\sin_{\wedge} x$ and $\sin_{\vee} x$ mean evaluate the sine of x using the geometric product or antiproduct, respectively. In all cases, the product symbols are not inserted when the quantities involved in a calculation are nothing more than real numbers because the operations are applied in the conventional manner.

We begin by calculating the square of a dual number $s\mathbf{1} + t\mathbb{1}$ under the geometric product, which is a simple matter of applying the rules

$$\mathbf{1} \wedge \mathbf{1} = \mathbf{1}, \quad \mathbf{1} \wedge \mathbb{1} = \mathbb{1}, \quad \mathbb{1} \wedge \mathbf{1} = \mathbb{1}, \quad \text{and} \quad \mathbb{1} \wedge \mathbb{1} = 0. \tag{3.21}$$

The square of $s\mathbf{1} + t\mathbb{1}$ under the product is thus given by

$$(s\mathbf{1} + t\mathbb{1})^2_{\wedge} = (s\mathbf{1} + t\mathbb{1}) \wedge (s\mathbf{1} + t\mathbb{1}) = s^2\mathbf{1} + 2st\mathbb{1}. \tag{3.22}$$

Under the geometric antiproduct, the rules change to

$$\mathbf{1} \vee \mathbf{1} = 0, \quad \mathbf{1} \vee \mathbb{1} = \mathbf{1}, \quad \mathbb{1} \vee \mathbf{1} = \mathbf{1}, \quad \text{and} \quad \mathbb{1} \vee \mathbb{1} = \mathbb{1}. \tag{3.23}$$

Using these rules for the antiproduct, the square of $s\mathbf{1} + t\mathbb{1}$ is given by

$$(s\mathbf{1} + t\mathbb{1})^2_{\vee} = (s\mathbf{1} + t\mathbb{1}) \vee (s\mathbf{1} + t\mathbb{1}) = 2st\mathbf{1} + t^2\mathbb{1}. \tag{3.24}$$

This shows an obvious symmetry between the product and antiproduct in which the behaviors of the scalar and antiscalar components trade places. The same type of symmetry appears in all the operations we can perform, so we won't bother deriving any further results for both the product and antiproduct. The formulas for both products are still listed in Table 3.4.

A dual number $s\mathbf{1} + t\mathbb{1}$ can be raised to an arbitrary power under the geometric product with the formula

$$(s\mathbf{1} + t\mathbb{1})^n_{\wedge} = s^n\mathbf{1} + ns^{n-1}t\mathbb{1}. \tag{3.25}$$

This can be demonstrated for positive integer powers inductively by first assuming Equation (3.22) is true for $n = 2$ and then observing

$$\begin{aligned}(s\mathbf{1} + t\mathbb{1})^n_{\wedge} \wedge (s\mathbf{1} + t\mathbb{1}) &= \left(s^n\mathbf{1} + ns^{n-1}t\mathbb{1}\right) \wedge (s\mathbf{1} + t\mathbb{1}) \\ &= s^{n+1}\mathbf{1} + (n+1)s^n t\mathbb{1}.\end{aligned} \tag{3.26}$$

Equation (3.25) can be used to derive formulas for the inverse and for square roots as well, but we haven't proven that negative or noninteger values of n are valid, so we will derive inverses and square roots with alternate methods.

The inverse of a dual number $s\mathbf{1} + t\mathbb{1}$ with respect to the geometric product is the dual number $x\mathbf{1} + y\mathbb{1}$ with the property that

$$(s\mathbf{1} + t\mathbb{1}) \wedge (x\mathbf{1} + y\mathbb{1}) = \mathbf{1}. \tag{3.27}$$

Multiplying this out with the rules given by Equation (3.21), we must have $sx = 1$ and $sy + tx = 0$, which tells us that $x = 1/s$ and $y = -t/s^2$. The inverse of $s\mathbf{1} + t\mathbb{1}$ is therefore given by

$$\frac{1}{s\mathbf{1} + t\mathbb{1}}_{\wedge} = \frac{1}{s}\mathbf{1} - \frac{t}{s^2}\mathbb{1}. \tag{3.28}$$

A dual number does not have an inverse if $s = 0$, even if the number as a whole is not zero.

We use a similar approach to find the square root of a dual number $s\mathbf{1} + t\mathbb{1}$ with respect to the geometric product because it must be the dual number $x\mathbf{1} + y\mathbb{1}$ with the property that

$$(x\mathbf{1} + y\mathbb{1})^2_{⟑} = s\mathbf{1} + t\mathbb{1}. \tag{3.29}$$

Applying Equation (3.22) to the square, we see that we must have $x^2 = s$ and $2xy = t$. This tells us that $x = \sqrt{s}$ and $y = t/2\sqrt{s}$, so the square root of $s\mathbf{1} + t\mathbb{1}$ is given by

$$\sqrt[⟑]{s\mathbf{1} + t\mathbb{1}} = \sqrt{s}\,\mathbf{1} + \frac{t}{2\sqrt{s}}\mathbb{1}. \tag{3.30}$$

In order to calculate the exponential of a dual number with the geometric product, which we denote by $\exp_{⟑}(s\mathbf{1} + t\mathbb{1})$, we make use of its power series. Writing out the first few terms, we have

$$\exp_{⟑}(s\mathbf{1} + t\mathbb{1}) = \mathbf{1} + (s\mathbf{1} + t\mathbb{1}) + \frac{1}{2!}(s\mathbf{1} + t\mathbb{1})^2_{⟑} + \frac{1}{3!}(s\mathbf{1} + t\mathbb{1})^3_{⟑} + \frac{1}{4!}(s\mathbf{1} + t\mathbb{1})^4_{⟑} + \cdots. \tag{3.31}$$

Each of the powers of $s\mathbf{1} + t\mathbb{1}$ can be expanded with Equation (3.25), and that gives us

$$\exp_{⟑}(s\mathbf{1} + t\mathbb{1}) = \mathbf{1} + (s\mathbf{1} + t\mathbb{1}) + \frac{1}{2!}\left(s^2\mathbf{1} + 2st\mathbb{1}\right) + \frac{1}{3!}\left(s^3\mathbf{1} + 3s^2t\mathbb{1}\right) + \frac{1}{4!}\left(s^4\mathbf{1} + 4s^3t\mathbb{1}\right) + \cdots. \tag{3.32}$$

By looking at the scalar and antiscalar parts separately, we can see that the scalar part of the result is simply $\exp(s)$ and the antiscalar part of the result is $t\exp(s)$. This means the exponential of a dual number of given by

$$\exp_{⟑}(s\mathbf{1} + t\mathbb{1}) = \exp(s)\mathbf{1} + t\exp(s)\mathbb{1}. \tag{3.33}$$

Formulas for the sine, cosine, hyperbolic sine, and hyperbolic cosine functions under the geometric product are derived through power series manipulation in a manner identical to the exponential function, so we skip their derivations to spare ourselves the redundancy. The tangent and hyperbolic tangent functions are calculated through the ratio of their respective sine and cosine functions. In the case of the tangent function, we have

$$\tan_{⟑}(s\mathbf{1} + t\mathbb{1}) = \frac{\sin_{⟑}(s\mathbf{1} + t\mathbb{1})}{\cos_{⟑}(s\mathbf{1} + t\mathbb{1})}_{⟑} = \frac{(\sin s)\mathbf{1} + (t\cos s)\mathbb{1}}{(\cos s)\mathbf{1} - (t\sin s)\mathbb{1}}_{⟑}. \tag{3.34}$$

When we apply Equation (3.28) to take care of the division, we arrive at

$$\tan_{⟑}(s\mathbf{1} + t\mathbb{1}) = (\tan s)\mathbf{1} + t\left(1 + \tan^2 s\right)\mathbb{1}. \tag{3.35}$$

Note that the division needs to be performed with respect to the same product for which the sine and cosine functions are being calculated.

Math Library Notes

- The `DualNum` class implements dual numbers. It stores a scalar component `s` and an antiscalar component `t` as floating-point values.
- The geometric product and antiproduct are implemented with the `WedgeDot()` and `AntiwedgeDot()` functions. The multiplication and division operators `*` and `/` perform the geometric antiproduct because that product is more useful in the context of rigid transformations.
- Operations appearing in Table 3.4 can be performed on `DualNum` instances with respect to the geometric product using the `Inverse()`, `Sqrt()`, `InverseSqrt()`, `Exp()`, `Sin()`, `Cos()`, and `Tan()` functions. The same operations can be performed with respect to the geometric antiproduct using the `AntiInverse()`, `AntiSqrt()`, `AntiInverseSqrt()`, `AntiExp()`, `AntiSin()`, `AntiCos()`, and `AntiTan()` functions.

Operation	Geometric Product	Geometric Antiproduct
General power	$(s\mathbf{1}+t\mathbb{1})^n_{\wedge} = s^n\mathbf{1} + ns^{n-1}t\mathbb{1}$	$(s\mathbf{1}+t\mathbb{1})^n_{\vee} = nt^{n-1}s\mathbf{1} + t^n\mathbb{1}$
Square	$(s\mathbf{1}+t\mathbb{1})^2_{\wedge} = s^2\mathbf{1} + 2st\mathbb{1}$	$(s\mathbf{1}+t\mathbb{1})^2_{\vee} = 2st\mathbf{1} + t^2\mathbb{1}$
Inverse	$\dfrac{1}{s\mathbf{1}+t\mathbb{1}}{}^{\wedge} = \dfrac{1}{s}\mathbf{1} - \dfrac{t}{s^2}\mathbb{1}$	$\dfrac{1}{s\mathbf{1}+t\mathbb{1}}{}^{\vee} = -\dfrac{s}{t^2}\mathbf{1} + \dfrac{1}{t}\mathbb{1}$
Square root	$\sqrt[\wedge]{s\mathbf{1}+t\mathbb{1}} = \sqrt{s}\,\mathbf{1} + \dfrac{t}{2\sqrt{s}}\mathbb{1}$	$\sqrt[\vee]{s\mathbf{1}+t\mathbb{1}} = \dfrac{s}{2\sqrt{t}}\mathbf{1} + \sqrt{t}\,\mathbb{1}$
Inverse square root	$\dfrac{1}{\sqrt[\wedge]{s\mathbf{1}+t\mathbb{1}}}{}^{\wedge} = \dfrac{1}{\sqrt{s}}\mathbf{1} - \dfrac{t}{2s\sqrt{s}}\mathbb{1}$	$\dfrac{1}{\sqrt[\vee]{s\mathbf{1}+t\mathbb{1}}}{}^{\vee} = -\dfrac{s}{2t\sqrt{t}}\mathbf{1} + \dfrac{1}{\sqrt{t}}\mathbb{1}$
Exponential	$\exp_{\wedge}(s\mathbf{1}+t\mathbb{1}) = \exp(s)\mathbf{1} + t\exp(s)\mathbb{1}$	$\exp_{\vee}(s\mathbf{1}+t\mathbb{1}) = s\exp(t)\mathbf{1} + \exp(t)\mathbb{1}$
Sine	$\sin_{\wedge}(s\mathbf{1}+t\mathbb{1}) = (\sin s)\mathbf{1} + (t\cos s)\mathbb{1}$	$\sin_{\vee}(s\mathbf{1}+t\mathbb{1}) = (s\cos t)\mathbf{1} + (\sin t)\mathbb{1}$
Cosine	$\cos_{\wedge}(s\mathbf{1}+t\mathbb{1}) = (\cos s)\mathbf{1} - (t\sin s)\mathbb{1}$	$\cos_{\vee}(s\mathbf{1}+t\mathbb{1}) = -(s\sin t)\mathbf{1} + (\cos t)\mathbb{1}$
Tangent	$\tan_{\wedge}(s\mathbf{1}+t\mathbb{1}) = (\tan s)\mathbf{1} + t(1+\tan^2 s)\mathbb{1}$	$\tan_{\vee}(s\mathbf{1}+t\mathbb{1}) = s(1+\tan^2 t)\mathbf{1} + (\tan t)\mathbb{1}$
Hyperbolic sine	$\sinh_{\wedge}(s\mathbf{1}+t\mathbb{1}) = (\sinh s)\mathbf{1} + (t\cosh s)\mathbb{1}$	$\sinh_{\vee}(s\mathbf{1}+t\mathbb{1}) = (s\cosh t)\mathbf{1} + (\sinh t)\mathbb{1}$
Hyperbolic cosine	$\cosh_{\wedge}(s\mathbf{1}+t\mathbb{1}) = (\cosh s)\mathbf{1} + (t\sinh s)\mathbb{1}$	$\cosh_{\vee}(s\mathbf{1}+t\mathbb{1}) = (s\sinh t)\mathbf{1} + (\cosh t)\mathbb{1}$
Hyperbolic tangent	$\tanh_{\wedge}(s\mathbf{1}+t\mathbb{1}) = (\tanh s)\mathbf{1} + t(1-\tanh^2 s)\mathbb{1}$	$\tanh_{\vee}(s\mathbf{1}+t\mathbb{1}) = s(1-\tanh^2 t)\mathbf{1} + (\tanh t)\mathbb{1}$

Table 3.4. This table summarizes the results obtained when common operations are applied to a dual number $s\mathbf{1}+t\mathbb{1}$ under the geometric product $\wedge$ and geometric antiproduct $\vee$. When the antiscalar part t is zero, each of the formulas for the geometric product reduces to the conventional operation on a real number s. Likewise, when the scalar part s is zero, each of the formulas for the geometric antiproduct reduces to the conventional operation on a real number t.

3.3 Reflection and Rotation

The transformative powers of the geometric product start with its ability to perform a reflection, and this is something we can observe with an experiment involving vector division. Suppose that $\mathbf{a}$ and $\mathbf{v}_0$ are vectors, and set $\mathbf{m} = \mathbf{a} \wedge \mathbf{v}_0$. If we divide $\mathbf{m}$ by $\mathbf{v}_0$, then we get $\mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{v}_0^{-1}$, which just gives us $\mathbf{a}$ right back. That is not very interesting, but if we instead divide $\mathbf{m}$ by $\mathbf{a}$, then something important happens. As shown in Figure 3.2, the quantity $\mathbf{m} \wedge \mathbf{a}^{-1}$ must be equal to some value $\mathbf{v}_1$ such that $\mathbf{v}_1 \wedge \mathbf{a} = \mathbf{m}$, which means that

$$\mathbf{v}_1 = \mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}^{-1}. \tag{3.36}$$

By decomposing $\mathbf{m}$ into its scalar and bivector parts, we can rewrite this as

$$\mathbf{v}_1 = (\mathbf{a} \bullet \mathbf{v}_0) \wedge \mathbf{a}^{-1} + (\mathbf{a} \wedge \mathbf{v}_0) \wedge \mathbf{a}^{-1}. \tag{3.37}$$

We can reverse the order of multiplication of $\mathbf{a}$ and $\mathbf{v}_0$ so we have terms that resemble the projection and rejection formulas given in Equation (3.19). Reversing the factors of a dot product has no effect, but reversing vector factors of a wedge product causes the result to change sign. The expression for $\mathbf{v}_1$ thus becomes

$$\mathbf{v}_1 = (\mathbf{v}_0 \bullet \mathbf{a}) \wedge \mathbf{a}^{-1} - (\mathbf{v}_0 \wedge \mathbf{a}) \wedge \mathbf{a}^{-1}. \tag{3.38}$$

The first term is the component of $\mathbf{v}_0$ that is parallel to $\mathbf{a}$, and the second term is the *negation* of the component of $\mathbf{v}_0$ that is perpendicular to $\mathbf{a}$. This tells us that $\mathbf{v}_1$ must be the reflection of the vector $\mathbf{v}_0$ across the vector $\mathbf{a}$, as shown in Figure 3.2. The geometric reason why $\mathbf{v}_1$ must be the reflection of $\mathbf{v}_0$ across $\mathbf{a}$ makes itself known when we consider that $\mathbf{m}$ is equal to both $\mathbf{a} \wedge \mathbf{v}_0$ and $\mathbf{v}_1 \wedge \mathbf{a}$. Both of these quantities must correspond to parallelograms sharing the same side $\mathbf{a}$ and possessing the same area, the same orientation, and the same interior angles, as shown in the figure, because this information is stored in the scalar and bivector parts of $\mathbf{m}$.

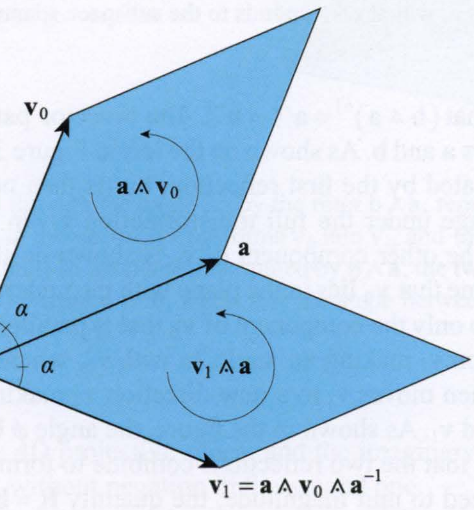


Figure 3.2. The vector $\mathbf{v}_1$ is the reflection of the vector $\mathbf{v}_0$ across the vector $\mathbf{a}$. Because $\mathbf{v}_1$ is the same length as $\mathbf{v}_0$ and makes the same angle with $\mathbf{a}$ as $\mathbf{v}_0$ does, the geometric product $\mathbf{v}_1 \wedge \mathbf{a}$ yields the same scalar and bivector as the geometric product $\mathbf{a} \wedge \mathbf{v}_0$.

The sandwich product $\mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}^{-1}$ gives us the reflection of the vector $\mathbf{v}_0$ across the vector $\mathbf{a}$, and this preserves the component of $\mathbf{v}_0$ that's parallel to $\mathbf{a}$ while negating the component of $\mathbf{v}_0$ that's perpendicular to $\mathbf{a}$. Through the simple addition of a minus sign, we can change this into a slightly different kind of reflection. As shown in Figure 3.3, the vector given by $-\mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}^{-1}$ corresponds to the reflection of $\mathbf{v}_0$ across the plane $\overline{\mathbf{a}}$, where the complement represents all directions perpendicular to $\mathbf{a}$. These two types of reflection have significantly different meanings in ordinary 3D space, but when we start reflecting things in a projective space where everything is homogeneous, the extra minus sign will have only the effect of inverting the orientation of the object being reflected without changing its attitude.

Let's now consider what happens when a reflection across a vector $\mathbf{a}$ is followed by another reflection across a vector $\mathbf{b}$. The first reflection transforms an arbitrary vector $\mathbf{v}_0$ into a new vector $\mathbf{v}_1$ through the formula $\mathbf{v}_1 = \mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}^{-1}$. The second reflection transforms $\mathbf{v}_1$ into another new vector $\mathbf{v}_2$ through the formula

$$\mathbf{v}_2 = \mathbf{b} \wedge \mathbf{v}_1 \wedge \mathbf{b}^{-1} = (\mathbf{b} \wedge \mathbf{a}) \wedge \mathbf{v}_0 \wedge \left(\mathbf{a}^{-1} \wedge \mathbf{b}^{-1}\right). \tag{3.39}$$

In Figure 3.4, these two steps are illustrated separately and in combination. If we set $\mathbf{R} = \mathbf{b} \wedge \mathbf{a}$, then we can write

$$\boxed{\mathbf{v}_2 = \mathbf{R} \wedge \mathbf{v}_0 \wedge \mathbf{R}^{-1}} \tag{3.40}$$

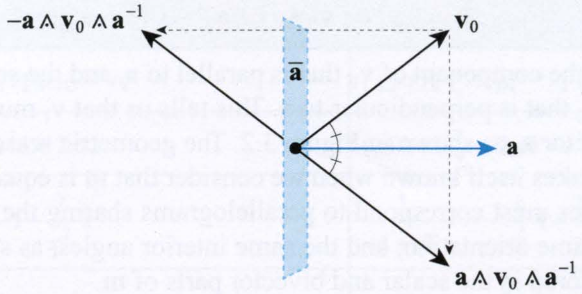


Figure 3.3. The sandwich product $\mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}^{-1}$ reflects the vector $\mathbf{v}_0$ across the vector $\mathbf{a}$. Its negation is the reflection of $\mathbf{v}_0$ across $\overline{\mathbf{a}}$, which corresponds to the subspace spanned by all directions perpendicular to $\mathbf{a}$.

after recognizing that $(\mathbf{b} \wedge \mathbf{a})^{-1} = \mathbf{a}^{-1} \wedge \mathbf{b}^{-1}$. The bivector part of $\mathbf{R}$ is oriented in the plane determined by the vectors $\mathbf{a}$ and $\mathbf{b}$. As shown on the left in Figure 3.4, the component of $\mathbf{v}_0$ perpendicular to this plane is negated by the first reflection, but is then negated again by the second reflection, so it does not change under the full transformation given by Equation (3.40). The effect of the transformation on the other component of $\mathbf{v}_0$ is shown on the right in Figure 3.4. For the sake of simplicity, we assume that $\mathbf{v}_0$ lies in the plane with the understanding that the following explanation otherwise applies to only the component of $\mathbf{v}_0$ that is parallel to the plane. The first reflection moves $\mathbf{v}_0$ to a new direction $\mathbf{v}_1$ making an angle 2α with $\mathbf{v}_0$, where α is the angle between $\mathbf{a}$ and $\mathbf{v}_0$. The second reflection then moves $\mathbf{v}_1$ to a new direction $\mathbf{v}_2$ making an angle 2β with $\mathbf{v}_1$, where β is the angle between $\mathbf{b}$ and $\mathbf{v}_1$. As shown in the figure, the angle ϕ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to $\alpha + \beta$. We conclude that the two reflections combine to form a rotation through a total angle 2ϕ.

When normalized to unit magnitude, the quantity $\mathbf{R} = \mathbf{b} \wedge \mathbf{a}$ is an operator called a *rotor*. The sandwich product given by Equation (3.40) rotates vectors through an angle 2ϕ in the direction from $\mathbf{a}$ to $\mathbf{b}$ parallel to the bivector $\mathbf{b} \wedge \mathbf{a}$, where ϕ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Note that the direction of rotation is the opposite of the winding direction associated with the bivector $\mathbf{b} \wedge \mathbf{a}$. For this reason, an operator $\mathbf{R}$ that rotates in the direction from $\mathbf{a}$ to $\mathbf{b}$ would have to be written as

$$\mathbf{R} = \mathbf{a} \bullet \mathbf{b} - \mathbf{a} \wedge \mathbf{b} \tag{3.41}$$

to keep the vectors $\mathbf{a}$ and $\mathbf{b}$ in the order corresponding to the rotation direction.

The rotation operation in Equation (3.40) is valid in any number of dimensions, but it is usually 3D space that matters to us. In three dimensions, the wedge product $\mathbf{a} \wedge \mathbf{b}$ can be interpreted as the complement of an axis of rotation given by the cross product $\mathbf{a} \times \mathbf{b}$. If $\mathbf{a}$ and $\mathbf{b}$ have unit length, then Equation (3.41) can be written as

$$\mathbf{R} = \cos\phi - \sin\phi\,\overline{\mathbf{n}}, \tag{3.42}$$

where $\mathbf{n}$ is a unit vector pointing in the same direction as $\mathbf{a} \times \mathbf{b}$, and the complement is taken with respect to the volume element $\mathbf{e}_{123}$. This operator rotates through an angle of 2ϕ, so the operator that rotates through an angle ϕ is given by

$$\boxed{\mathbf{R} = \cos\frac{\phi}{2} - \left(\sin\frac{\phi}{2}\right)\overline{\mathbf{n}}.} \tag{3.43}$$

Upon comparison with Equation (1.63), it's now clear that rotors in three dimensions are equivalent to the set of quaternions. Due to the subtraction of the bivector part in Equation (3.43), the imaginary units i, j, and k of a quaternion are equated with the negated bivector basis elements such that $i = -\mathbf{e}_{23}$, $j = -\mathbf{e}_{31}$, and $k = -\mathbf{e}_{12}$. Later in this chapter, we will see that quaternions actually have two

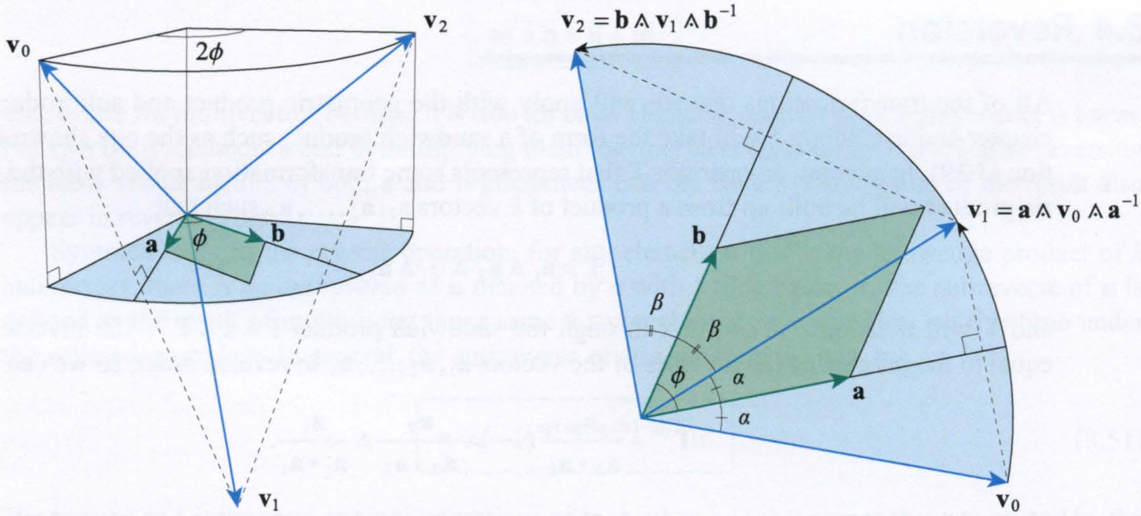


Figure 3.4. A vector $\mathbf{v}_0$ is rotated through the angle 2ϕ by the rotor $\mathbf{b} \wedge \mathbf{a}$, represented by the green parallelogram. (Left) The reflection across the vector $\mathbf{a}$ transforms $\mathbf{v}_0$ into $\mathbf{v}_1$, and the second reflection across the vector $\mathbf{b}$ transforms $\mathbf{v}_1$ into $\mathbf{v}_2$. (Right) In the plane determined by $\mathbf{b} \wedge \mathbf{a}$, the two reflections combine to form a rotation through the angle $2\alpha + 2\beta$ from $\mathbf{a}$ to $\mathbf{b}$, where α is the angle between $\mathbf{a}$ and $\mathbf{v}_0$, and β is the angle between $\mathbf{b}$ and $\mathbf{v}_1$.

different representations in the 4D projective space, and the imaginary units i, j, and k map to the basis bivectors $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, and $\mathbf{e}_{43}$ without negation in the second one.

As we mentioned in Section 1.4, it wasn't quite correct to call a quaternion the sum of a scalar part and a vector part, and we now know that a quaternion is really the sum of a scalar part and *bivector* part. Whenever we calculated the sandwich product $\mathbf{q}\mathbf{v}\mathbf{q}^*$ to rotate a vector $\mathbf{v}$ with a quaternion $\mathbf{q}$, we were actually treating $\mathbf{v}$ as a bivector, which works in three dimensions because it has the same number of components as a vector. The use of the quaternion conjugate arises from the fact that the inverse of $\mathbf{b} \wedge \mathbf{a}$ is just $\mathbf{a} \wedge \mathbf{b}$ when $\mathbf{a}$ and $\mathbf{b}$ have unit length, and reversing the order of the factors in the geometric product of two vectors has the effect of negating the bivector part of the result.

The rotation performed by the rotor $\mathbf{R}$ in Equation (3.40) is a linear transformation that applies to grade-one vectors. The sandwich product $\mathbf{R} \wedge \mathbf{v} \wedge \mathbf{R}^{-1}$ is also an exomorphism that preserves the structure of the algebra under the wedge product, and this means that a rotor doesn't only perform rotations on vectors, but on elements of any grade. We can understand why this must be true when we consider a pair of orthogonal basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$. The geometric product $\mathbf{e}_i \wedge \mathbf{e}_j$ is equal to the wedge product $\mathbf{e}_i \wedge \mathbf{e}_j$ because the dot product between the two basis vectors is zero. Thus, the sandwich product $\mathbf{R} \wedge \mathbf{e}_{ij} \wedge \mathbf{R}^{-1}$ can be written entirely with the geometric product as

$$\mathbf{R} \wedge \mathbf{e}_{ij} \wedge \mathbf{R}^{-1} = \mathbf{R} \wedge \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{R}^{-1}. \tag{3.44}$$

By inserting the product $\mathbf{R}^{-1} \wedge \mathbf{R}$ between the factors $\mathbf{e}_i$ and $\mathbf{e}_j$, we can split this into two sandwich products to obtain

$$\mathbf{R} \wedge \mathbf{e}_{ij} \wedge \mathbf{R}^{-1} = \left(\mathbf{R} \wedge \mathbf{e}_i \wedge \mathbf{R}^{-1}\right) \wedge \left(\mathbf{R} \wedge \mathbf{e}_j \wedge \mathbf{R}^{-1}\right). \tag{3.45}$$

The geometric product outside parentheses on the right side has been replaced by a wedge product because the rotations of $\mathbf{e}_i$ and $\mathbf{e}_j$ must still be orthogonal, so the geometric product between them has no scalar part. An arbitrary bivector is simply a sum of scalar multiples of the basis bivectors $a\mathbf{e}_{ij} + b\mathbf{e}_{kl} + \cdots$, and each term in this sum is rotated separately by Equation (3.45).

3.4 Reversion

All of the transformations that we will apply with the geometric product and antiproduct in this chapter and in Chapter 5 will take the form of a sandwich product such as the one shown in Equation (3.39). In general, an operator $\mathbf{T}$ that represents some transformation applied with the geometric product will be built up from a product of k vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ such that

$$\mathbf{T} = \mathbf{a}_1 \mathbin{⟑} \mathbf{a}_2 \mathbin{⟑} \cdots \mathbin{⟑} \mathbf{a}_k, \tag{3.46}$$

and $\mathbf{T}$ will transform an object $\mathbf{x}$ through the sandwich product $\mathbf{T} \mathbin{⟑} \mathbf{x} \mathbin{⟑} \mathbf{T}^{-1}$. The inverse of $\mathbf{T}$ is equal to the product of the inverses of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in reverse order, so we can write

$$\mathbf{T}^{-1} = \frac{\mathbf{a}_k}{\mathbf{a}_k \bullet \mathbf{a}_k} \mathbin{⟑} \cdots \mathbin{⟑} \frac{\mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2} \mathbin{⟑} \frac{\mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}. \tag{3.47}$$

If all of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are bulk normalized (which means $\mathbf{a}_i \bullet \mathbf{a}_i = 1$), then the inverse of $\mathbf{T}$ is simply the geometric product of the vectors themselves multiplied in reverse order, and it differs from $\mathbf{T}$ only by a possible change in sign. This arises so often that we define a special unary operation to handle it.

3.4.1 Reverse and Antireverse

For any basis element $\mathbf{u}$ that is the wedge product of k basis vectors, the *reverse* of $\mathbf{u}$ is denoted by $\tilde{\mathbf{u}}$, with a tilde above it, and defined as the result of multiplying those same k basis vectors in reverse order.[2] The overall effect is that the reverse of $\mathbf{u}$ is given by

Reverse

$$\boxed{\tilde{\mathbf{u}} = (-1)^{\operatorname{gr}(\mathbf{u})(\operatorname{gr}(\mathbf{u})-1)/2}\,\mathbf{u}.} \tag{3.48}$$

This relationship holds because the number of individual transpositions necessary to reverse the vector factors of a grade-k basis element $\mathbf{u}$ is the sum of $k-1$ for the first factor, $k-2$ for the second factor, and so on. The total is the $(k-1)$-th triangular number, which is given by $k(k-1)/2$. This is odd when the grade k is equal to 2 or 3 modulo 4, so these are the grades for which the reverse operation $\tilde{\mathbf{u}}$ flips the sign of $\mathbf{u}$. Otherwise, when k is equal to 0 or 1 modulo 4, the reverse operation has no effect.

Just as we did for complements in Section 2.2, we extend the reverse operation to all elements of an exterior algebra by requiring that it is a linear operation. That is, for any scalar s and basis elements $\mathbf{a}$ and $\mathbf{b}$, we have

$$\widetilde{s\mathbf{a}} = s\tilde{\mathbf{a}} \quad \text{and} \quad \widetilde{\mathbf{a}+\mathbf{b}} = \tilde{\mathbf{a}} + \tilde{\mathbf{b}}. \tag{3.49}$$

When the reverse operation is applied to a multivector that has components of different grades, some of those components could be negated while others are not. For example, taking the reverse of a quaternion $\mathbf{q} = w\mathbf{1} + x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12}$ has the effect of negating only the three bivector components without changing the scalar component so that $\tilde{\mathbf{q}} = w\mathbf{1} - x\mathbf{e}_{23} - y\mathbf{e}_{31} - z\mathbf{e}_{12}$. This is the origin of the conjugate operation defined by Equation (1.48).

The reverse operation distributes over the geometric product between two arbitrary multivectors $\mathbf{m}$ and $\mathbf{n}$ through the rule

[2] There are two notations for the reverse operation that are commonly found in the literature. One is the tilde notation that we use in this book, and the other is a dagger notation in which the reverse of $\mathbf{u}$ is denoted by $\mathbf{u}^\dagger$. The dagger option is not well suited for a symmetric notation corresponding to the reverse's anti-operation.

$$\boxed{\widetilde{\mathbf{m} \wedge \mathbf{n}} = \tilde{\mathbf{n}} \wedge \tilde{\mathbf{m}}.} \tag{3.50}$$

This is true for multivectors because it is true for basis elements, and the geometric product is linear. For two basis elements $\mathbf{a}$ and $\mathbf{b}$, multiplying them together in reverse order as $\tilde{\mathbf{b}} \wedge \tilde{\mathbf{a}}$ after reversing the basis vector factors of both $\mathbf{a}$ and $\mathbf{b}$ guarantees that the basis vector factors of the result also appear in reverse order.

Symmetrically to the reverse operation, for any element $\mathbf{u}$ that is the antiwedge product of k antivectors, there is an *antireverse* of $\mathbf{u}$ denoted by $\underset{\sim}{\mathbf{u}}$ with a tilde below it. The antireverse of $\mathbf{u}$ is defined as the result of multiplying those same k antivectors in reverse order, but this time under the antiwedge product. In general, the antireverse of an element $\mathbf{u}$ is given by

Antireverse

$$\boxed{\underset{\sim}{\mathbf{u}} = (-1)^{\operatorname{ag}(\mathbf{u})(\operatorname{ag}(\mathbf{u})-1)/2}\, \mathbf{u}.} \tag{3.51}$$

The reverse and antireverse are anti-operations of each other, and that means they are related by the De Morgan laws

$$\underset{\sim}{\mathbf{u}} = \underline{\left(\tilde{\bar{\mathbf{u}}}\right)} \quad \text{and} \quad \underset{\sim}{\mathbf{u}} = \overline{\left(\tilde{\underline{\mathbf{u}}}\right)}, \tag{3.52}$$

where the complement is applied before the reverse operation inside the parentheses in both cases. We can determine how the reverse and antireverse of a grade-k element $\mathbf{u}$ in n dimensions are related through a change of sign by combining the powers of -1 in Equations (3.48) and (3.51). This gives us the equation

$$\underset{\sim}{\mathbf{u}} = (-1)^{k(k-1)/2+(n-k)(n-k-1)/2}\, \tilde{\mathbf{u}}. \tag{3.53}$$

The exponent simplifies to $k(k-n)+n(n-1)/2$, and since we can negate any term without changing the resulting power of -1, we can write the relationship between reverse and antireverse as

$$\underset{\sim}{\mathbf{u}} = (-1)^{\operatorname{gr}(\mathbf{u})\operatorname{ag}(\mathbf{u})+n(n-1)/2}\, \tilde{\mathbf{u}}. \tag{3.54}$$

The differences between $\tilde{\mathbf{u}}$ and $\underset{\sim}{\mathbf{u}}$ in the 4D projective algebra can be seen in Table 3.5 where they are listed for all 16 basis elements.

An important property of the reverse operation is that for any two arbitrary multivectors $\mathbf{m}$ and $\mathbf{n}$, the scalar part of $\mathbf{m} \wedge \tilde{\mathbf{n}}$ is equal to the dot product $\mathbf{m} \bullet \mathbf{n}$, a relationship that can be written more succinctly as

$$\boxed{(\mathbf{m} \wedge \tilde{\mathbf{n}})_1\, \mathbf{1} = \mathbf{m} \bullet \mathbf{n}.} \tag{3.55}$$

To see why this equality holds, we must first realize that for any two *different* basis elements $\mathbf{a}$ and $\mathbf{b}$, the geometric product $\mathbf{a} \wedge \mathbf{b}$ cannot be a scalar because either $\mathbf{a}$ or $\mathbf{b}$ must have a basis vector factor that the other does not have. This means that every contribution to the scalar part of $\mathbf{m} \wedge \tilde{\mathbf{n}}$

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$\mathbb{1}$
$\underset{\sim}{\mathbf{u}}$	$\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbb{1}$

Table 3.5. For each of the 16 basis elements $\mathbf{u}$ in the 4D geometric algebra, this table lists the reverse $\tilde{\mathbf{u}}$ and the antireverse $\underset{\sim}{\mathbf{u}}$.

comes from the product of a component of $\mathbf{m}$ and a component of $\mathbf{n}$ with the same basis element. For any basis element $\mathbf{a} = \mathbf{e}_i \wedge \mathbf{e}_j \wedge \cdots$, we know that

$$\mathbf{a} \mathbin{\dot{\wedge}} \tilde{\mathbf{a}} = (\mathbf{e}_i \bullet \mathbf{e}_i)(\mathbf{e}_j \bullet \mathbf{e}_j)\cdots. \tag{3.56}$$

But by the determinant expansion of the dot product given by Equation (2.83), we also know that

$$\mathbf{a} \bullet \mathbf{a} = (\mathbf{e}_i \wedge \mathbf{e}_j \wedge \cdots) \bullet (\mathbf{e}_i \wedge \mathbf{e}_j \wedge \cdots) = (\mathbf{e}_i \bullet \mathbf{e}_i)(\mathbf{e}_j \bullet \mathbf{e}_j)\cdots, \tag{3.57}$$

so we can conclude that $\mathbf{a} \mathbin{\dot{\wedge}} \tilde{\mathbf{a}} = \mathbf{a} \bullet \mathbf{a}$. Since the geometric product and dot product are both linear, the scalar contributions from matching components of $\mathbf{m}$ and $\mathbf{n}$ simply add to yield the complete value of $\mathbf{m} \bullet \mathbf{n}$.

As always, the relationship in Equation (3.55) has a dual counterpart, and we can derive it by taking complements of both sides. After taking the right complement and applying De Morgan laws to the geometric product and dot product, we have

$$(\overline{\mathbf{m}} \mathbin{\dot{\vee}} \overline{\tilde{\mathbf{n}}})_{\mathbb{1}} \mathbb{1} = \overline{\mathbf{m}} \circ \overline{\mathbf{n}}. \tag{3.58}$$

Since $\mathbf{m}$ and $\mathbf{n}$ are arbitrary multivectors, so are their complements, so we can just take their left complements to essentially relabel them without changing the meaning of the equation. For three out of the four operands in Equation (3.58), the right and left complements simply cancel out. In the fourth case, we are taking the reverse of $\underline{\mathbf{n}}$ and following it with the right complement, but Equation (3.52) tells us that those operations produce $\underset{\sim}{\mathbf{n}}$. The dual version of Equation (3.55), which should be no big surprise, is then given by

$$\boxed{(\mathbf{m} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{n}})_{\mathbb{1}} \mathbb{1} = \mathbf{m} \circ \mathbf{n}.} \tag{3.59}$$

Math Library Notes

- The reverse and antireverse operations are implemented by the `Reverse()` and `Antireverse()` functions. In cases when no sign change occurs, these functions perform no operation and simply return a reference to the value passed to them.

Reverse

$\tilde{\mathbf{u}}$

- Reverses the order in which the vector factors of $\mathbf{u}$ are multiplied under the wedge product.
- Changes sign when the grade of $\mathbf{u}$ is 2 or 3 modulo 4.

DUALITY

Antireverse

$\underset{\sim}{\mathbf{u}}$

- Reverses the order in which the antivector factors of $\mathbf{u}$ are multiplied under the antiwedge product.
- Changes sign when the antigrade of $\mathbf{u}$ is 2 or 3 modulo 4.

3.4.2 Dual Identities

We can express bulk and weight duals using the two geometric products and two reverse operations. By substituting the same basis element $\mathbf{u}$ for both operands in the identity given by Equation (2.156), we get the relationship $\mathbf{u} \wedge \mathbf{u}^{\star} = (\mathbf{u} \bullet \mathbf{u})\,\mathbb{1}$ telling us that the product of $\mathbf{u}$ with its bulk dual is the volume element multiplied by the square of $\mathbf{u}$. Since $\mathbf{u}$ and $\mathbf{u}^{\bigstar}$ don't have any factors in common (assuming the metric is diagonal), the wedge product $\mathbf{u} \wedge \mathbf{u}^{\bigstar}$ is the same as the geometric product $\mathbf{u} \mathbin{\dot\wedge} \mathbf{u}^{\bigstar}$, and we can write

$$\mathbf{u} \mathbin{\dot\wedge} \mathbf{u}^{\bigstar} = (\mathbf{u} \bullet \mathbf{u})\,\mathbb{1}. \tag{3.60}$$

If we multiply both sides by $\tilde{\mathbf{u}}$ on the left, then we have

$$\tilde{\mathbf{u}} \mathbin{\dot\wedge} \mathbf{u} \mathbin{\dot\wedge} \mathbf{u}^{\bigstar} = \tilde{\mathbf{u}} \mathbin{\dot\wedge} (\mathbf{u} \bullet \mathbf{u})\,\mathbb{1}. \tag{3.61}$$

Recognizing that $\tilde{\mathbf{u}} \mathbin{\dot\wedge} \mathbf{u} = \mathbf{u} \bullet \mathbf{u}$ and assuming $\mathbf{u} \bullet \mathbf{u} \neq 0$ for the moment, we can cancel $\mathbf{u} \bullet \mathbf{u}$ on both sides, leaving us with

$$\boxed{\mathbf{u}^{\bigstar} = \tilde{\mathbf{u}} \mathbin{\dot\wedge} \mathbb{1}.} \tag{3.62}$$

In the case that $\mathbf{u} \bullet \mathbf{u} = 0$, we also know that $\mathbf{u}^{\bigstar} = 0$ because both are made zero by the product $\mathbf{Gu}$ with the extended metric $\mathbf{G}$, and it must further be true that $\tilde{\mathbf{u}} \mathbin{\dot\wedge} \mathbb{1} = 0$ because the volume element contains all the factors of $\mathbf{u}$. Thus, Equation (3.62) holds in general, and it is extended to all elements $\mathbf{u}$ in the algebra through linearity.

Following a similar process for weight duals that begins with the identity $\mathbf{u} \mathbin{\dot\vee} \mathbf{u}^{\star} = (\mathbf{u} \circ \mathbf{u}) \mathbin{\dot\vee} \mathbf{1}$ given by Equation (2.120), we obtain

$$\boxed{\mathbf{u}^{\star} = \utilde{\mathbf{u}} \mathbin{\dot\vee} \mathbf{1}.} \tag{3.63}$$

Analogs to Equations (3.62) and (3.63) for left bulk and weight duals simply have the order of multiplication reversed on the right side, so we also have the identities

$$\mathbf{u}_{\bigstar} = \mathbb{1} \mathbin{\dot\wedge} \tilde{\mathbf{u}} \quad \text{and} \quad \mathbf{u}_{\star} = \mathbf{1} \mathbin{\dot\vee} \utilde{\mathbf{u}}. \tag{3.64}$$

The only difference in the definitions of right and left duals is whether the right or left complement is applied after multiplying by the metric or antimetric. Consequently, if $\mathbf{u}$ is composed of components all having even grade or all having odd grade, then $\tilde{\mathbf{u}} \mathbin{\dot\wedge} \mathbb{1} = \pm\mathbb{1} \mathbin{\dot\wedge} \tilde{\mathbf{u}}$ and $\utilde{\mathbf{u}} \mathbin{\dot\vee} \mathbf{1} = \pm\mathbf{1} \mathbin{\dot\vee} \utilde{\mathbf{u}}$, where a sign change depends on the dimension of the algebra and whether $\mathbf{u}$ has even or odd grades in it.

3.4.3 Geometric Constraint

Returning to the operator $\mathbf{T}$ described at the beginning of this section, we can use the reverse operation to write the inverse given by Equation (3.47) as

$$\mathbf{T}^{-1} = \frac{\tilde{\mathbf{T}}}{(\mathbf{a}_1 \bullet \mathbf{a}_1)(\mathbf{a}_2 \bullet \mathbf{a}_2)\cdots(\mathbf{a}_k \bullet \mathbf{a}_k)}. \tag{3.65}$$

In the case that all of the dot products in the denominator are one, we simply have $\mathbf{T}^{-1} = \tilde{\mathbf{T}}$. Otherwise, since $\mathbf{T} \mathbin{\dot\wedge} \mathbf{T}^{-1} = \mathbf{1}$, we can write

$$\mathbf{T} \mathbin{\dot\wedge} \tilde{\mathbf{T}} = (\mathbf{a}_1 \bullet \mathbf{a}_1)(\mathbf{a}_2 \bullet \mathbf{a}_2)\cdots(\mathbf{a}_k \bullet \mathbf{a}_k). \tag{3.66}$$

This is a scalar value, and Equation (3.55) tells us that the scalar part of $\mathbf{T} \mathbin{\dot\wedge} \tilde{\mathbf{T}}$ must be equal to the dot product $\mathbf{T} \bullet \mathbf{T}$. We can now generalize the vector inverse formula given by Equation (3.16) to include any multivector $\mathbf{T}$ that is a geometric product of vectors by stating

$$\mathbf{T}^{-1} = \frac{\tilde{\mathbf{T}}}{\mathbf{T} \bullet \mathbf{T}}. \tag{3.67}$$

Remember that we have only shown Equation (3.67) to be true for a multivector $\mathbf{T}$ having the form given by Equation (3.46), and it is not true for arbitrary multivectors. In general, the geometric product $\mathbf{T} ⟑ \tilde{\mathbf{T}}$ creates higher-grade components in addition to a scalar component. However, all of the geometric objects in an exterior algebra can be constructed with a wedge product of vectors corresponding to points they contain, and it will be the case that all of the transformation operators in a geometric algebra are constructed with a geometric product of vectors (or a geometric antiproduct of antivectors) corresponding to a sequence of reflections. A multivector quantity $\mathbf{u}$ representing a geometric object or transformation operator satisfies the property

Geometric constraint

$$\boxed{\mathbf{u} ⟑ \tilde{\mathbf{u}} = \mathbf{u} \bullet \mathbf{u}.} \tag{3.68}$$

This relationship is called the *geometric constraint*, and it has the effect of putting some restrictions on the components of any multivector quantity $\mathbf{u}$ that has geometric significance. For example, if we consider the representation of a line $\boldsymbol{l}$ defined by Equation (2.36), then the geometric product with the reverse gives us

$$\boldsymbol{l} ⟑ \tilde{\boldsymbol{l}} = \left(l_{mx}^2 + l_{my}^2 + l_{mz}^2\right)\mathbf{1} + 2\left(l_{vx}l_{mx} + l_{vy}l_{my} + l_{vz}l_{mz}\right)\mathbb{1}. \tag{3.69}$$

In order to satisfy the geometric constraint, this must be equal to $\boldsymbol{l} \bullet \boldsymbol{l}$, so the antiscalar component must always be zero. This means that $l_{vx}l_{mx} + l_{vy}l_{my} + l_{vz}l_{mz} = 0$ for all lines, which is a conclusion that we reached by a different route in Section 2.4.2. We will encounter similar constraints for the operators discussed later in this chapter. For some of the geometric objects introduced in Chapter 4, the constraints will be more complex.

There is a dual counterpart to the geometric constraint stated in Equation (3.68) that applies to any quantity that can be expressed as a geometric antiproduct of antivectors. The dual form of the geometric constraint is easy to guess, but we'll take a moment to derive it anyway. Suppose that $\mathbf{u}$ can be expressed as

$$\mathbf{u} = \mathbf{a}_1 ⟇ \mathbf{a}_2 ⟇ \cdots ⟇ \mathbf{a}_k, \tag{3.70}$$

where each $\mathbf{a}_i$ is an antivector. If we take the right complement of both sides and apply the De Morgan law, then we have

$$\overline{\mathbf{u}} = \overline{\mathbf{a}}_1 ⟑ \overline{\mathbf{a}}_2 ⟑ \cdots ⟑ \overline{\mathbf{a}}_k. \tag{3.71}$$

Since this is now a geometric product of k vectors, $\overline{\mathbf{u}}$ must satisfy $\overline{\mathbf{u}} ⟑ \tilde{\overline{\mathbf{u}}} = \overline{\mathbf{u}} \bullet \overline{\mathbf{u}}$. Applying the De Morgan laws to both sides gives us

$$\mathbf{u} ⟇ \underline{\left(\tilde{\overline{\mathbf{u}}}\right)} = \mathbf{u} \circ \mathbf{u}. \tag{3.72}$$

Using Equation (3.52) to apply a De Morgan law one last time to the factor containing the reverse, we obtain the property

$$\boxed{\mathbf{u} ⟇ \underset{\sim}{\mathbf{u}} = \mathbf{u} \circ \mathbf{u}.} \tag{3.73}$$

This version of the geometric constraint applies to lines just as well as the original version in Equation (3.68) because any line can be expressed as the intersection of two planes. When we calculate the geometric antiproduct of a line $\boldsymbol{l}$ with its antireverse, we get

$$\boldsymbol{l} ⟇ \underset{\sim}{\boldsymbol{l}} = 2\left(l_{vx}l_{mx} + l_{vy}l_{my} + l_{vz}l_{mz}\right)\mathbf{1} + \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2\right)\mathbb{1}. \tag{3.74}$$

This time, the scalar component must be zero in order to satisfy $\boldsymbol{l} ⟇ \underset{\sim}{\boldsymbol{l}} = \boldsymbol{l} \circ \boldsymbol{l}$, so we end up with the same constraint of $l_{vx}l_{mx} + l_{vy}l_{my} + l_{vz}l_{mz} = 0$ that we had before.

In the case that a bivector $\boldsymbol{l}$ does not satisfy the geometric constraint but is interpreted as a line anyway, Equations (3.69) and (3.74) both contain exactly the information that we need to tell how far out of whack it actually is. Both quantities are dual numbers, and we can divide by their square roots with respect to the appropriate product to normalize the bivector and establish the geometric constraint at the same time. In the case of the geometric product, the inverse square root of Equation (3.69) is given by

$$\frac{1}{\sqrt[\wedge]{\boldsymbol{l} \wedge \tilde{\boldsymbol{l}}}}^{\wedge} = \frac{1}{\sqrt{\boldsymbol{l}_{\mathbf{m}}^2}}\left(\mathbf{1} - \frac{\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathbf{m}}}{\boldsymbol{l}_{\mathbf{m}}^2}\mathbb{1}\right), \tag{3.75}$$

which follows from the formula listed in Table 3.4. Multiplying the original line $\boldsymbol{l}$ by this quantity will clearly have the effect of dividing out the magnitude of the moment $\boldsymbol{l}_{\mathbf{m}}$, but there is another effect due to the presence of the extra value $\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathbf{m}} / \boldsymbol{l}_{\mathbf{m}}^2$ coming from the antiscalar part of $\boldsymbol{l} \wedge \tilde{\boldsymbol{l}}$. It corresponds to the projection of the direction $\boldsymbol{l}_{\mathrm{v}}$ onto the moment $\boldsymbol{l}_{\mathbf{m}}$, which is exactly the vector that we would need to subtract from the line's direction to make it perpendicular to the moment as required by the geometric constraint. When we multiply $\boldsymbol{l}$ by Equation (3.75), the result is

$$\boxed{\frac{\boldsymbol{l}}{\sqrt[\wedge]{\boldsymbol{l} \wedge \tilde{\boldsymbol{l}}}}^{\wedge} = \frac{1}{\sqrt{\boldsymbol{l}_{\mathbf{m}}^2}}\left[\boldsymbol{l} - \frac{\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathbf{m}}}{\boldsymbol{l}_{\mathbf{m}}^2}\left(l_{mx}\mathbf{e}_{41} + l_{my}\mathbf{e}_{42} + l_{mz}\mathbf{e}_{43}\right)\right]}, \tag{3.76}$$

and it does indeed orthogonalize the direction and moment by subtracting the projection of $\boldsymbol{l}_{\mathrm{v}}$ onto $\boldsymbol{l}_{\mathbf{m}}$ from $\boldsymbol{l}_{\mathrm{v}}$ before bulk normalizing the whole line. We can repeat this process for the geometric antiproduct by calculating the inverse square root of Equation (3.74) to get

$$\frac{1}{\sqrt[\vee]{\boldsymbol{l} \vee \underset{\sim}{\boldsymbol{l}}}}^{\vee} = \frac{1}{\sqrt{\boldsymbol{l}_{\mathrm{v}}^2}}\left(\frac{\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathbf{m}}}{\boldsymbol{l}_{\mathrm{v}}^2}\mathbf{1} + \mathbb{1}\right). \tag{3.77}$$

This time, the projection of the moment $\boldsymbol{l}_{\mathbf{m}}$ onto the direction $\boldsymbol{l}_{\mathrm{v}}$ appears, and the magnitude of the direction shows up in the denominator. Multiplying by $\boldsymbol{l}$ gives us

$$\boxed{\frac{\boldsymbol{l}}{\sqrt[\vee]{\boldsymbol{l} \vee \underset{\sim}{\boldsymbol{l}}}}^{\vee} = \frac{1}{\sqrt{\boldsymbol{l}_{\mathrm{v}}^2}}\left[\boldsymbol{l} - \frac{\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathbf{m}}}{\boldsymbol{l}_{\mathrm{v}}^2}\left(l_{vx}\mathbf{e}_{23} + l_{vy}\mathbf{e}_{31} + l_{vz}\mathbf{e}_{12}\right)\right]}, \tag{3.78}$$

which leaves the direction alone, orthogonalizes the moment by subtracting its projection onto the direction, and weight normalizes the whole line. There will be occasions in Section 3.6 when a line needs to be fixed up, and Equation (3.78) will be preferred over Equation (3.76) because it unitizes the line and does not alter its direction.

3.5 Euclidean Isometries

The main goal of this chapter is to develop a set of operators that are able to perform all Euclidean isometries in 3D space. A *Euclidean isometry* is a transformation that preserves distances such that the size and shape of an object does not change, but it can be moved around and possibly turned inside out. When an object undergoes a Euclidean isometry, the distance between every pair of points $\mathbf{p}_1$ and $\mathbf{p}_2$ belonging to the object is the same after the transformation as is was before the transformation. This implies that for any third point $\mathbf{p}_0$ belonging to the object, the angle between any two vectors $\mathbf{p}_1 - \mathbf{p}_0$ and $\mathbf{p}_2 - \mathbf{p}_0$ is also preserved by the transformation. Transformations that preserve distances and angles are called *rigid* transformations, and they are a subset of the less restricted *conformal* transformations, which only preserve angles but not necessarily distances, that we will encounter in Chapter 5.

Every Euclidean isometry is either the identity transformation or a transformation that can be constructed as a sequence of k reflections across k distinct planes. The types of isometries are broadly classified into two groups, those for which k is even and those for which k is odd. In the case that k is even, we call the transformation a *proper isometry*, and in the case that k is odd, we call the transformation an *improper isometry*. The identity is considered a proper isometry because it doesn't reflect across any planes at all, and thus $k = 0$.

As illustrated in Figure 3.5, proper isometries include rotations and translations, and they correspond to continuous motions through space. Every proper isometry can be expressed as a *screw motion* comprising a rotation through an angle ϕ about an arbitrary line and a translation by a distance δ along the direction in which that same line runs. (This fact is known as Chasles' theorem.) All simpler motions are special cases of the general screw motion. In the case that the angle of rotation ϕ is zero, the motion we are left with is purely a translation along some direction. This happens when we reflect across two planes that are parallel to each other. In the case that the translation distance δ is zero, the motion we are left with is purely a rotation about some line. This happens when we reflect across two planes that are not parallel to each other, and the line serving as the axis of rotation is precisely where the two planes intersect. We will see that a translation can be understood in a projective space as a rotation about a line at infinity, so rotations and translations are really just two different manifestations of the same thing.

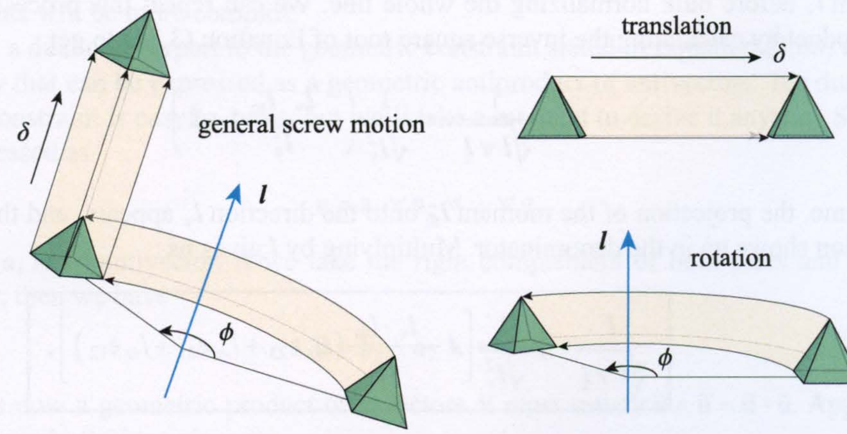


Figure 3.5. All proper isometries in 3D space can be expressed as a screw motion comprising a rotation through an angle ϕ about some line $\boldsymbol{l}$ and a translation by a distance δ along the direction in which that line runs. Pure translation and pure rotation are special cases.

As illustrated in Figure 3.6, all improper isometries are characterized by an extra reflection that causes the object undergoing the transformation to be mirrored or turned inside out. Every improper isometry can be expressed as a *rotoreflection* comprising a rotation through an angle ϕ about an arbitrary line and a reflection through a plane perpendicular to that same line. Similar to how proper isometries are all special cases of a screw motion, all simpler improper isometries are special cases of the general rotoreflection. In the case that the angle of rotation ϕ is zero, the isometry is purely a reflection across some plane. If the angle ϕ is 180 degrees, then the rotoreflection becomes an *inversion* through a single point. Finally, if the axis of rotation is a line at infinity, then the isometry is a *transflection* in which a reflection across a plane is combined with a translation along a direction that's parallel to that same plane.

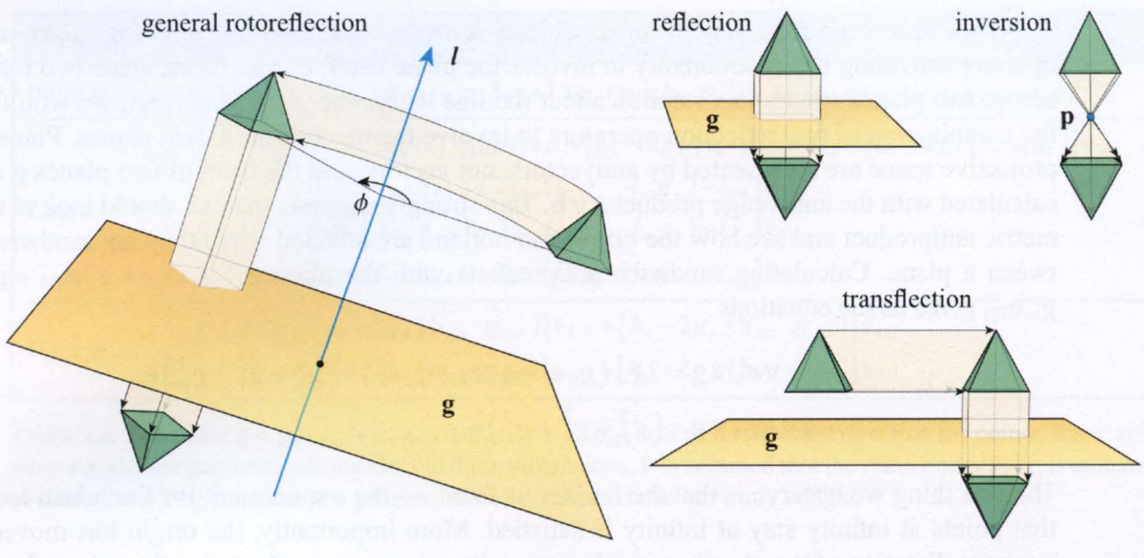


Figure 3.6. All improper isometries in 3D space can be expressed as a rotoreflection comprising a rotation through an angle ϕ about some line $\boldsymbol{l}$ and a reflection across a plane $\mathbf{g}$ perpendicular to that line. Reflections, inversions, and transflections are special cases.

3.5.1 Reflection

Since all Euclidean isometries can be conceptually built with a sequence of planar reflections, we would certainly like to find an operator that can reflect any geometric object that we have in our projective space across any given plane. This one operator could then be applied repeatedly to construct all of the possible isometries that we have discussed. An important property that any operator performing a rigid transformation must have is that it preserves the horizon. That is, any point lying at infinity before the transformation still lies at infinity after the transformation, but possibly in a different direction. Also, any point not originally lying at infinity does not somehow end up in the horizon after the transformation.

As we previously demonstrated in Figure 3.3, the sandwich product $-\mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}^{-1}$ reflects the vector $\mathbf{v}_0$ across the plane perpendicular to the vector $\mathbf{a}$. If we know $\mathbf{a}$ is normalized to unit length such that $\mathbf{a} \bullet \mathbf{a} = 1$, then $\mathbf{a}$ is its own inverse, and we can use the simpler expression $-\mathbf{a} \wedge \mathbf{v}_0 \wedge \mathbf{a}$. This formula reflects vectors and higher-grade quantities in a nonprojective environment where there is only direction and no position. To find out what happens in the 4D projective algebra, we can apply the same formula to the origin $\mathbf{e}_4$ and the horizon $\mathbf{e}_{321}$ by attempting to reflect them across the complement of a vector representing a point $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$. Since everything is homogeneous here, we don't care about any scaling that occurs if $\mathbf{p}$ is not bulk normalized, so we don't bother dividing by $\mathbf{p} \bullet \mathbf{p}$ to obtain $\mathbf{p}^{-1}$ and instead just calculate a sandwich product with $\mathbf{p}$ on both sides. Transforming the origin and horizon in this way gives us the equations

$$\begin{aligned}
-\mathbf{p} \wedge \mathbf{e}_4 \wedge \mathbf{p} &= \left(p_x^2 + p_y^2 + p_z^2\right)\mathbf{e}_4 \\
-\mathbf{p} \wedge \mathbf{e}_{321} \wedge \mathbf{p} &= 2p_w\left(p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} + p_z\mathbf{e}_{412}\right) - \left(p_x^2 + p_y^2 + p_z^2\right)\mathbf{e}_{321}.
\end{aligned} \tag{3.79}$$

The first equation tells us that the origin is fixed by the sandwich product. There is no point $\mathbf{p}$ that would move the origin somewhere else, so we couldn't possibly use this formula to reflect points across arbitrary planes. The second equation tells us the horizon does not stay put unless $\mathbf{p}$ itself lies at infinity (because that would mean $p_w = 0$). This is a problem because all Euclidean isometries leave the horizon at infinity where it belongs. We are forced to conclude that repeated applications of the formula $-\mathbf{p} \wedge \mathbf{v}_0 \wedge \mathbf{p}$ cannot perform Euclidean isometries.

If we're looking for a way to reflect objects across a plane, then it's only logical to expect an operator providing this functionality to involve the plane itself. Furthermore, since two reflections across two planes results in a rotation about the line where the planes intersect, we would expect the combination of two reflection operators to involve the meet of those two planes. Planes in the projective space are represented by antivectors, not vectors, and the meet of two planes $\mathbf{g}$ and $\mathbf{h}$ is calculated with the antiwedge product $\mathbf{g} \vee \mathbf{h}$. This strongly suggests that we should look at the geometric antiproduct and see how the origin and horizon are affected when they are sandwiched between a plane. Calculating sandwich antiproducts with the plane $\mathbf{g} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} + g_w \mathbf{e}_{321}$ gives us the equations

$$\begin{aligned} -\mathbf{g} \vee \mathbf{e}_4 \vee \mathbf{g} &= 2g_w\left(g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3\right) - \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_4 \\ -\mathbf{g} \vee \mathbf{e}_{321} \vee \mathbf{g} &= \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_{321}. \end{aligned} \tag{3.80}$$

The first thing we observe is that the horizon is fixed, so the requirement for Euclidean isometries that points at infinity stay at infinity is satisfied. More importantly, the origin has moved in the opposite direction of the plane's normal vector $g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3$ by twice the value of g_w, which is the plane's weighted distance from the origin. (The direction is opposite the plane's normal because the $\mathbf{e}_4$ term has been negated.) This transforms the origin to a location on the other side of the plane at exactly the same distance from the plane, so we have achieved a reflection. To verify that this works universally for any point $\mathbf{p}$, we calculate

$$-\mathbf{g} \vee \mathbf{p} \vee \mathbf{g} = 2\left(p_x g_x + p_y g_y + p_z g_z + p_w g_w\right)\left(g_x\mathbf{e}_1 + g_y\mathbf{e}_2 + g_z\mathbf{e}_3\right) - \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{p}. \tag{3.81}$$

The conventional 4D dot product $\mathbf{p} \cdot \mathbf{g}$ is visible in this equation, and it corresponds to the weighted distance between the point $\mathbf{p}$ and the plane $\mathbf{g}$. Moving by twice that distance in the opposite direction of the plane's normal vector does indeed reflect $\mathbf{p}$ across $\mathbf{g}$ in the same way that Equation (1.35) did in Chapter 1, but it's generalized here for points and planes having arbitrary weights. We can now be certain that any plane $\mathbf{g}$ is directly able to act as a reflection operator under the geometric antiproduct.

The similarities between Equations (3.79) and (3.80) should be rather obvious. Duality is once again showing itself through the symmetric effects that the geometric product and antiproduct have on the origin and horizon. These two products are both performing two operations at once just as the wedge and antiwedge products do, as discussed in Section 2.6. Sandwiching with the geometric *antiproduct* rigidly transforms the *space* of an object, performing a Euclidean isometry, and sandwiching with the geometric *product* rigidly transforms the *antispace* of an object. Simultaneously, there is a different set of transformations that happen in antispace for the geometric antiproduct and in regular space for the geometric product that are not Euclidean isometries, and we will find out what those are in Section 3.9.

The sandwich antiproduct with a plane $\mathbf{g}$ not only reflects a point across $\mathbf{g}$, but it reflects any type of geometric object $\mathbf{u}$ across $\mathbf{g}$. The exact calculations when $\mathbf{u}$ is a point $\mathbf{p}$, a line $\boldsymbol{l}$, and another plane $\mathbf{h}$ are listed in Table 3.6 for a unitized reflection plane $\mathbf{g}$. The minus sign in front of the expression $-\mathbf{g} \vee \mathbf{u} \vee \mathbf{g}$ has the effect of negating the orientation of the object $\mathbf{u}$ being reflected. This can be annoying in the case of points because a point starting with a positive w coordinate ends up with a negative w coordinate after the reflection. Since the projective space is homogeneous, dropping the minus sign does not change the meaning of the geometric object, so it may be convenient to remove it from a calculation. However, keeping the minus sign may be desirable when reflecting lines and planes because it causes the direction of a line and the normal of a plane to be reflected in the expected way.

Type	Reflection Formula
Point $\mathbf{p}$	$-\mathbf{g} \vee \mathbf{p} \vee \mathbf{g} = [2g_x(\mathbf{p} \cdot \mathbf{g}) - p_x]\mathbf{e}_1 + [2g_y(\mathbf{p} \cdot \mathbf{g}) - p_y]\mathbf{e}_2 + [2g_z(\mathbf{p} \cdot \mathbf{g}) - p_z]\mathbf{e}_3 - p_w\mathbf{e}_4$
Line $\boldsymbol{l}$	$-\mathbf{g} \vee \boldsymbol{l} \vee \mathbf{g} = [l_{vx} - 2g_x(\boldsymbol{l}_\mathbf{v} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{41} + [l_{vy} - 2g_y(\boldsymbol{l}_\mathbf{v} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{42} + [l_{vz} - 2g_z(\boldsymbol{l}_\mathbf{v} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{43}$ $+ [2g_x(\boldsymbol{l}_\mathbf{m} \cdot \mathbf{g}_{xyz}) + 2g_w(l_{vy}g_z - l_{vz}g_y) - l_{mx}]\mathbf{e}_{23}$ $+ [2g_y(\boldsymbol{l}_\mathbf{m} \cdot \mathbf{g}_{xyz}) + 2g_w(l_{vz}g_x - l_{vx}g_z) - l_{my}]\mathbf{e}_{31}$ $+ [2g_z(\boldsymbol{l}_\mathbf{m} \cdot \mathbf{g}_{xyz}) + 2g_w(l_{vx}g_y - l_{vy}g_x) - l_{mz}]\mathbf{e}_{12}$
Plane $\mathbf{h}$	$-\mathbf{g} \vee \mathbf{h} \vee \mathbf{g} = [h_x - 2g_x(\mathbf{h}_{xyz} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{423} + [h_y - 2g_y(\mathbf{h}_{xyz} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{431}$ $+ [h_z - 2g_z(\mathbf{h}_{xyz} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{412} + [h_w - 2g_w(\mathbf{h}_{xyz} \cdot \mathbf{g}_{xyz})]\mathbf{e}_{321}$

Table 3.6. The plane $\mathbf{g} = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$ acts as a reflection operator for points, lines, and planes under the geometric antiproduct in three dimensions. It is assumed that the reflection plane $\mathbf{g}$ is unitized so that $g_x^2 + g_y^2 + g_z^2 = 1$, but the geometries being reflected can have any weight.

3.5.2 Rotation

We mentioned earlier that rotations and translations involve a pair of reflections across two different planes, so we'd like to know exactly what happens when we reflect a point $\mathbf{p}$ across one plane $\mathbf{g}$ followed by another plane $\mathbf{h}$. For the moment, we assume that $\mathbf{g}$ and $\mathbf{h}$ are not parallel and meet at an angle ϕ as shown in Figure 3.7. The first reflection is performed by the sandwich antiproduct $-\mathbf{g} \vee \mathbf{p} \vee \mathbf{g}$, and the second reflection sandwiches this result between two antiproducts with the plane $\mathbf{h}$. The minus signs appearing in the two sandwich products cancel out, and the transformed point $\mathbf{p}'$ after both reflections is given by

$$\mathbf{p}' = \mathbf{h} \vee (\mathbf{g} \vee \mathbf{p} \vee \mathbf{g}) \vee \mathbf{h}. \tag{3.82}$$

We can reassociate the antiproducts so that the planes are multiplied together first, after which $\mathbf{p}$ is sandwiched between the quantity $\mathbf{h} \vee \mathbf{g}$ and its antireverse $\mathbf{g} \vee \mathbf{h}$. Defining $\mathbf{R} = \mathbf{h} \vee \mathbf{g}$ lets us write

$$\mathbf{p}' = \mathbf{R} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{R}}. \tag{3.83}$$

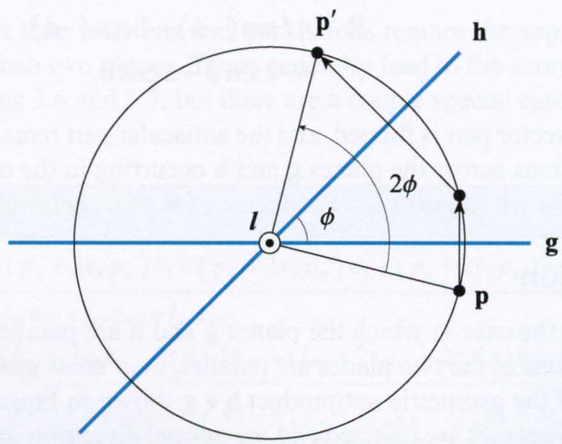


Figure 3.7. When the point $\mathbf{p}$ is reflected first across the plane $\mathbf{g}$ and then across the plane $\mathbf{h}$, the result is a rotation about the line $\boldsymbol{l}$ (pointing out of the page) where the two planes intersect. This rotation moves $\mathbf{p}$ to the point $\mathbf{p}'$ by rotating through the angle 2ϕ, where ϕ is the angle between the two planes.

Our task now is to determine the exact nature of the quantity **R**, and this task can be accomplished by taking a closer look at the geometric antiproduct $\mathbf{h} \vee \mathbf{g}$. If we write out all of the components, then we have

$$\begin{aligned}\mathbf{h} \vee \mathbf{g} = &(g_y h_z - g_z h_y)\,\mathbf{e}_{41} + (g_z h_x - g_x h_z)\,\mathbf{e}_{42} + (g_x h_y - g_y h_x)\,\mathbf{e}_{43} \\ &+ (g_w h_x - g_x h_w)\,\mathbf{e}_{23} + (g_w h_y - g_y h_w)\,\mathbf{e}_{31} + (g_w h_z - g_z h_w)\,\mathbf{e}_{12} \\ &+ (g_x h_x + g_y h_y + g_z h_z)\,\mathbb{1}.\end{aligned} \tag{3.84}$$

The six bivector components are exactly what we get from the meet $\mathbf{h} \vee \mathbf{g}$ listed in Table 2.7, but with the planes **h** and **g** multiplied in the reverse order. This bivector thus corresponds to the line where the two planes intersect, which is the axis of rotation. The remaining component is an antiscalar whose value is the dot product between the normal directions of the two planes. If the planes are both unitized, then this dot product is equal to the cosine of the angle ϕ between them. Furthermore, the $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, and $\mathbf{e}_{43}$ components making up the line's weight have a magnitude equal to the sine of the angle ϕ because they hold the coordinates of the cross product between the planes' normal directions. For a unitized axis of rotation, we define the line $\boldsymbol{l}$ by

$$\boldsymbol{l} = \frac{\mathbf{h} \vee \mathbf{g}}{\left\| \mathbf{h} \vee \mathbf{g} \right\|_{\circ}} = \frac{\mathbf{h} \vee \mathbf{g}}{\sin \phi}. \tag{3.85}$$

Then the operator **R** becomes

Rotation operator

$$\boxed{\mathbf{R} = \boldsymbol{l} \sin \phi + \mathbb{1} \cos \phi.} \tag{3.86}$$

This is a general operator that performs a rotation about the unitized line $\boldsymbol{l}$. As shown in Figure 3.7, the angle through which this operator rotates is *twice* the angle ϕ. This doubling of the angle is the same thing that happened with quaternions in Section 1.4.2, and similar doublings will continue appearing as we discover more operators. The operator **R** can be applied to any type of geometry in the algebra, but a direct implementation of the calculation $\mathbf{R} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{R}}$ is rather inefficient. Optimal calculations for transforming points, lines, and planes with an even more general operator **Q** are provided in Section 3.6.5 below, and specialized versions for the operator **R** can easily be derived from them by setting $Q_{mw} = 0$.

The inverse of the operator **R** must correspond to a rotation through the same angle about the same line, but in the opposite direction. Negating the angle ϕ in Equation (3.86) gives us

$$\begin{aligned}\mathbf{R}^{-1} &= \boldsymbol{l} \sin(-\phi) + \mathbb{1} \cos(-\phi) \\ &= -\boldsymbol{l} \sin \phi + \mathbb{1} \cos \phi.\end{aligned} \tag{3.87}$$

The sign of the bivector part is flipped, and the antiscalar part remains unchanged. This corresponds to a pair of reflections across the planes **g** and **h** occurring in the opposite order, and it thus should be unsurprising that we can express the inverse of **R** as its antireverse such that $\mathbf{R}^{-1} = \underset{\sim}{\mathbf{R}}$.

3.5.3 Translation

Now we consider the case in which the planes **g** and **h** are parallel, as shown in Figure 3.8. Since the normal directions of the two planes are parallel, their cross product is zero, and the $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, and $\mathbf{e}_{43}$ components of the geometric antiproduct $\mathbf{h} \vee \mathbf{g}$ shown in Equation (3.84) are therefore zero as well. Assuming the planes are unitized and the normal directions are oriented the same way, the dot product in the antiscalar term is one, and we are left with

$$\mathbf{h} \vee \mathbf{g} = (g_w h_x - g_x h_w)\,\mathbf{e}_{23} + (g_w h_y - g_y h_w)\,\mathbf{e}_{31} + (g_w h_z - g_z h_w)\,\mathbf{e}_{12} + \mathbb{1}. \tag{3.88}$$

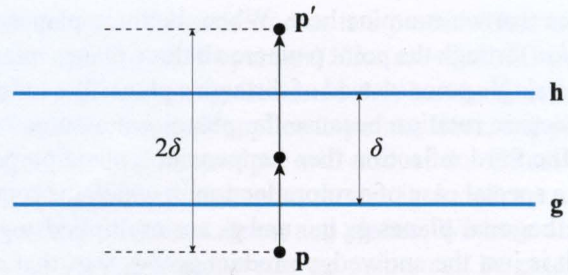


Figure 3.8. When a point $\mathbf{p}$ is reflected across two parallel planes $\mathbf{g}$ and $\mathbf{h}$, it is translated to the point $\mathbf{p}'$ in the direction perpendicular to the planes by twice the distance δ between the planes.

Since the normal directions are equal, we can make the substitutions $h_x = g_x$, $h_y = g_y$, and $h_z = g_z$. This allows us to factor out the difference $g_w - h_w$ from the first three terms to get

$$\mathbf{h} \vee \mathbf{g} = (g_w - h_w)(g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12}) + \mathbb{1}. \tag{3.89}$$

The values g_w and h_w represent the distances between the origin and the planes $\mathbf{g}$ and $\mathbf{h}$, so their difference is the distance δ between the two planes. As illustrated in Figure 3.8, reflecting a point $\mathbf{p}$ across the plane $\mathbf{g}$ and then across the plane $\mathbf{h}$ causes it to move in the normal direction by twice the distance δ. It is a translation by twice the vector $\boldsymbol{\tau} = (\delta g_x, \delta g_y, \delta g_z)$. Thus, a general translation operator $\mathbf{T}$ has the form

Translation operator

$$\boxed{\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1},} \tag{3.90}$$

and the sandwich antiproduct $\mathbf{T} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{T}}$ translates a point $\mathbf{p}$ by the displacement vector $2\boldsymbol{\tau}$. Comparing against Equation (3.86), this can be interpreted as a rotation through an infinitesimal angle about a line that's infinitely far away. The exact calculations involved in translating a point $\mathbf{p}$, a line $\boldsymbol{l}$, and a plane $\mathbf{g}$ with the operator $\mathbf{T}$ are listed in Table 3.7.

Like the rotation operator, the inverse of a translation operator $\mathbf{T}$ corresponds to reflections across the planes $\mathbf{g}$ and $\mathbf{h}$ in the opposite order with the effect of translating by the vector $-2\boldsymbol{\tau}$. Thus, the inverse of $\mathbf{T}$ is also given by its antireverse such that $\mathbf{T}^{-1} = \underset{\sim}{\mathbf{T}}$.

3.5.4 Inversion

Isometries more complex than rotations and translations require the application of more than two reflections across more than two planes. These generally lead to the screw motions and rotoreflections discussed in Sections 3.6 and 3.7, but there are a couple special cases involving the reflection

Type	Translation Formula
Point $\mathbf{p}$	$\mathbf{T} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{T}} = (p_x + 2\tau_x p_w)\mathbf{e}_1 + (p_y + 2\tau_y p_w)\mathbf{e}_2 + (p_z + 2\tau_z p_w)\mathbf{e}_3 + p_w \mathbf{e}_4$
Line $\boldsymbol{l}$	$\mathbf{T} \vee \boldsymbol{l} \vee \underset{\sim}{\mathbf{T}} = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + (l_{mx} + 2\tau_y l_{vz} - 2\tau_z l_{vy})\mathbf{e}_{23} + (l_{my} + 2\tau_z l_{vx} - 2\tau_x l_{vz})\mathbf{e}_{31} + (l_{mz} + 2\tau_x l_{vy} - 2\tau_y l_{vx})\mathbf{e}_{12}$
Plane $\mathbf{g}$	$\mathbf{T} \vee \mathbf{g} \vee \underset{\sim}{\mathbf{T}} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} + (g_w - 2\tau_x g_x - 2\tau_y g_y - 2\tau_z g_z)\mathbf{e}_{321}$

Table 3.7. The multivector $\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$ acts as a translation operator for points, lines, and planes under the geometric antiproduct in three dimensions. These formulas translate by the displacement vector $2\boldsymbol{\tau}$. The operator $\mathbf{T}$ is always unitized, and the geometries being translated can have any weight.

across three planes that we examine here. When the three planes are all mutually orthogonal, the result is an *inversion* through the point $\mathbf{p}$ where all three planes intersect. An inversion is a reflection that goes through a single point instead of through a plane. The reflections across the first two planes amount to a 180 degree rotation because the planes are assumed to make an angle of 90 degrees with each other. The third reflection then happens in a plane perpendicular to the axis of rotation, so an inversion is a special case of a rotoreflection in which the rotation angle is 180 degrees. When three mutually orthogonal planes $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ are multiplied together under the geometric antiproduct, the result is just the antiwedge product $\mathbf{g}_1 \vee \mathbf{g}_2 \vee \mathbf{g}_3$ that gives the point $\mathbf{p}$ where they intersect, and information about the original attitudes of the planes is not needed. Thus, the sandwich antiproduct $-\mathbf{p} \veebar \mathbf{u} \veebar \underset{\sim}{\mathbf{p}}$ performs an inversion operation. The exact calculations when $\mathbf{u}$ is another point $\mathbf{q}$, a line $\boldsymbol{l}$, and a plane $\mathbf{g}$ are listed in Table 3.8 for an inversion point with unit weight.

Type	Inversion Formula
Point $\mathbf{q}$	$-\mathbf{p} \veebar \mathbf{q} \veebar \underset{\sim}{\mathbf{p}} = (q_x - 2q_w p_x)\,\mathbf{e}_1 + (q_y - 2q_w p_y)\,\mathbf{e}_2 + (q_z - 2q_w p_z)\,\mathbf{e}_3 - q_w\,\mathbf{e}_4$
Line $\boldsymbol{l}$	$-\mathbf{p} \veebar \boldsymbol{l} \veebar \underset{\sim}{\mathbf{p}} = -l_{vx}\,\mathbf{e}_{41} - l_{vy}\,\mathbf{e}_{42} - l_{vz}\,\mathbf{e}_{43}$ $+ (l_{mx} + 2p_z l_{vy} - 2p_y l_{vz})\,\mathbf{e}_{23} + (l_{my} + 2p_x l_{vz} - 2p_z l_{vx})\,\mathbf{e}_{31} + (l_{mz} + 2p_y l_{vx} 2p_x l_{vy})\,\mathbf{e}_{12}$
Plane $\mathbf{g}$	$-\mathbf{p} \veebar \mathbf{g} \veebar \underset{\sim}{\mathbf{p}} = -g_x\,\mathbf{e}_{423} - g_y\,\mathbf{e}_{431} - g_z\,\mathbf{e}_{412} + (2p_x g_x + 2p_y g_y + 2p_z g_z + g_w)\,\mathbf{e}_{321}$

Table 3.8. The point $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_4$ acts as an inversion operator for points, lines, and planes under the geometric antiproduct in three dimensions. This operator $\mathbf{p}$ is always unitized, and the geometries being inverted can have any weight.

3.5.5 Transflection

The last special case that we look at is a *transflection*, which is also known as a *glide reflection*. A transflection is built from reflections across three planes configured such that two planes are parallel to each other and the third plane is perpendicular to the first two. The resulting isometry is the combination of the translation produced by the parallel planes and the reflection across another plane to which the translation is parallel. The order in which these two transformations occur does not matter, so we can multiply the translation operator $\mathbf{T}$ given by Equation (3.90) by a unitized plane $\mathbf{g}$ in the order $\mathbf{g} \veebar \mathbf{T}$ or $\mathbf{T} \veebar \mathbf{g}$ to obtain a transflection operator $\mathbf{J}$. In both cases, we get

Transflection operator

$$\boxed{\mathbf{J} = \tau_x\mathbf{e}_1 + \tau_y\mathbf{e}_2 + \tau_z\mathbf{e}_3 + \mathbf{g}.} \tag{3.91}$$

This operator does not translate along the direction $\boldsymbol{\tau}$ as might be expected, but it instead translates along the direction given by the cross product $\mathbf{g}_{xyz} \times \boldsymbol{\tau}$. As with the other operators, the translation distance is twice the magnitude of $\boldsymbol{\tau}$. Since we assumed the plane $\mathbf{g}$ is perpendicular to the other two planes responsible for the translation, $\boldsymbol{\tau}$ is perpendicular to $\mathbf{g}_{xyz}$. This means that $\mathbf{g}_{xyz} \times \boldsymbol{\tau}$ is also perpendicular to $\mathbf{g}_{xyz}$ and is still a translation parallel to the plane. Optimal calculations for transforming points, lines, and planes with a more general operator $\mathbf{F}$ are provided in Section 3.7.3 below, and specialized versions for the operator $\mathbf{J}$ can easily be derived from them by setting $F_{pw} = 0$.

3.6 Motors

In 3D space, a combination of any number of rotations and translations can ultimately be reduced to a single rotation about a specific axis combined with a translation along the direction of that same axis. The operator that performs this general motion, and thus performs any proper isometry in 3D

Euclidean space, is called a *motor*. The name motor is a portmanteau of "motion operator" or, in some historical contexts, "moment-vector operator". The set of all 3D motors is equivalent to the set of dual quaternions briefly introduced in Section 1.4.4, but the functionality is properly generalized in projective geometric algebra so it applies to points, lines, and planes. Motors can also be formulated in any number of dimensions, and we will discuss the 2D case in Section 3.8.

None of the operators described in this chapter, motors included, are actually necessary in order to perform rigid transformations of points, lines, and planes. We have already seen in Section 2.7 that any 4×4 transformation matrix that operates on points can be extended to an exomorphism matrix that operates on all types of geometries. Motors can perform only a limited subset of the transformations that a 4×4 matrix can perform, and they can often be more expensive when it comes to raw computation. Motors have advantages that include storage size, ease of parameterization, and quality of interpolation that we will certainly highlight throughout this section, but a grounded perspective demands that we be candid about their disadvantages as well.

3.6.1 Motion Operator

The general form of a motor $\mathbf{Q}$ is revealed when we use the geometric antiproduct to combine the rotation operator $\mathbf{R}$ given by Equation (3.86) with the translation operator $\mathbf{T}$ given by Equation (3.90), but under the condition that the displacement vector $\boldsymbol{\tau}$ is parallel to the direction $\boldsymbol{l}_{\mathbf{v}}$ of the rotation axis. These operators commute when $\boldsymbol{\tau}$ and $\boldsymbol{l}_{\mathbf{v}}$ are parallel, so the result is the same whether we calculate $\mathbf{Q} = \mathbf{R} \mathbin{\vee\!\!\!\!\!\lor} \mathbf{T}$ or $\mathbf{Q} = \mathbf{T} \vee \mathbf{R}$. Setting to $\boldsymbol{\tau} = \delta\,(l_{vx}, l_{vy}, l_{vz})$ and assuming the line $\boldsymbol{l}$ is unitized, we have

$$\begin{aligned}\mathbf{Q} = {} & (l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43} + l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12})\sin\phi \\ & + (l_{vx}\,\mathbf{e}_{23} + l_{vy}\,\mathbf{e}_{31} + l_{vz}\,\mathbf{e}_{12})\,\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi.\end{aligned} \tag{3.92}$$

This operator has eight components, and they all have even grades. The bivector and antiscalar components of the rotation operator $\mathbf{R}$ are present, but there are additional terms that modify the line's moment and add a new scalar component. Upon recognizing that $\boldsymbol{l}^{\star} = -l_{vx}\,\mathbf{e}_{23} - l_{vy}\,\mathbf{e}_{31} - l_{vz}\,\mathbf{e}_{12}$, the operator $\mathbf{Q}$ can be written more compactly as

$$\boxed{\mathbf{Q} = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi.} \tag{3.93}$$

As shown in Figure 3.9, the operation $\mathbf{Q} \vee \mathbf{u} \vee \widetilde{\mathbf{Q}}$ rotates the object $\mathbf{u}$ through the angle 2ϕ about the line $\boldsymbol{l}$ and translates it by the distance 2δ along the line's direction $\boldsymbol{l}_{\mathbf{v}}$.

Negating the values of both δ and ϕ causes the rotation and translation to occur in the opposite direction, which is exactly the operation that the inverse of the motor $\mathbf{Q}$ would perform. Since the cosine function is even, the antiscalar component of $\mathbf{Q}$ does not change when ϕ is negated. Since the sine function is odd, two negations happen in the scalar component, so it also doesn't change. Only a single sign change affects the six bivector components, however, so they are all negated when $\mathbf{Q}$ is inverted. Considering that a motor is conceptually the composition of reflections across planes, these particular negations are expected because it must be the case that $\mathbf{Q}^{-1} = \widetilde{\mathbf{Q}}$.

A motor $\mathbf{Q}$ can be divided into two four-component parts and written generically as

Rigid motor (3D)

$$\mathbf{Q} = \underbrace{Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}}_{\text{Moment and Displacement}}, \tag{3.94}$$

where the two-letter subscripts identify specific components. The subscripts *vx*, *vy*, *vz*, *mx*, *my*, and *mz* correspond to the same bivector components as they do for a line, where they are assigned to

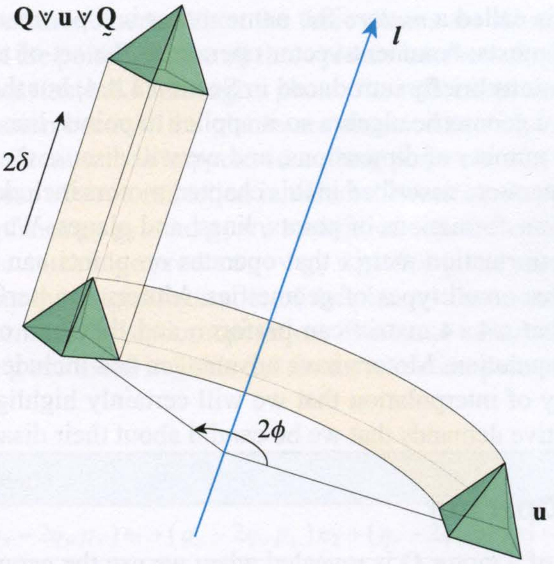


Figure 3.9. A motor **Q** represents a proper Euclidean isometry, which can always be regarded as a rotation about a line ***l*** and a displacement along the same line.

the direction and moment. For a motor, these three-dimensional parts are extended to four dimensions by adding a *w* coordinate to each one. The antiscalar component is labeled *vw*, and it is associated with the line's direction. The scalar component is labeled *mw*, and it is associated with the line's moment. The shorthand notation $\mathbf{Q_v}$ means the 4D vector $(Q_{vx}, Q_{vy}, Q_{vz}, Q_{vw})$, and $\mathbf{Q_m}$ means the 4D vector $(Q_{mx}, Q_{my}, Q_{mz}, Q_{mw})$. These four-dimensional parts are the bulk and weight of **Q**, which we identify as

$$\mathbf{Q}_\bullet = Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1} \tag{3.95}$$

and

$$\mathbf{Q}_\circ = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1}. \tag{3.96}$$

The weight $\mathbf{Q}_\circ$ contains information about the rotation performed by **Q**, and it can be regarded as a quaternion in which we equate $i = \mathbf{e}_{41}$, $j = \mathbf{e}_{42}$, and $k = \mathbf{e}_{43}$. The bulk $\mathbf{Q}_\bullet$ contains a mixture of information about the position of the rotation axis and the displacement along the axis. If the bulk is zero, then the operation performed by the motor is a rotation about an axis through the origin without any translation. The set of all motors with a bulk of zero is equivalent to the set of quaternions.

The fact that every proper Euclidean isometry must be the result of an even number of reflections across planes means that every motor must be the geometric antiproduct of an even number of antivectors representing those planes. That means the geometric constraint given by Equation (3.73) applies, and it must therefore always be true that

$$\mathbf{Q} \vee \mathbf{Q} = \mathbf{Q} \circ \mathbf{Q}. \tag{3.97}$$

When we expand the geometric antiproduct $\mathbf{Q} \vee \mathbf{Q}$, we find that

$$\mathbf{Q} \vee \mathbf{Q} = 2\left(Q_{vx}Q_{mx} + Q_{vy}Q_{my} + Q_{vz}Q_{mz} + Q_{vw}Q_{mw}\right)\mathbf{1} + \left(Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2\right)\mathbb{1}. \tag{3.98}$$

To satisfy the geometric constraint, the scalar term here must be zero, which requires that

Motor constraint

$$\boxed{Q_{vx}Q_{mx} + Q_{vy}Q_{my} + Q_{vz}Q_{mz} + Q_{vw}Q_{mw} = 0.} \tag{3.99}$$

This constraint is always satisfied by a motor having the form shown in Equation (3.93) as long as the line $\boldsymbol{l}$ satisfies the constraint for a line. The constraint for a motor is very similar to that of a line and only adds the additional term involving the w coordinates.

If the scalar component $\mathbf{Q}_{mw}$ is zero, then $\mathbf{Q}$ is called a *simple motor*. For this to be the case, we must have $\delta \sin \phi = 0$, so either the displacement distance δ is zero, which would make $\mathbf{Q}$ a pure rotation like the operator $\mathbf{R}$ in Equation (3.86), or the rotation angle ϕ is zero, which would make $\mathbf{Q}$ a pure translation like the operator $\mathbf{T}$ in Equation (3.90). An arbitrary motor $\mathbf{Q}$ can always be factored into a pair of simple motors corresponding to a rotation about an axis passing through the origin followed by a translation. This allows us to express $\mathbf{Q}$ as the product $\mathbf{Q} = \mathbf{T} \vee \mathbf{R}$. The rotation factor $\mathbf{R}$ is just the weight of $\mathbf{Q}$, shown in Equation (3.96). The translation factor $\mathbf{T}$ is found by multiplying $\mathbf{Q}$ by the inverse of $\mathbf{R}$, which is just $\underset{\sim}{\mathbf{Q}}_{\circ}$. Doing so gives us

$$\begin{aligned}\mathbf{T} = \mathbf{Q} \vee \underset{\sim}{\mathbf{Q}}_{\circ} &= (Q_{vy}Q_{mz} - Q_{vz}Q_{my} + Q_{mx}Q_{vw} - Q_{vx}Q_{mw})\,\mathbf{e}_{23} \\ &+ (Q_{vz}Q_{mx} - Q_{vx}Q_{mz} + Q_{my}Q_{vw} - Q_{vy}Q_{mw})\,\mathbf{e}_{31} \\ &+ (Q_{vx}Q_{my} - Q_{vy}Q_{mx} + Q_{mz}Q_{vw} - Q_{vz}Q_{mw})\,\mathbf{e}_{12} + \mathbb{1}. \qquad (3.100)\end{aligned}$$

A motor can always be constructed as the composition of a rotation and translation, and this is similar to how the equivalent 4×4 matrix is built from the rotation in its upper-left 3×3 portion and the translation in its fourth column. The particular details of both methods are shown side by side in Comparison Chart #3.

If $\mathbf{Q}$ is a simple motor, then the six bivector components are exactly the components of the line about which a rotation occurs, and they can be read off directly. This is true when $\delta = 0$ because the adjustment to the moment shown in Equation (3.92) is zero. In this case, the extracted line would need to be unitized because it is multiplied by $\sin \phi$. If $\mathbf{Q}$ is a simple motor because $\phi = 0$, then the three bivector components that exist can be interpreted as a rotation about the line at infinity given by the term $-\boldsymbol{l}^{\star}\delta$ in Equation (3.93). If $\mathbf{Q}$ is *not* a simple motor, then the six bivector components still represent the line about which the rotation occurs, but the line is not unitized, and it does not satisfy the geometric constraint. In this case, we need to apply Equation (3.78) to fix everything up. The orthogonalization of the moment has the effect of removing the adjustment of $-\boldsymbol{l}^{\star}\delta \cos \phi$ in Equation (3.93), and we recover the original line $\boldsymbol{l}$.

The bulk norm and weight norm of a motor $\mathbf{Q}$ are given by

$$\|\mathbf{Q}\|_{\bullet} = \mathbf{1}\sqrt{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2} \qquad (3.101)$$

and

$$\|\mathbf{Q}\|_{\circ} = \mathbb{1}\sqrt{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}. \qquad (3.102)$$

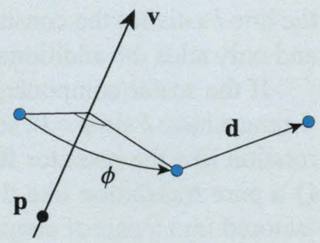
For flat geometric objects, the geometric norm given by the ratio of the bulk norm to the weight norm is equal to the distance between the object and the origin. The geometric norm of an operator has a different interpretation, but once we find out what it means, it will make sense to apply that meaning to flat objects as well. First, given an arbitrary motor $\mathbf{Q}$, we can factor out the weight so we're able to assume that $\mathbf{Q}$ is unitized. That allows us to put $\mathbf{Q}$ in the form of Equation (3.92) and assume that the line $\boldsymbol{l}$ is unitized because $\left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2\right)\sin^2 \phi + \cos^2 \phi = 1$. Now we only need to worry about deciphering the meaning of the bulk norm given by Equation (3.101). When we substitute the component values from Equation (3.92) corresponding to Q_{mx}, Q_{my}, Q_{mz}, and Q_{mw} into the squared bulk norm, we have

$$\begin{aligned}\|\mathbf{Q}\|_{\bullet}^2 &= (l_{mx} \sin \phi + l_{vx}\delta \cos \phi)^2 + (l_{my} \sin \phi + l_{vy}\delta \cos \phi)^2 + (l_{mz} \sin \phi + l_{vz}\delta \cos \phi)^2 \\ &+ \delta^2 \sin^2 \phi. \qquad (3.103)\end{aligned}$$

Comparison Chart #3

Rotation-Translation Operator

Construct an operator that performs a rotation through the angle ϕ about the line containing the point $\mathbf{p}$ and running parallel to the normalized direction vector $\mathbf{v}$ followed by a translation along the displacement vector $\mathbf{d}$.



Conventional Methods	Geometric Algebra
Break the operation into three parts that will each have an associated 4×4 matrix: (a) Translate by $-\mathbf{p}$ so the line passes through the origin. (b) Rotate through the angle ϕ about the direction $\mathbf{v}$. (c) Translate by $\mathbf{p} + \mathbf{d}$ to move the line back to its original position and apply the displacement $\mathbf{d}$.	Calculate $\boldsymbol{l} = \mathbf{p} \wedge \mathbf{v}$ as the line about which the rotation occurs.
Let $\mathbf{R}$ be the upper-left 3×3 portion of the transformation matrix corresponding to part (b). When an arbitrary vector $\mathbf{u}$ is rotated about the direction $\mathbf{v}$, the component $\mathbf{u}_{\parallel\mathbf{v}}$ parallel to $\mathbf{v}$ is unchanged. The rotation is applied only to the perpendicular component $\mathbf{u}_{\perp\mathbf{v}}$, and the transformed vector $\mathbf{u}'$ is given by $\mathbf{u}' = \mathbf{u}_{\parallel\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}} \cos\phi + (\mathbf{v} \times \mathbf{u}_{\perp\mathbf{v}}) \sin\phi.$ Using $\mathbf{u}_{\parallel\mathbf{v}} = (\mathbf{u} \cdot \mathbf{v})\,\mathbf{v}$ and $\mathbf{u}_{\perp\mathbf{v}} = \mathbf{u} - \mathbf{u}_{\parallel\mathbf{v}}$, this expands to $\mathbf{u}' = \mathbf{u} \cos\phi + (\mathbf{u} \cdot \mathbf{v})\,\mathbf{v}\,(1 - \cos\phi) + (\mathbf{v} \times \mathbf{u}) \sin\phi,$ and the equivalent matrix operating on $\mathbf{u}$ is $\mathbf{R} = \mathbf{I} \cos\phi + \mathbf{v}\mathbf{v}^{\mathrm{T}}\,(1 - \cos\phi) + [\mathbf{v}]_{\times} \sin\phi.$	The rotation through the angle ϕ about the line $\boldsymbol{l}$ is performed by the operator $\mathbf{R} = \boldsymbol{l} \sin\frac{\phi}{2} + \mathbb{1} \cos\frac{\phi}{2}.$ The translation by the displacement vector $\mathbf{d}$ is performed by the operator $\mathbf{T} = \frac{d_x}{2}\mathbf{e}_{23} + \frac{d_y}{2}\mathbf{e}_{31} + \frac{d_z}{2}\mathbf{e}_{12} + \mathbb{1}.$
The full 4×4 transformation $\mathbf{M}$ is then $\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{p}+\mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p}+\mathbf{d}-\mathbf{R}\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix},$ where the product corresponds to parts (a), (b), and (c) from right to left.	The full transformation is then $\mathbf{Q} = \mathbf{T} \vee \mathbf{R}$.

Expanding all of the squared quantities and applying the properties $\boldsymbol{l}_{\mathrm{v}}^2 = 1$ and $\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathrm{m}} = 0$ possessed by the line simplifies this expression to

$$\|\mathbf{Q}\|_{\bullet}^2 = \boldsymbol{l}_{\mathrm{m}}^2 \sin^2\phi + \delta^2 \tag{3.104}$$

We can recognize right away that the term δ^2 is the square of half the translation distance along the axis of rotation, but figuring out what distance the term $\boldsymbol{l}_{\mathrm{m}}^2 \sin^2\phi$ represents takes a little more work. The value of $\boldsymbol{l}_{\mathrm{m}}^2$ is the squared distance between the axis of rotation and the origin, so we really just need to make sense out of the $\sin^2\phi$ factor. We start by employing the trigonometric identity

$$\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi) = \frac{1}{4}(2 - 2\cos 2\phi), \tag{3.105}$$

and then we use a small trick to transform the rightmost expression into something easier to interpret. We replace the standalone value of 2 with $\sin^2 2\phi + \cos^2 2\phi + 1$, and that lets us write

$$\begin{aligned}\sin^2\phi &= \frac{1}{4}\left(\sin^2 2\phi + \cos^2 2\phi + 1 - 2\cos 2\phi\right) \\ &= \frac{1}{4}\left[\sin^2 2\phi + (1 - \cos 2\phi)^2\right].\end{aligned} \tag{3.106}$$

As shown in Figure 3.10, if we set $r = \sqrt{\boldsymbol{l}_{\mathrm{m}}^2}$, then $r \sin 2\phi$ and $r(1 - \cos 2\phi)$ are the side lengths in a right triangle for which the length of the hypotenuse is the distance that the origin $\mathbf{o}$ is moved by a rotation through the angle 2ϕ about the line $\boldsymbol{l}$. The value of $\boldsymbol{l}_{\mathrm{m}}^2 \sin^2\phi$ is therefore the square of half that distance due to the factor of 1/4. This corresponds to half the distance the origin is moved in the directions perpendicular to $\boldsymbol{l}$, and the other term δ^2 appearing in Equation (3.104) corresponds to half the distance the origin is moved in the direction parallel to $\boldsymbol{l}$. We come to the conclusion that the geometric norm of a motor $\mathbf{Q}$ is equal to half the total distance that the origin $\mathbf{o}$ is moved by the operation $\mathbf{Q} \vee \mathbf{o} \vee \tilde{\mathbf{Q}}$.

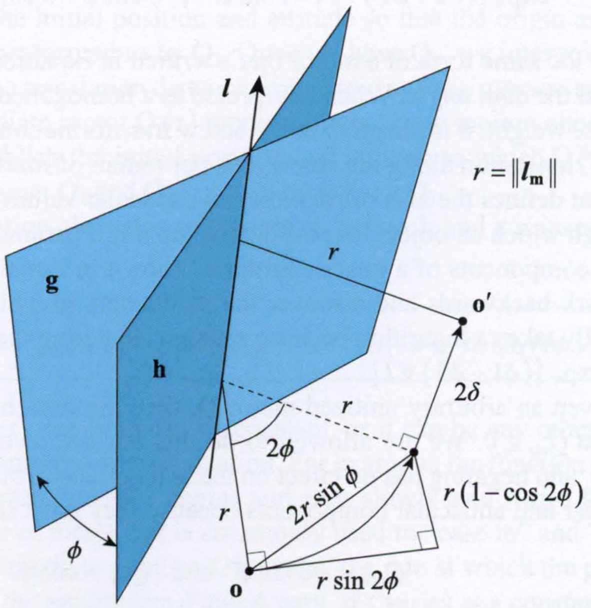


Figure 3.10. A rotation through the angle 2ϕ about the unitized line $\boldsymbol{l}$ moves the origin along the hypotenuse of the right triangle having side lengths $r \sin 2\phi$ and $r(1 - \cos 2\phi)$, where $r = \|\boldsymbol{l}_{\mathrm{m}}\|$.

If we plug the angle $\phi = \pi/2$ into the operator $\mathbf{R} = \boldsymbol{l} \sin\phi + \mathbb{1}\cos\phi$, then the antiscalar term vanishes, and we are left with $\mathbf{R} = \boldsymbol{l}$. Since the rotation goes through twice the angle ϕ, a line by itself with no additional terms corresponds to a 180-degree rotation. We mentioned earlier that the interpretation of the geometric norm of an operator would also make sense if it were to be applied to a flat geometry. In a 180-degree rotation, the origin would be moved to the opposite side of the line, and the total distance that it moved would be twice the distance between the line and the origin. The same thing happens when the origin is reflected across a plane or inverted through a point. Thus, we can say that the geometric norm of any flat geometry can be interpreted both as the distance between the origin and the geometry and as half the distance that the origin is moved when the geometry is used as an operator in a sandwich antiproduct.

3.6.2 Parameterization

A motor $\mathbf{Q}$ can be expressed as the exponential of a unitized line $\boldsymbol{l}$ multiplied by the dual number $\delta\mathbf{1} + \phi\mathbb{1}$, where ϕ continues to represent half the angle of rotation about the line $\boldsymbol{l}$, and δ is half the displacement distance along the line $\boldsymbol{l}$. The exponential is evaluated with respect to the geometric antiproduct, which can be written as

$$\exp_{\vee}[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \boldsymbol{l}] = \cos_{\vee}(\delta\mathbf{1} + \phi\mathbb{1}) + \sin_{\vee}(\delta\mathbf{1} + \phi\mathbb{1}) \vee \boldsymbol{l}. \tag{3.107}$$

This expansion into sine and cosine terms can be verified by looking at the power series for the exponential and using $\boldsymbol{l} \vee \boldsymbol{l} = -\mathbb{1}$. (Keep in mind that any quantity raised to the zero power is the multiplicative identity, which is $\mathbb{1}$ in this case.) Using the formulas for sine and cosine of a dual number with respect to the antiproduct in Table 3.4, this expands to

$$\exp_{\vee}[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \boldsymbol{l}] = -\delta\mathbf{1}\sin\phi + \mathbb{1}\cos\phi + (\delta\mathbf{1}\cos\phi + \mathbb{1}\sin\phi) \vee \boldsymbol{l}. \tag{3.108}$$

By Equation (3.63), we know that the weight dual of $\boldsymbol{l}$ is equal to $\underset{\sim}{\boldsymbol{l}} \vee \mathbf{1}$. Since the antireverse of $\boldsymbol{l}$ is its negative and $\boldsymbol{l}$ commutes with scalars under the geometric antiproduct, we have $\mathbf{1} \vee \boldsymbol{l} = -\boldsymbol{l}^{\star}$. With this substitution, the exponential now produces

$$\exp_{\vee}[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \boldsymbol{l}] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi, \tag{3.109}$$

which is exactly the same form of a motor that's written in Equation (3.93). The quantity $\delta\mathbf{1} + \phi\mathbb{1}$ is sometimes called the *dual angle*. When interpreted as a homogeneous magnitude in which $\delta\mathbf{1}$ is the bulk and $\phi\mathbb{1}$ is the weight, it is the *pitch* of the screw transformation performed by the motor, which is the amount of translation along the screw axis per radian of rotation.

The line $\boldsymbol{l}$ that defines the axis of rotation and the scalar values δ and ϕ that define the distance and angle through which an object is moved make up a full parameterization of a motion operator. Given the eight components of a generic motor as shown in Equation (3.94), we would often like to be able to work backwards and discover the parameters to which the motor corresponds. This process essentially takes a logarithm because we are calculating the information in the exponent of the expression $\exp_{\vee}[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \boldsymbol{l}]$.

If we are given an arbitrary unitized motor $\mathbf{Q}$, then the first thing we do is negate all terms if necessary so that $Q_{vw} \geq 0$. We are allowed to do this because everything is homogeneous in the projective space, and negating has no effect on the actual transformation that $\mathbf{Q}$ performs. Next, we examine the scalar and antiscalar components because they must satisfy

$$Q_{mw} = -\delta\sin\phi \quad\text{and}\quad Q_{vw} = \cos\phi. \tag{3.110}$$

In the case that $Q_{vw} = 1$, the angle ϕ is zero, and the motor represents a pure translation. When this happens, we simply read the half-displacement vector $\boldsymbol{\tau}$ off of the $\mathbf{e}_{23}$, $\mathbf{e}_{31}$ and $\mathbf{e}_{12}$ components.

Otherwise, if $Q_{vw} < 1$, then there must be some rotation involved. We can assume that the angle ϕ is in the range $(0, \pi/2]$ because a motor rotates through the angle 2ϕ, and a rotation through a negative angle about the line $\boldsymbol{l}$ is equivalent to a rotation through a positive angle of the same size about the line $-\boldsymbol{l}$. We can now calculate $s = \sin\phi = \sqrt{1 - Q_{vw}^2}$, which cannot be zero because $Q_{vw} < 1$, and then determine the values of δ and ϕ using the relationships

$$\delta = -\frac{Q_{mw}}{s} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{s}{Q_{vw}}\right). \tag{3.111}$$

If $Q_{vw} = 0$, then we assign $\phi = \pi/2$ as the angle having an infinite tangent. The components of line's direction $\boldsymbol{l}_{\mathbf{v}}$ are easy to recover because they are simply multiplied by $\sin\phi$ in Equation (3.109), so we have

$$\boldsymbol{l}_{\mathbf{v}} = \frac{1}{s}\mathbf{Q}_{vxyz}. \tag{3.112}$$

The line's moment $\boldsymbol{l}_{\mathbf{m}}$ needs to be orthogonalized with Equation (3.78), but we can use $\boldsymbol{l}_{\mathbf{v}}^2 = 1/s^2$ and the motor constraint given by Equation (3.99) to write it as

$$\boldsymbol{l}_{\mathbf{m}} = \frac{1}{s}\left(\mathbf{Q}_{mxyz} + \frac{Q_{vw}Q_{mw}}{s^2}\mathbf{Q}_{vxyz}\right). \tag{3.113}$$

The ability to extract the parameters δ, ϕ, and $\boldsymbol{l}$ from a motor $\mathbf{Q}$ gives us everything we need to continuously interpolate between two motors. The rotation and translation information contained in a motor can be used to describe the position and attitude of an object in space just like a 4×4 matrix can, so motor interpolation provides a method for smoothly moving from one position and attitude to another. Given that we want to interpolate from a motor $\mathbf{Q}_1$ to a motor $\mathbf{Q}_2$, we don't start with the parameters of $\mathbf{Q}_1$ and interpolate directly to the parameters of $\mathbf{Q}_2$. Instead, we calculate

$$\mathbf{Q}_0 = \mathbf{Q}_2 \vee \mathbf{Q}_1^{-1} = \mathbf{Q}_2 \vee \underset{\sim}{\mathbf{Q}_1} \tag{3.114}$$

and determine the parameters δ, ϕ, and $\boldsymbol{l}$ for the motor $\mathbf{Q}_0$. Multiplying by the inverse of $\mathbf{Q}_1$ has the effect of factoring out the initial position and attitude so that the origin and coordinate axes start where they would be transformed to by $\mathbf{Q}_1$. Once we have $\mathbf{Q}_0$, we interpolate from $\mathbb{1}$ (the identity) to $\mathbf{Q}_0$ by interpolating the translation distance from 0 to δ and the rotation angle from 0 to ϕ in order to construct an intermediate motor $\mathbf{Q}(t)$ representing a screw motion about the line $\boldsymbol{l}$. Finally, we multiply by $\mathbf{Q}_1$ to reestablish the initial position and attitude to which $\mathbf{Q}(t)$ is relative so that the intermediate motor between $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is given by $\mathbf{Q}(t) \vee \mathbf{Q}_1$.

During the interpolation, the value of t ranges from 0 to 1, and it is inserted into the exponential form of a motor to give us

$$\boxed{\mathbf{Q}(t) = \exp_{\vee}\left[t(\delta\mathbf{1} + \phi\mathbb{1}) \vee \boldsymbol{l}\right] = \boldsymbol{l}\sin(t\phi) - \boldsymbol{l}^{\star} t\delta\cos(t\phi) - t\delta\sin(t\phi) + \mathbb{1}\cos(t\phi).} \tag{3.115}$$

The rate at which t varies over time can be constant, or it can be any other function that produces the desired motion for some specific application. For example, the function $t(u) = 3u^2 - 2u^3$, which is known as the *smoothstep* function, begins and ends slowly as u varies from 0 to 1 at a constant rate. This function is one of many that is commonly used to "ease in" and "ease out" an animation from one position and attitude to another. Of course, the rate at which the parameter t varies determines the rate at which the parameters δ and ϕ vary. If t varies at a constant rate, then the interpolated motion between two motors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ consists of a rotation and translation at a constant rate as well.

While the method of interpolation just described produces perfect results, the work involved in using Equations (3.111), (3.112), and (3.113) to extract the parameters of $\mathbf{Q}_2 \vee \underset{\sim}{\mathbf{Q}_1}$ and then using Equation (3.115) construct an intermediate motor can be computationally expensive. One of the most common applications of motor interpolation appears in real-time 3D graphics and is called *dual quaternion skinning*. Character models are often constructed out of a tree hierarchy of rigid nodes called bones covered by a polygonal mesh that could contain many thousands of vertices. Each bone has a transform representing its position and attitude relative to the root of the model tree, and each vertex in the mesh has a position that is influenced by the current transform of one or more bones. As the model is animated and the bones move, every vertex position needs to be recalculated over and over using a weighted blend among potentially several different bones, and this requires interpolating among several different bone transforms. This interpolation can be performed very cheaply by blending the entries of 4×4 matrices, but this often leads to visual artifacts because the intermediate matrices can stray very far from a state in which the upper-left 3×3 portion is orthogonal. Motor interpolation does not suffer from this problem because the motor generated by Equation (3.115) is always unitized and always satisfies the geometric constraint. Interpolated motors tend to preserve volume in cases where interpolated matrices cause a mesh to collapse.

The computational expense of interpolating motors thousands of times can be reduced significantly if we're willing to sacrifice the exact proportionality between the interpolation parameter t and the motor parameters δ and ϕ. A simple interpolation of the individual components of two unitized motors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ produces an intermediate motor that is approximately correct. To make sure the interpolation follows the shortest path, it is standard practice to first negate either $\mathbf{Q}_1$ or $\mathbf{Q}_2$, if necessary, so that $\mathbf{Q}_1 \circ \mathbf{Q}_2$ is nonnegative. The approximate motor $\mathbf{Q}(t)$ is then given by

$$\mathbf{Q}(t) = (1-t)\,\mathbf{Q}_1 + t\mathbf{Q}_2. \tag{3.116}$$

Unless $t = 0$ or $t = 1$, this motor is not unitized, and it does not satisfy the geometric constraint given by Equation (3.99), so we need to make some adjustments to get it into a valid form matching Equation (3.93). The process is virtually identical to the one used in Section 3.4.3 to fix a line, except this time, we divide by the square root of

$$\mathbf{Q} \vee \underset{\sim}{\mathbf{Q}} = 2(\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m})\,\mathbf{1} + \mathbf{Q}_\mathbf{v}^2\,\mathbb{1}. \tag{3.117}$$

This dual number contains exactly the information needed to unitize the motor and adjust $\mathbf{Q}_\mathbf{m}$ so its orthogonal to $\mathbf{Q}_\mathbf{v}$ as a four-dimensional vector. The inverse square root of $\mathbf{Q} \vee \underset{\sim}{\mathbf{Q}}$ is

$$\frac{1}{\sqrt[\vee]{\mathbf{Q} \vee \underset{\sim}{\mathbf{Q}}}}{}^{\vee} = \frac{1}{\|\mathbf{Q}_\mathbf{v}\|}\left(-\frac{\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2}\mathbf{1} + \mathbb{1}\right), \tag{3.118}$$

and multiplying $\mathbf{Q}$ by it gives us

$$\boxed{\frac{\mathbf{Q}}{\|\mathbf{Q}_\mathbf{v}\|} \vee \left(-\frac{\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2}\mathbf{1} + \mathbb{1}\right) = \frac{1}{\|\mathbf{Q}_\mathbf{v}\|}\left[\mathbf{Q} - \frac{\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m}}{\mathbf{Q}_\mathbf{v}^2}(Q_{vx}\mathbf{e}_{23} + Q_{vy}\mathbf{e}_{31} + Q_{vz}\mathbf{e}_{12} + Q_{vw})\right].} \tag{3.119}$$

If we unitize the interpolated $\mathbf{Q}(t)$ first, then we just need to subtract

$$(\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m})(Q_{vx}\mathbf{e}_{23} + Q_{vy}\mathbf{e}_{31} + Q_{vz}\mathbf{e}_{12} + Q_{vw}) \tag{3.120}$$

from the four bulk components, and we're good to go.

An interpolated motor $\mathbf{Q}(t)$ calculated with Equation (3.116) and fixed up with Equation (3.119) is exactly the same as an interpolated motor calculated with Equation (3.115) when $t = 0$,

$t = 1$, and $t = 1/2$. Otherwise, the interpolation is a little slow near $t = 0$ and $t = 1$, and it's a little fast near the middle where $t = 1/2$. The exact match at $t = 1/2$ gives us an efficient way to calculate the square root of a motor $\mathbf{Q}$ because we can use Equation (3.119) to fix up a motor that is halfway between the identity and $\mathbf{Q}$. We can even drop the halves in the expression $\frac{1}{2}\mathbf{Q} + \frac{1}{2}\mathbb{1}$ because any scale will be divided out by Equation (3.119).

Suppose $\mathbf{Q}$ is a unitized motor that satisfies the geometric constraint, and let $\mathbf{R} = \mathbf{Q} + \mathbb{1}$. Then the square root of $\mathbf{Q}$ with respect to the geometric antiproduct must be given by

$$\sqrt[\vee]{\mathbf{Q}} = \frac{\mathbf{R}}{\|\mathbf{R}_\mathbf{v}\|} \vee \left(-\frac{\mathbf{R}_\mathbf{v} \cdot \mathbf{R}_\mathbf{m}}{\mathbf{R}_\mathbf{v}^2} \mathbf{1} + \mathbb{1} \right). \tag{3.121}$$

Since $\mathbf{Q}_\mathbf{v}^2 = 1$, the value of $\mathbf{R}_\mathbf{v}^2$ is given by

$$\mathbf{R}_\mathbf{v}^2 = Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + (Q_{vw} + 1)^2 = 2 + 2Q_{vw}, \tag{3.122}$$

and since $\mathbf{Q}_\mathbf{v} \cdot \mathbf{Q}_\mathbf{m} = 0$, the value of $\mathbf{R}_\mathbf{v} \cdot \mathbf{R}_\mathbf{m}$ is given by

$$\mathbf{R}_\mathbf{v} \cdot \mathbf{R}_\mathbf{m} = Q_{vx}Q_{mx} + Q_{vy}Q_{my} + Q_{vz}Q_{mz} + (Q_{vw} + 1)Q_{mw} = Q_{mw}. \tag{3.123}$$

When we plug these into Equation (3.121), we have

$$\sqrt[\vee]{\mathbf{Q}} = \frac{\mathbf{R}}{\sqrt{2 + 2Q_{vw}}} \vee \left(-\frac{Q_{mw}}{2 + 2Q_{vw}} \mathbf{1} + \mathbb{1} \right). \tag{3.124}$$

Substituting $\mathbf{Q} + \mathbb{1}$ for $\mathbf{R}$ and using the compact notation Q_1 and $Q_{\mathbb{1}}$ for the scalar and antiscalar coordinates, we can write the formula for the square root of a general motor $\mathbf{Q}$ as

Square root of general motor

$$\boxed{\sqrt[\vee]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} \vee \left(\mathbb{1} - \frac{Q_1}{2 + 2Q_{\mathbb{1}}} \mathbf{1} \right).} \tag{3.125}$$

If $\mathbf{Q}$ is a simple motor, then $Q_1 = 0$, and this reduces to

Square root of simple motor

$$\boxed{\sqrt[\vee]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\|\mathbf{Q} + \mathbb{1}\|_\circ},} \tag{3.126}$$

where we have replaced $\sqrt{2 + 2Q_{\mathbb{1}}}$ by the weight norm of $\mathbf{R}$. This equation tells us that we can find the square root of a simple motor by just adding $\mathbb{1}$ and unitizing the result.

3.6.3 Line to Line Motion

A line is not only a geometric object but also a motion operator that performs a 180-degree rotation about itself. We can find an operator that transforms a line $\mathbf{k}$ into a line $\boldsymbol{l}$ by first considering the quotient $\boldsymbol{l} \vee \mathbf{k}$ because this is the operator that interpolates between a rotation about $\mathbf{k}$ and a rotation about $\boldsymbol{l}$. When we calculate $\boldsymbol{l} \vee \mathbf{k}$, we get

$$\begin{aligned} \boldsymbol{l} \vee \underset{\sim}{\mathbf{k}} = {} & (l_{vz}k_{vy} - l_{vy}k_{vz})\,\mathbf{e}_{41} + (l_{vz}k_{my} - l_{vy}k_{mz} + l_{mz}k_{vy} - l_{my}k_{vz})\,\mathbf{e}_{23} \\ & + (l_{vx}k_{vz} - l_{vz}k_{vx})\,\mathbf{e}_{42} + (l_{vx}k_{mz} - l_{vz}k_{mx} + l_{mx}k_{vz} - l_{mz}k_{vx})\,\mathbf{e}_{31} \\ & + (l_{vy}k_{vx} - l_{vx}k_{vy})\,\mathbf{e}_{43} + (l_{vy}k_{mx} - l_{vx}k_{my} + l_{my}k_{vx} - l_{mx}k_{vy})\,\mathbf{e}_{12} \\ & + (l_{vx}k_{mx} + l_{vy}k_{my} + l_{vz}k_{mz} + l_{mx}k_{vx} + l_{my}k_{vy} + l_{mz}k_{vz})\,\mathbf{1} + (l_{vx}k_{vx} + l_{vy}k_{vy} + l_{vz}k_{vz})\,\mathbb{1}. \end{aligned} \tag{3.127}$$

Comparing against the standard form of a motor shown in Equation (3.93) tells us several things about this operator. Assuming that the lines are both unitized, the antiscalar coordinate $\boldsymbol{l}_\mathbf{v} \cdot \mathbf{k}_\mathbf{v}$ is the cosine of the angle ϕ between the lines' directions, and this matches the antiscalar term in Equation (3.93). The scalar coordinate $\boldsymbol{l}_\mathbf{v} \cdot \mathbf{k}_\mathbf{m} + \boldsymbol{l}_\mathbf{m} \cdot \mathbf{k}_\mathbf{v}$ shows up in the formula for the distance δ between two lines, listed in Table 2.15, and it must be scaled by the sine of the angle between the lines. This matches the scalar term in Equation (3.93). Finally, the direction stored in the $\mathbf{e}_{41}$, $\mathbf{e}_{42}$, and $\mathbf{e}_{43}$ terms is equal to $\mathbf{k} \times \boldsymbol{l}$. This is the direction in which the axis of rotation runs, and it must be perpendicular to the directions of both $\mathbf{k}$ and $\boldsymbol{l}$.

Let $\mathbf{f}$ be the line representing the axis of rotation for the motor $\boldsymbol{l} \vee \underset{\sim}{\mathbf{k}}$. To determine the position of $\mathbf{f}$, we need to extract the bivector terms from $\boldsymbol{l} \vee \underset{\sim}{\mathbf{k}}$ and apply Equation (3.78) to unitize them and enforce the geometric constraint. The line $\mathbf{f}$ is then given by

$$\mathbf{f} = \frac{1}{\sqrt{\mathbf{b}_\mathbf{v}^2}} \left[\mathbf{b} - \frac{\mathbf{b}_\mathbf{v} \cdot \mathbf{b}_\mathbf{m}}{\mathbf{b}_\mathbf{v}^2} \left(b_{vx}\mathbf{e}_{23} + b_{vy}\mathbf{e}_{31} + b_{vz}\mathbf{e}_{12} \right) \right], \tag{3.128}$$

where we have assigned $\mathbf{b} = \langle \boldsymbol{l} \vee \underset{\sim}{\mathbf{k}} \rangle_2$. A tedious calculation that we do not reproduce here demonstrates that $\mathbf{f} \wedge \mathbf{k} = 0$ and $\mathbf{f} \wedge \boldsymbol{l} = 0$, which means that the distance between the line $\mathbf{f}$ and each of the lines $\mathbf{k}$ and $\boldsymbol{l}$ is zero. Therefore, $\mathbf{f}$ intersects both lines, and because the $\mathbf{f}$ also runs perpendicular to both lines, it can only pass through the points of closest approach lying on $\mathbf{k}$ and $\boldsymbol{l}$, as shown in Figure 3.11.

The information that we've extracted from Equation (3.127) reveals that $\boldsymbol{l} \vee \underset{\sim}{\mathbf{k}}$ is an operator that performs a rotation through an angle 2ϕ about the line $\mathbf{f}$ combined with a translation along that line by a distance 2δ. This is exactly double the motion necessary to move the line $\mathbf{k}$ so that it is coincident with the line $\boldsymbol{l}$. We conclude that the operator that transforms a line $\mathbf{k}$ into the line $\boldsymbol{l}$ is given by the square root of $\boldsymbol{l} \vee \underset{\sim}{\mathbf{k}}$, which has the same axis of rotation $\mathbf{f}$, but only rotates through the angle ϕ and translates by the distance δ.

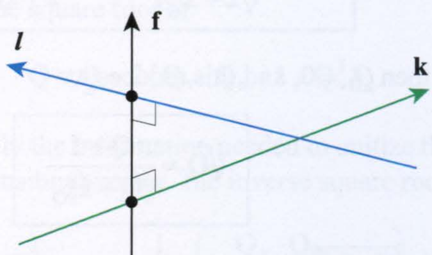


Figure 3.11. The line $\mathbf{f}$ is the axis of rotation for the operator $\boldsymbol{l} \vee \underset{\sim}{\mathbf{k}}$, given by Equation (3.128). It is perpendicular to both the lines $\mathbf{k}$ and $\boldsymbol{l}$, and it passes through the points of closest approach lying on $\mathbf{k}$ and $\boldsymbol{l}$.

3.6.4 Matrix Conversion

The sandwich antiproduct $\mathbf{Q} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{Q}}$ is a linear transformation of the point $\mathbf{p}$ by the motor $\mathbf{Q}$, and expressing it as a matrix-vector product is a very straightforward process. All we have to do is write each component the transformed point in terms of its original coordinates and read off the coefficients multiplying $p_x\mathbf{e}_1$, $p_y\mathbf{e}_2$, $p_z\mathbf{e}_3$, and $p_w\mathbf{e}_4$. Since $\mathbf{Q}^{-1} = \underset{\sim}{\mathbf{Q}}$, the only difference between a motor and its inverse is that the bivector components are negated, so the matrices corresponding to $\mathbf{Q}$ and $\mathbf{Q}^{-1}$ are very similar. In fact, we can construct two matrices for any motor $\mathbf{Q}$ in such a way that their sum corresponds to $\mathbf{Q}$ and their difference corresponds to $\mathbf{Q}^{-1}$.

Given a specific unitized motor $\mathbf{Q}$, we define the matrices $\mathbf{A}_\mathbf{Q}$ and $\mathbf{B}_\mathbf{Q}$ as

$$\mathbf{A_Q} = \begin{bmatrix} 1-2\left(Q_{vy}^2+Q_{vz}^2\right) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2\left(Q_{vy}Q_{mz}-Q_{vz}Q_{my}\right) \\ 2Q_{vx}Q_{vy} & 1-2\left(Q_{vz}^2+Q_{vx}^2\right) & 2Q_{vy}Q_{vz} & 2\left(Q_{vz}Q_{mx}-Q_{vx}Q_{mz}\right) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1-2\left(Q_{vx}^2+Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my}-Q_{vy}Q_{mx}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.129}$$

and

$$\mathbf{B_Q} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2\left(Q_{vw}Q_{mx}-Q_{vx}Q_{mw}\right) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2\left(Q_{vw}Q_{my}-Q_{vy}Q_{mw}\right) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2\left(Q_{vw}Q_{mz}-Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.130}$$

where we have used the fact that $Q_{vx}^2+Q_{vy}^2+Q_{vz}^2+Q_{vw}^2=1$ in the diagonal entries of $\mathbf{A_Q}$. Then the 4×4 matrix $\mathbf{M_Q}$ that transforms a point $\mathbf{p}$, regarded as a 4×1 column matrix, in the same way that it would be transformed by the motor $\mathbf{Q}$ is given by

$$\mathbf{M_Q} = \mathbf{A_Q} + \mathbf{B_Q}. \tag{3.131}$$

That is, $\mathbf{M_Q p} = \mathbf{Q} \vee \mathbf{p} \vee \mathbf{Q}$. The inverse of $\mathbf{M_Q}$ is given by

$$\mathbf{M_Q^{-1}} = \mathbf{A_Q} - \mathbf{B_Q}, \tag{3.132}$$

and this is related to $\mathbf{Q}$ by the equation $\mathbf{M_Q^{-1}p} = \mathbf{Q} \vee \mathbf{p} \vee \mathbf{Q}$.

The matrix $\mathbf{M_Q}$ and its inverse are 4×4 matrices each having a fourth row equal to $[0\ \ 0\ \ 0\ \ 1]$, which is to be expected since $\mathbf{Q}$ performs a rotation and translation. These matrices can be extended to full 16×16 exomorphism matrices as described in Section 2.7 in order to transform lines and planes. In the case of planes, however, the third compound matrix of $\mathbf{M_Q}$ is always equal to its inverse transpose, so a plane $\mathbf{g}$, regarded as a 1×4 row matrix, can simply be transformed through right multiplication by $\mathbf{M_Q^{-1}}$. That is $\mathbf{gM_Q^{-1}} = \mathbf{Q} \vee \mathbf{g} \vee \tilde{\mathbf{Q}}$.

Going the other way and converting a 4×4 matrix $\mathbf{M}$ into a motor $\mathbf{Q}$ is more difficult. The method described here is similar to the conversion from a 3×3 matrix to a quaternion described at the end of Section 1.4.2, but it is extended to include the calculation of the four additional components of $\mathbf{Q_m}$. We start by assuming that $\mathbf{M}$ represents nothing more than a rotation and translation, which means it has the form

$$\mathbf{M} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.133}$$

where the upper-left 3×3 portion is orthogonal, and the determinant is +1. By equating the entries of $\mathbf{M}$ to the entries of $\mathbf{A_Q}+\mathbf{B_Q}$ given by Equations (3.129) and (3.130), we can construct many relationships that can be used to determine the components of the corresponding motor $\mathbf{Q}$. For the diagonal entries of $\mathbf{M}$, we have the four relationships

$$\begin{aligned} M_{00} - M_{11} - M_{22} + 1 &= 4Q_{vx}^2 \\ M_{11} - M_{22} - M_{00} + 1 &= 4Q_{vy}^2 \\ M_{22} - M_{00} - M_{11} + 1 &= 4Q_{vz}^2 \\ M_{00} + M_{11} + M_{22} + 1 &= 4\left(1 - Q_{vx}^2 - Q_{vy}^2 - Q_{vz}^2\right) = 4Q_{vw}^2, \end{aligned} \tag{3.134}$$

and for the off-diagonal entries of **M**, we have the six additional relationships

$$\begin{aligned}
M_{21} + M_{12} &= 4Q_{vy}Q_{vz} \\
M_{02} + M_{20} &= 4Q_{vz}Q_{vx} \\
M_{10} + M_{01} &= 4Q_{vx}Q_{vy} \\
M_{21} - M_{12} &= 4Q_{vx}Q_{vw} \\
M_{02} - M_{20} &= 4Q_{vy}Q_{vw} \\
M_{10} - M_{01} &= 4Q_{vz}Q_{vw}.
\end{aligned} \tag{3.135}$$

If $M_{00} + M_{11} + M_{22} \geq 0$, then we calculate

$$Q_{vw} = \pm\frac{1}{2}\sqrt{M_{00} + M_{11} + M_{22} + 1}, \tag{3.136}$$

where either sign can be chosen. In this case, we know $|Q_{vw}|$ is at least $1/2$, so we can safely divide by $4Q_{vw}$ in the last three off-diagonal relationships shown in Equation (3.135) to solve for Q_{vx}, Q_{vy}, and Q_{vz}. Otherwise, if $M_{00} + M_{11} + M_{22} < 0$, then we select one of the first three diagonal relationships in Equation (3.134) based on which diagonal entry M_{00}, M_{11}, or M_{22} has the greatest value and calculate Q_{vx}, Q_{vy}, or Q_{vz}. The result is plugged into two of the first three off-diagonal relationships to solve for the other two values of Q_{vx}, Q_{vy}, and Q_{vz}. Finally, we plug it into one of the last three off-diagonal relationships to solve for Q_{vw}.

After the four components of $\mathbf{Q_v}$ have been calculated, determining the four remaining components of $\mathbf{Q_m}$ is relatively easy. The values of Q_{mx}, Q_{my}, Q_{mz}, and Q_{mw} are given by

$$\begin{aligned}
Q_{mx} &= \tfrac{1}{2}\left(Q_{vw}t_x + Q_{vz}t_y - Q_{vy}t_z\right) \\
Q_{my} &= \tfrac{1}{2}\left(Q_{vw}t_y + Q_{vx}t_z - Q_{vz}t_x\right) \\
Q_{mz} &= \tfrac{1}{2}\left(Q_{vw}t_z + Q_{vy}t_x - Q_{vx}t_y\right) \\
Q_{mw} &= -\tfrac{1}{2}\left(Q_{vx}t_x + Q_{vy}t_y + Q_{vz}t_z\right),
\end{aligned} \tag{3.137}$$

where $t_x = M_{03}$, $t_y = M_{13}$, and $t_z = M_{23}$ are the entries of **M** corresponding to the translation.

3.6.5 Implementation

The reasons that one would use motors instead of the equivalent 4×4 matrices include the smaller storage size, the ease of parameterization, and trivial invertibility. A 4×4 matrix requires storage for 12 floating-point numbers because its fourth row is always $[0 \;\; 0 \;\; 0 \;\; 1]$. Because we are considering matrices that have equivalent motor representations, we can assume that the upper-left 3×3 portion of the matrix is orthogonal and has a determinant of $+1$. If minimum storage space is our goal, then we don't need to store the third column of such a matrix because it must be equal to the cross product of the first two columns. We can get away with storing only 9 floating-point numbers, and the third column can be reconstituted whenever we need to perform calculations. By comparison, a motor requires storage for 8 floating-point numbers, but this can also be reduced by considering unitization and the geometric constraint. Since **Q** and **–Q** represent the same transformation, we are free to negate so that $Q_{vw} \geq 0$. Then, we don't need to store the value of Q_{vw} for a unitized motor **Q** because $\mathbf{Q}_\mathbf{v}^2 = 1$, and the value of Q_{vw} can always be calculated with

$$Q_{vw} = \sqrt{1 - Q_{vx}^2 - Q_{vy}^2 - Q_{vz}^2}. \tag{3.138}$$

The geometric constraint requires that $\mathbf{Q_v} \cdot \mathbf{Q_m} = 0$, so we also don't need to store the value of Q_{mw} because it can always be calculated with

$$Q_{mw} = -\frac{Q_{vx}Q_{mx} + Q_{vy}Q_{my} + Q_{vz}Q_{mz}}{Q_{vw}}. \tag{3.139}$$

Thus, it is possible to store only 6 floating-point numbers for motor, which is two-thirds the space required for the equivalent matrix. However, the cost of reconstituting the Q_{vw} and Q_{mw} is rather significant due to the square root and division involved.

Computational performance is a major consideration for most applications, and this is unfortunately an area in which motors do not shine. There are tricks that we can use to optimize a direct implementation of the sandwich antiproduct $\mathbf{Q} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{Q}}$, but the best results we can achieve will still be more than twice as expensive as a conventional matrix-vector multiplication. Due to this limitation and the fact that matrices are capable of providing a much larger set of linear transformations, motors should not be viewed as a wholesale replacement for matrices, but only as a specialized tool for purposes that benefit from their advantages.

A close examination of the sandwich antiproduct $\mathbf{Q} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{Q}}$ reveals some redundancies that we can exploit to minimize the total number of floating-point operations necessary to implement it. When a point $\mathbf{p}$ is transformed into the point $\mathbf{p}'$ by the motor $\mathbf{Q}$, the point's w coordinate is not modified, and the transformation of its x, y, and z coordinates can be written as

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left[Q_{vw}\left(\mathbf{v} \times \mathbf{p}_{xyz}\right) + \left(\mathbf{v} \cdot \mathbf{p}_{xyz}\right)\mathbf{v} - \mathbf{v}^2\mathbf{p}_{xyz} + p_w\left(\mathbf{v} \times \mathbf{m} + Q_{vw}\mathbf{m} - Q_{mw}\mathbf{v}\right)\right]. \tag{3.140}$$

Here, we have assigned $\mathbf{v} = (Q_{vx}, Q_{vy}, Q_{vz})$ and $\mathbf{m} = (Q_{mx}, Q_{my}, Q_{mz})$ for convenience, and all bold quantities are being treated as ordinary 3D vectors. We first apply the vector triple product identity

$$(\mathbf{v} \cdot \mathbf{p}_{xyz})\mathbf{v} - \mathbf{v}^2\mathbf{p}_{xyz} = \mathbf{v} \times (\mathbf{v} \times \mathbf{p}_{xyz}) \tag{3.141}$$

in order to expose a reusable cross product. Equation (3.140) now becomes

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left[Q_{vw}\left(\mathbf{v} \times \mathbf{p}_{xyz}\right) + \mathbf{v} \times \left(\mathbf{v} \times \mathbf{p}_{xyz}\right) + p_w\left(\mathbf{v} \times \mathbf{m} + Q_{vw}\mathbf{m} - Q_{mw}\mathbf{v}\right)\right]. \tag{3.142}$$

After a little refactoring, we can rewrite this as

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left[Q_{vw}\left(\mathbf{v} \times \mathbf{p}_{xyz} + p_w\mathbf{m}\right) + \mathbf{v} \times \left(\mathbf{v} \times \mathbf{p}_{xyz} + p_w\mathbf{m}\right) - Q_{mw}p_w\mathbf{v}\right], \tag{3.143}$$

which shows that we can take advantage of an even larger reusable subcalculation given by

$$\mathbf{a} = \mathbf{v} \times \mathbf{p}_{xyz} + p_w\mathbf{m}. \tag{3.144}$$

The entire transformation is now reduced to

Motor-point transformation

$$\boxed{\begin{aligned} \mathbf{p}'_{xyz} &= \mathbf{p}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a} - Q_{mw}p_w\mathbf{v}\right) \\ p'_w &= p_w. \end{aligned}} \tag{3.145}$$

Calculating the vector $\mathbf{a}$ requires 9 combined multiply-add operations, and then calculating $\mathbf{p}'_{xyz}$ with Equation (3.145) requires 16 more for a total of 25 multiply-add operations (or 22 multiplies and 21 separate adds if the multiplication by two is turned into an addition). When it is known that $p_w = 1$, four multiplies are eliminated, and the total computational cost is 21 multiply-add operations. The matrix product $\mathbf{M_Q p}$, however, requires only 12 multiply-add operations in general, and this is reduced to 9 when $p_w = 1$. If many points are to be transformed by a motor, then it is much better for performance to convert the motor to a matrix first and use matrix multiplication.

For the transformation of a line $\boldsymbol{l}$, we look at the direction and moment of the sandwich antiproduct $\mathbf{Q} \vee \boldsymbol{l} \vee \underset{\sim}{\mathbf{Q}}$ separately. The direction of the transformed line $\boldsymbol{l}'$ is given by

$$\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left[Q_{vw}\left(\mathbf{v}\times\boldsymbol{l}_{\mathbf{v}}\right) + \left(\mathbf{v}\cdot\boldsymbol{l}_{\mathbf{v}}\right)\mathbf{v} - \mathbf{v}^2\boldsymbol{l}_{\mathbf{v}}\right], \tag{3.146}$$

where $\mathbf{v}$ and $\mathbf{m}$ continue to have the same meanings as above. This is the same as Equation (3.140) with $\mathbf{p}_{xyz} = \boldsymbol{l}_{\mathbf{v}}$ and $p_w = 0$, so we can apply the same vector triple product identity and jump straight to the optimal calculation

Motor-line transformation (direction)

$$\boxed{\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v}\times\mathbf{a}\right),} \tag{3.147}$$

where $\mathbf{a} = \mathbf{v}\times\boldsymbol{l}_{\mathbf{v}}$. The moment of the transformed line $\boldsymbol{l}'$ is given by

$$\begin{aligned}\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\Big[&Q_{mw}\left(\mathbf{v}\times\boldsymbol{l}_{\mathbf{v}}\right) + Q_{vw}\left(\mathbf{m}\times\boldsymbol{l}_{\mathbf{v}} + \mathbf{v}\times\boldsymbol{l}_{\mathbf{m}}\right) + \left(\mathbf{v}\cdot\boldsymbol{l}_{\mathbf{m}}\right)\mathbf{v} - \mathbf{v}^2\boldsymbol{l}_{\mathbf{m}}\\ &+ \left(\mathbf{m}\cdot\boldsymbol{l}_{\mathbf{v}}\right)\mathbf{v} + \left(\mathbf{v}\cdot\boldsymbol{l}_{\mathbf{v}}\right)\mathbf{m} - 2\left(\mathbf{v}\cdot\mathbf{m}\right)\boldsymbol{l}_{\mathbf{v}}\Big].\end{aligned} \tag{3.148}$$

To simplify this formula, we apply the vector triple product identity three times, allowing us to make the substitutions

$$\begin{aligned}\left(\mathbf{v}\cdot\boldsymbol{l}_{\mathbf{m}}\right)\mathbf{v} - \mathbf{v}^2\boldsymbol{l}_{\mathbf{m}} &= \mathbf{v}\times\left(\mathbf{v}\times\boldsymbol{l}_{\mathbf{m}}\right)\\ \left(\mathbf{m}\cdot\boldsymbol{l}_{\mathbf{v}}\right)\mathbf{v} - \left(\mathbf{v}\cdot\mathbf{m}\right)\boldsymbol{l}_{\mathbf{v}} &= \mathbf{m}\times\left(\mathbf{v}\times\boldsymbol{l}_{\mathbf{v}}\right)\\ \left(\mathbf{v}\cdot\boldsymbol{l}_{\mathbf{v}}\right)\mathbf{m} - \left(\mathbf{v}\cdot\mathbf{m}\right)\boldsymbol{l}_{\mathbf{v}} &= \mathbf{v}\times\left(\mathbf{m}\times\boldsymbol{l}_{\mathbf{v}}\right).\end{aligned} \tag{3.149}$$

It is now possible to reuse the cross products $\mathbf{a} = \mathbf{v}\times\boldsymbol{l}_{\mathbf{v}}$, $\mathbf{b} = \mathbf{v}\times\boldsymbol{l}_{\mathbf{m}}$, and $\mathbf{c} = \mathbf{m}\times\boldsymbol{l}_{\mathbf{v}}$, and we can write the transformed moment as

Motor-line transformation (moment)

$$\boxed{\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\left[Q_{mw}\mathbf{a} + Q_{vw}\left(\mathbf{b}+\mathbf{c}\right) + \mathbf{v}\times\left(\mathbf{b}+\mathbf{c}\right) + \mathbf{m}\times\mathbf{a}\right].} \tag{3.150}$$

Calculating $\boldsymbol{l}'_{\mathbf{v}}$ requires 12 multiply-adds (or 9 multiplies and 12 separate adds), and calculating $\boldsymbol{l}'_{\mathbf{m}}$ requires 24 multiply-adds (or 21 multiplies and 24 separate adds). Thus, transforming a line with a motor requires a total of 36 multiply-add operations. The 6×6 matrix that performs the same transformation requires only 27 multiply-adds, however, because nine of its entries are always zero.

For the transformation of a plane $\mathbf{g}$, we look at the normal and position of the sandwich antiproduct $\mathbf{Q} \vee \mathbf{g} \vee \mathbf{Q}$ separately. The normal of the transformed plane $\mathbf{g}'$ is given by

$$\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2\left[Q_{vw}\left(\mathbf{v}\times\mathbf{g}_{xyz}\right) + \left(\mathbf{v}\cdot\mathbf{g}_{xyz}\right)\mathbf{v} - \mathbf{v}^2\mathbf{g}_{xyz}\right]. \tag{3.151}$$

Once again applying the vector triple product identity $\left(\mathbf{v}\cdot\mathbf{g}_{xyz}\right)\mathbf{v} - \mathbf{v}^2\mathbf{g}_{xyz} = \mathbf{v}\times\left(\mathbf{v}\times\mathbf{g}_{xyz}\right)$, we can now write

Motor-plane transformation (normal)

$$\boxed{\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v}\times\mathbf{a}\right),} \tag{3.152}$$

where $\mathbf{a} = \mathbf{v}\times\mathbf{g}_{xyz}$. No special tricks can be applied to the transformed w coordinate. After grouping terms, it is given by

Motor-plane transformation (position)

$$\boxed{g'_w = g_w + 2\left[\left(\mathbf{m}\times\mathbf{g}_{xyz} + Q_{mw}\mathbf{g}_{xyz}\right)\cdot\mathbf{v} - Q_{vw}\left(\mathbf{m}\cdot\mathbf{g}_{xyz}\right)\right].} \tag{3.153}$$

Calculating $\mathbf{g}'_{xyz}$ requires 12 multiply-adds (or 9 multiplies and 12 separate adds), and calculating g'_w requires 17 multiply-adds (or 16 multiplies and 10 separate adds). The total count for transforming a plane with a motor is thus 29 multiply-add operations. However, the same transformation can be accomplished using a 4×4 matrix with only 13 multiply-adds.

Aside from the computational cost of transforming points, lines, and planes with a motor, we are interested in the cost of composing multiple operations. The geometric antiproduct of two motors $\mathbf{Q}$ and $\mathbf{R}$ is given by

$$\begin{aligned}
\mathbf{Q} \vee \mathbf{R} = &(Q_{vw}R_{vx} + Q_{vx}R_{vw} + Q_{vy}R_{vz} - Q_{vz}R_{vy})\,\mathbf{e}_{41} \\
&+ (Q_{vw}R_{vy} - Q_{vx}R_{vz} + Q_{vy}R_{vw} + Q_{vz}R_{vx})\,\mathbf{e}_{42} \\
&+ (Q_{vw}R_{vz} + Q_{vx}R_{vy} - Q_{vy}R_{vx} + Q_{vz}R_{vw})\,\mathbf{e}_{43} \\
&+ (Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz})\,\mathbb{1} \\
&+ (Q_{mw}R_{vx} + Q_{mx}R_{vw} + Q_{my}R_{vz} - Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} + Q_{vy}R_{mz} - Q_{vz}R_{my})\,\mathbf{e}_{23} \\
&+ (Q_{mw}R_{vy} - Q_{mx}R_{vz} + Q_{my}R_{vw} + Q_{mz}R_{vx} + Q_{vw}R_{my} - Q_{vx}R_{mz} + Q_{vy}R_{mw} + Q_{vz}R_{mx})\,\mathbf{e}_{31} \\
&+ (Q_{mw}R_{vz} + Q_{mx}R_{vy} - Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} + Q_{vx}R_{my} - Q_{vy}R_{mx} + Q_{vz}R_{mw})\,\mathbf{e}_{12} \\
&+ (Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} - Q_{vz}R_{mz})\,\mathbf{1}. \qquad (3.154)
\end{aligned}$$

Calculating the eight components of the result requires 48 multiply-add operations. This is significantly higher than the cost of multiplying the equivalent 4×4 matrices together, which requires only 36 multiply-add operations. This can be reduced to 33 by exploiting the orthogonality of the matrices and computing the third column of the result with the cross product of the first two columns instead of the three dot products involved in the matrix multiplication.

Generalizing to an n-dimensional projective algebra corresponding to an $(n-1)$-dimensional Euclidean space, a motor has 2^{n-1} components over the basis elements with even antigrade. The product of two motors thus involves the multiplication of 2^{2n-2} individual pairings of scalar coefficients, where one comes from the first motor and the other comes from the second. Exactly one quarter of these individual multiplications can be skipped because the result is identically zero. This is due to half of the components of each motor not having a factor of the projective basis vector $\mathbf{e}_n$ and the antiproduct of two such components being zero. The product of two motors thus requires exactly $\frac{3}{4}(2^{2n-2}) = 3 \cdot 2^{2n-4}$ multiply-add operations. By comparison, the product of two $n \times n$ matrices having a bottom row of $[0 \;\; 0 \;\; \cdots \;\; 1]$ requires at most $n(n-1)^2$ multiply-add operations because the calculation of each entry in the $(n-1) \times n$ portion that excludes the bottom row requires $n-1$ multiply-adds. The orthogonality of the upper-left $(n-1) \times (n-1)$ portions of the matrices allows further optimizations in Euclidean spaces of two and three dimensions where $n = 3$ and $n = 4$. The second column of any 2×2 rotation matrix with first column (x, y) must be $(-y, x)$, so it does not need to be calculated separately, and four multiply-adds can be saved in the $n = 3$ case. As already mentioned, three multiply-adds can be saved in the $n = 4$ case because the third column of a 3×3 rotation matrix must be equal to the cross product of the first two columns. Table 3.9 shows a comparison between the computational costs of composing motors and the equivalent matrices in projective algebras having two to five dimensions, which corresponds to Euclidean spaces having one to four dimensions. The cost of composing matrices is *always* less than the cost of composing motors.

Math Library Notes

- The `Motor3D` class stores the eight components of a 3D motion operator as a pair of `Quaternion` objects named `v` and `m`.
- The * operator can multiply motors by other motors or by quaternions with the geometric antiproduct.
- The `Transform()` function takes a `FlatPoint3D`, `Point3D`, `Line3D`, or `Plane3D` object as its first parameter and a `Motor3D` object as its second parameter. It performs the motor transformation with the optimized formula and returns a result having the same type as the first parameter.

- The `Motor3D` class has a `GetTransformMatrix()` member function that calculates a 4×4 matrix with Equation (3.131) and returns it as a `Transform3D` object.
- The `Motor3D` class has a `SetTransformMatrix()` member function that accepts a `Transform3D` object as its parameter. It converts the matrix to a motor using the method described in this section. The code assumes that the upper-left 3×3 portion of the input is a true rotation matrix, meaning that it is orthogonal and has a determinant of $+1$.

Projective Dimension	Motor Composition	Matrix Composition
$n = 2$	3	2
$n = 3$	12	8
$n = 4$	48	33
$n = 5$	192	80

Table 3.9. This is a comparison of the number of multiply-add operations required by motor composition and equivalent matrix composition in n-dimensional projective algebras corresponding to $(n-1)$-dimension Euclidean spaces. The motor-motor product requires $3 \cdot 2^{2n-4}$ multiply-adds, and the matrix-matrix product requires at most $n(n-1)^2$ multiply-adds. Four multiply-adds are subtracted for $n = 3$ because the second column of a 2×2 rotation matrix with first column (x, y) must be $(-y, x)$, and it does not need to be calculated separately. Three multiply-adds are subtracted for $n = 4$ because the third column of a 3×3 rotation matrix can always be calculated as the cross product of the first two columns.

3.7 Flectors

Motors correspond to transformations that can be built from an even number of reflections across planes and thus perform a proper Euclidean isometry. Transformations built from an odd number of reflections across planes perform improper Euclidean isometries because they each have an extra reflection that causes a mirroring in some way. In 3D space, every improper isometry can ultimately be reduced to a single rotation about a specific axis combined with a reflection across a plane perpendicular to that same axis. The operator that performs this general motion, and thus performs any improper isometry in 3D Euclidean space, is called a *flector*. As the name motor is derived from "motion operator", the name flector is a portmanteau of "reflection operator".

3.7.1 Reflection Operator

When we introduced motors, we combined the rotation operator $\mathbf{R}$ from Equation (3.86) with a translation along the direction of the rotation axis. To construct a flector, the translation is replaced with a reflection across a plane $\mathbf{g}$ that's perpendicular to the rotation axis $\boldsymbol{l}$, and this is illustrated in Figure 3.12. A comparison of Figures 3.9 and 3.12 highlights how choosing a translation or reflection aligned to the rotation axis is the only thing that differentiates between proper and improper isometries in 3D space. The operators $\mathbf{R}$ and $\mathbf{g}$ commute when $\mathbf{g}$ is perpendicular to the line $\boldsymbol{l}$, so it doesn't matter in which order they are multiplied to construct a flector $\mathbf{F}$. When we equate $\mathbf{g}_{xyz} = \boldsymbol{l}_{\mathbf{v}}$ and assume that both $\mathbf{R}$ and $\mathbf{g}$ are unitized, the product $\mathbf{R} \vee \mathbf{g}$ yields

$$\mathbf{F} = (\mathbf{g} \vee \boldsymbol{l}) \sin \phi + \mathbf{g} \cos \phi. \tag{3.155}$$

The antiwedge product $\mathbf{g} \vee \boldsymbol{l}$ is the point of intersection where the plane and line meet, so we can express a flector as

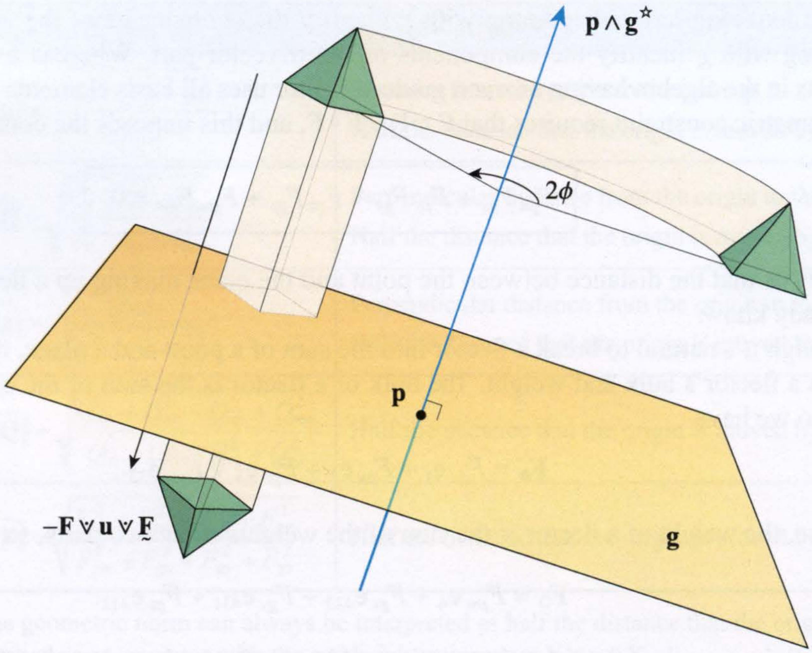


Figure 3.12. A flector represents an improper Euclidean isometry, which can always be regarded as a rotation about a line and a reflection across a plane perpendicular to the same line.

$$\boxed{\mathbf{F} = \mathbf{p} \sin \phi + \mathbf{g} \cos \phi,} \tag{3.156}$$

where $\mathbf{p}$ is a unitized point lying in the plane $\mathbf{g}$. As shown in Figure 3.12, the operation $-\mathbf{F} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{F}}$ reflects the object $\mathbf{u}$ across the plane $\mathbf{g}$ and rotates it through the angle 2ϕ about the line that's perpendicular to $\mathbf{g}$ and passes through $\mathbf{p}$. The line $\boldsymbol{l}$ about which the rotation occurs is the weight expansion of $\mathbf{p}$ onto $\mathbf{g}$, which is given by

$$\boldsymbol{l} = \mathbf{p} \wedge \mathbf{g}^{\star}. \tag{3.157}$$

Negating the angle ϕ has the effect of inverting the flector $\mathbf{F}$ because it reverses the direction of rotation, and the reflection operation is its own inverse. Because the sine function is odd, the vector part of $\mathbf{F}$ is negated when we take an inverse, and this makes sense because it must be the case that

$$\boxed{\mathbf{F}^{-1} = \underset{\sim}{\mathbf{F}}.} \tag{3.158}$$

A reflection across a plane and an inversion across a point are special cases of the general flector $\mathbf{F}$ given by Equation (3.156). If the angle ϕ is zero, then the vector part of $\mathbf{F}$ vanishes, and it is just a reflection across a plane $\mathbf{g}$. If $\phi = \pi/2$, which means the rotation goes through 180 degrees, then the trivector part of $\mathbf{F}$ vanishes, and it is just an inversion across the point $\mathbf{p}$. In both of these cases, no line can be calculated with Equation (3.157) because there is no unique axis of rotation.

A flector $\mathbf{F}$ can be divided into a four-component vector part and a four-component trivector part that we write generically as

Rigid flector (3D)

$$\mathbf{F} = \underbrace{F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{pw}\mathbf{e}_4}_{\text{Point}} + \underbrace{F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412} + F_{gw}\mathbf{e}_{321}}_{\text{Plane}}. \tag{3.159}$$

The two-letter subscripts beginning with p identify the components of the vector part, and those beginning with g identify the components of the trivector part. Whereas a motor uses all basis elements in the algebra having an even grade, a flector uses all basis elements having an odd grade. The geometric constraint requires that $\mathbf{F} \vee \underset{\sim}{\mathbf{F}} = \mathbf{F} \circ \mathbf{F}$, and this imposes the condition

Flector constraint

$$\boxed{F_{px}F_{gx} + F_{py}F_{gy} + F_{pz}F_{gz} + F_{pw}F_{gw} = 0.} \tag{3.160}$$

This tells us that the distance between the point and the plane making up a flector must be zero, as we already know.

Though it's natural to break a flector into the sum of a point and a plane, these parts are not the same as a flector's bulk and weight. The bulk of a flector is the sum of the bulks of the point and plane, so we have

$$\mathbf{F}_\bullet = F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{gw}\mathbf{e}_{321}. \tag{3.161}$$

Likewise, the weight of a flector is the sum of the weights of its two parts, so we have

$$\mathbf{F}_\circ = F_{pw}\mathbf{e}_4 + F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412}. \tag{3.162}$$

If the bulk is zero, then the operation performed by the flector is a rotoreflection in which both the axis of rotation and reflection plane contain the origin.

The bulk norm and weight norm of a flector $\mathbf{F}$ are given by

$$\left\|\mathbf{F}\right\|_\bullet = \mathbf{1}\sqrt{F_{px}^2 + F_{py}^2 + F_{pz}^2 + F_{gw}^2} \tag{3.163}$$

and

$$\left\|\mathbf{F}\right\|_\circ = \mathbb{1}\sqrt{F_{pw}^2 + F_{gx}^2 + F_{gy}^2 + F_{gz}^2}. \tag{3.164}$$

A flector is unitized when its weight norm is $\mathbb{1}$. Assuming that a flector $\mathbf{F}$ is unitized, we can determine the meaning of the bulk norm given by Equation (3.163) by plugging the component values from Equation (3.155) corresponding to F_{px}, F_{py}, F_{pz}, and F_{gw} into it and using the fact that $\mathbf{g}_{xyz} = \boldsymbol{l}_\mathbf{v}$ because the plane $\mathbf{g}$ is perpendicular to the line $\boldsymbol{l}$. The squared bulk norm is then given by

$$\left\|\mathbf{F}\right\|_\bullet^2 = \boldsymbol{l}_\mathbf{m}^2 \sin^2\phi + g_w^2. \tag{3.165}$$

This is nearly identical to Equation (3.104) for motors, except that the translation half-distance δ has been replaced by g_w. We already know from our examination of a motor's geometric norm that $\boldsymbol{l}_\mathbf{m}^2 \sin^2\phi$ is the square of half the distance that the origin is moved by the rotation. The value of g_w is the distance between the origin and the plane $\mathbf{g}$, and that is half the distance that the origin is moved by the reflection. The square root of $\boldsymbol{l}_\mathbf{m}^2 \sin^2\phi + g_w^2$ is therefore the total distance that the origin is moved by any flector, and we see that the geometric norm has the same meaning for flectors as it does for motors. Because lines are also motors and both points and planes are also flectors, the interpretation that the geometric norm corresponds to half the distance that the origin is moved is valid for all objects in projective geometric algebra. This is summarized in Table 3.10 for all five types of objects that arise in the 4D algebra.

Due to their reflective nature, the transformations performed by flectors cannot be smoothly interpolated, and flectors have no exponential form that can be parameterized. However, for any flector that is not just a point, we can factor out the reflection in order to express the flector as a product of a simple motor $\mathbf{R}$ and a plane. Suppose that $\mathbf{F} = \mathbf{p} + \mathbf{g}$ is a unitized flector. Then the weight norm of $\mathbf{p}$ is $\sin\phi$, and the weight norm of $\mathbf{g}$ is $\mathbb{1}\cos\phi$. Multiplying $\mathbf{F}$ by a unit-weight version of the plane $\mathbf{g}$ undoes the reflection, so we calculate

Type	Geometric Norm	Interpretation
Point $\mathbf{p}$	$\widehat{\lVert\mathbf{p}\rVert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point $\mathbf{p}$. Half the distance that the origin is moved by the flector $\mathbf{p}$.
Line $\boldsymbol{l}$	$\widehat{\lVert\boldsymbol{l}\rVert} = \sqrt{\dfrac{l_{mx}^2 + l_{my}^2 + l_{mz}^2}{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$	Perpendicular distance from the origin to the line $\boldsymbol{l}$. Half the distance that the origin is moved by the motor $\boldsymbol{l}$.
Plane $\mathbf{g}$	$\widehat{\lVert\mathbf{g}\rVert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$	Perpendicular distance from the origin to the plane $\mathbf{g}$. Half the distance that the origin is moved by the flector $\mathbf{g}$.
Motor $\mathbf{Q}$	$\widehat{\lVert\mathbf{Q}\rVert} = \sqrt{\dfrac{Q_{mx}^2 + Q_{my}^2 + Q_{mz}^2 + Q_{mw}^2}{Q_{vx}^2 + Q_{vy}^2 + Q_{vz}^2 + Q_{vw}^2}}$	Half the distance that the origin is moved by the motor $\mathbf{Q}$.
Flector $\mathbf{F}$	$\widehat{\lVert\mathbf{F}\rVert} = \sqrt{\dfrac{F_{px}^2 + F_{py}^2 + F_{pz}^2 + F_{gw}^2}{F_{pw}^2 + F_{gx}^2 + F_{gy}^2 + F_{gz}^2}}$	Half the distance that the origin is moved by the flector $\mathbf{F}$.

Table 3.10. The geometric norm can always be interpreted as half the distance that the origin $\mathbf{o}$ is moved by an object $\mathbf{X}$ applied as an operator with the sandwich antiproduct $\mathbf{X} \vee \mathbf{o} \vee \underset{\sim}{\mathbf{X}}$.

$$\mathbf{R} = \mathbf{F} \vee \frac{\mathbf{g}}{\sqrt{1 - p_w^2}}, \tag{3.166}$$

where we are dividing by $\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - p_w^2}$. This expands to

$$\begin{aligned}\mathbf{R} = {} & \frac{1}{\sqrt{1 - p_w^2}} \left[(g_y p_z - g_z p_y)\,\mathbf{e}_{23} + (g_z p_x - g_x p_z)\,\mathbf{e}_{31} + (g_x p_y - g_y p_x)\,\mathbf{e}_{12} \right] \\ & - \frac{p_w}{\sqrt{1 - p_w^2}} (g_x \mathbf{e}_{41} + g_y \mathbf{e}_{42} + g_z \mathbf{e}_{43}) + \mathbb{1}\sqrt{1 - p_w^2},\end{aligned} \tag{3.167}$$

which is a unitized rotation operator. As expected, the six bivector components corresponding to the axis of rotation are given by the weight expansion of $\mathbf{p}$ onto $\mathbf{g}$, multiplied by $\sin\phi$. The original flector $\mathbf{F}$ can now be expressed as

$$\mathbf{F} = \mathbf{R} \vee \frac{\mathbf{g}}{\sqrt{1 - p_w^2}}, \tag{3.168}$$

which combines the rotation and reflection.

3.7.2 Matrix Conversion

As we did with motors, we can construct two matrices for any flector $\mathbf{F}$ in such a way that their sum corresponds to the transformation $-\mathbf{F} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{F}}$ and their difference corresponds to the inverse transformation $-\underset{\sim}{\mathbf{F}} \vee \mathbf{u} \vee \mathbf{F}$. Given a specific unitized flector $\mathbf{F}$, we define the matrices $\mathbf{A}_\mathbf{F}$ and $\mathbf{B}_\mathbf{F}$ as

$$\mathbf{A}_\mathbf{F} = \begin{bmatrix} 2(F_{gy}^2 + F_{gz}^2) - 1 & -2F_{gx}F_{gy} & -2F_{gz}F_{gx} & 2(F_{px}F_{pw} - F_{gx}F_{gw}) \\ -2F_{gx}F_{gy} & 2(F_{gz}^2 + F_{gx}^2) - 1 & -2F_{gy}F_{gz} & 2(F_{py}F_{pw} - F_{gy}F_{gw}) \\ -2F_{gz}F_{gx} & -2F_{gy}F_{gz} & 2(F_{gx}^2 + F_{gy}^2) - 1 & 2(F_{pz}F_{pw} - F_{gz}F_{gw}) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.169}$$

and

$$\mathbf{B_F} = \begin{bmatrix} 0 & 2F_{gz}F_{pw} & -2F_{gy}F_{pw} & 2\left(F_{gy}F_{pz} - F_{gz}F_{py}\right) \\ -2F_{gz}F_{pw} & 0 & 2F_{gx}F_{pw} & 2\left(F_{gz}F_{px} - F_{gx}F_{pz}\right) \\ 2F_{gy}F_{pw} & -2F_{gx}F_{pw} & 0 & 2\left(F_{gx}F_{py} - F_{gy}F_{px}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.170}$$

where we have used the fact that $F_{gx}^2 + F_{gy}^2 + F_{gz}^2 + F_{pw}^2 = 1$ in the diagonal entries of $\mathbf{A_F}$. Then the corresponding 4×4 matrix $\mathbf{M_F}$ that transforms a point $\mathbf{p}$, regarded as a 4×1 column matrix, in the same way that it would be transformed by the flector $\mathbf{F}$ is given by

$$\mathbf{M_F} = \mathbf{A_F} + \mathbf{B_F}. \tag{3.171}$$

That is, $\mathbf{M_F p} = -\mathbf{F} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{F}}$. The inverse of $\mathbf{M_F}$ is given by

$$\mathbf{M_F^{-1}} = \mathbf{A_F} - \mathbf{B_F}, \tag{3.172}$$

and this is related to $\mathbf{F}$ by the equation $\mathbf{M_F^{-1} p} = -\underset{\sim}{\mathbf{F}} \vee \mathbf{p} \vee \mathbf{F}$.

To go the other way and convert a 4×4 matrix $\mathbf{M}$ into a flector $\mathbf{F}$, we start by assuming that $\mathbf{M}$ truly represents a rotoreflection, which means that it has the form

$$\mathbf{M} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.173}$$

where the upper-left 3×3 is orthogonal, and the determinant is -1. Proceeding as we did for motors, we equate the entries of $\mathbf{M}$ to the entries of $\mathbf{A_F} + \mathbf{B_F}$ given by Equations (3.169) and (3.170). Then, for the diagonal entries of $\mathbf{M}$, we have the four relationships

$$\begin{aligned} 1 - M_{00} + M_{11} + M_{22} &= 4F_{gx}^2 \\ 1 - M_{11} + M_{22} + M_{00} &= 4F_{gy}^2 \\ 1 - M_{22} + M_{00} + M_{11} &= 4F_{gz}^2 \\ 1 - M_{00} - M_{11} - M_{22} &= 4\left(1 - F_{gx}^2 - F_{gy}^2 - F_{gz}^2\right) = 4F_{pw}^2, \end{aligned} \tag{3.174}$$

And for the off-diagonal entries of $\mathbf{M}$, we have the six additional relationships

$$\begin{aligned} M_{21} + M_{12} &= -4F_{gy}F_{gz} \\ M_{02} + M_{20} &= -4F_{gz}F_{gx} \\ M_{10} + M_{01} &= -4F_{gx}F_{gy} \\ M_{21} - M_{12} &= -4F_{gx}F_{pw} \\ M_{02} - M_{20} &= -4F_{gy}F_{pw} \\ M_{10} - M_{01} &= -4F_{gz}F_{pw}. \end{aligned} \tag{3.175}$$

If $M_{00} + M_{11} + M_{22} \le 0$, then we calculate

$$F_{pw} = \pm\frac{1}{2}\sqrt{1 - M_{00} - M_{11} - M_{22}}, \tag{3.176}$$

where either sign can be chosen. In this case, we know $|F_{pw}|$ is at least 1/2, so we can safely divide by $-4F_{pw}$ in the last three off-diagonal relationships shown in Equation (3.175) to solve for F_{gx}, F_{gy}, and F_{gz}. Otherwise, if $M_{00} + M_{11} + M_{22} > 0$, then we select one of the first three diagonal relationships in Equation (3.174) based on which diagonal entry M_{00}, M_{11}, or M_{22} has the least value and calculate F_{gx}, F_{gy}, or F_{gz}. The result is plugged into two of the first three off-diagonal relationships to solve for the other two values of F_{gx}, F_{gy}, and F_{gz}. Finally, we plug it into one of the last three off-diagonal relationships to solve for F_{pw}.

After the four components F_{gx}, F_{gy}, F_{gz}, and F_{pw} have been calculated, the values of the four remaining components F_{px}, F_{py}, F_{pz}, and F_{gw} are given by

$$\begin{aligned}
F_{px} &= \tfrac{1}{2}(F_{pw}t_x + F_{gz}t_y - F_{gy}t_z) \\
F_{py} &= \tfrac{1}{2}(F_{pw}t_y + F_{gx}t_z - F_{gz}t_x) \\
F_{pz} &= \tfrac{1}{2}(F_{pw}t_z + F_{gy}t_x - F_{gx}t_y) \\
F_{gw} &= -\tfrac{1}{2}(F_{gx}t_x + F_{gy}t_y + F_{gz}t_z),
\end{aligned} \tag{3.177}$$

where $t_x = M_{03}$, $t_y = M_{13}$, and $t_z = M_{23}$ are the entries of $\mathbf{M}$ corresponding to the translation.

3.7.3 Implementation

The storage and performance characteristics of a flector are identical to those of a motor. As with motors, only six components of any unitized flector need to be stored, and the remaining two components can be reconstituted on demand. Before storage, a flector $\mathbf{F}$ can be negated if necessary so that $F_{pw} \geq 0$, and this component can later be calculated with

$$F_{pw} = \sqrt{1 - F_{gx}^2 - F_{gy}^2 - F_{gz}^2}. \tag{3.178}$$

The geometric constraint can also be exploited to avoid storing the F_{gw} component because it must be given by

$$F_{gw} = -\frac{F_{px}F_{gx} + F_{py}F_{gy} + F_{pz}F_{gz}}{F_{pw}}. \tag{3.179}$$

Thus, only the x, y, and z coordinates of the point and plane making up any flector are absolutely necessary in situations where minimal storage size is important.

Optimal formulas for transforming points, lines, and planes with a flector can be derived in the same manner that they were for motors in Section 3.6.5. First, when a point $\mathbf{q}$ is transformed into the point $\mathbf{q}'$ by the flector $\mathbf{F}$ using the sandwich antiproduct $-\mathbf{F} \vee \mathbf{q} \vee \underset{\sim}{\mathbf{F}}$, the point's w coordinate is negated, and the transformation of its x, y, and z coordinates can be written as

$$\mathbf{q}'_{xyz} = \mathbf{q}_{xyz} + 2\left[F_{pw}(\mathbf{g} \times \mathbf{q}_{xyz}) + (\mathbf{g} \cdot \mathbf{q}_{xyz})\mathbf{g} - \mathbf{g}^2\mathbf{q}_{xyz} + q_w(\mathbf{p} \times \mathbf{g} + F_{gw}\mathbf{g} - F_{pw}\mathbf{p})\right], \tag{3.180}$$

where we have assigned $\mathbf{p} = (F_{px}, F_{py}, F_{pz})$ and $\mathbf{g} = (F_{gx}, F_{gy}, F_{gz})$ for convenience. Applying the vector triple product identity

$$(\mathbf{g} \cdot \mathbf{q}_{xyz})\mathbf{g} - \mathbf{g}^2\mathbf{q}_{xyz} = \mathbf{g} \times (\mathbf{g} \times \mathbf{q}_{xyz}), \tag{3.181}$$

we can rewrite this as

$$\mathbf{q}'_{xyz} = \mathbf{q}_{xyz} + 2[F_{pw}(\mathbf{g} \times \mathbf{q}_{xyz}) + \mathbf{g} \times (\mathbf{g} \times \mathbf{q}_{xyz}) + q_w(\mathbf{p} \times \mathbf{g} + F_{gw}\mathbf{g} - F_{pw}\mathbf{p})]. \tag{3.182}$$

The cross product $\mathbf{g} \times \mathbf{q}_{xyz}$ shows up twice, and this redundant subcalculation can be made larger by refactoring to get

$$\mathbf{q}'_{xyz} = \mathbf{q}_{xyz} + 2\left[F_{pw}\left(\mathbf{g} \times \mathbf{q}_{xyz} - q_w \mathbf{p}\right) + \mathbf{g} \times \left(\mathbf{g} \times \mathbf{q}_{xyz} - q_w \mathbf{p}\right) + F_{gw} q_w \mathbf{g}\right]. \tag{3.183}$$

We now set $\mathbf{a} = \mathbf{g} \times \mathbf{q}_{xyz} - q_w \mathbf{p}$, and the final transformation formula is then

Flector-point transformation

$$\boxed{\begin{aligned} \mathbf{q}'_{xyz} &= \mathbf{q}_{xyz} + 2\left(F_{pw}\mathbf{a} + \mathbf{g} \times \mathbf{a} + F_{gw} q_w \mathbf{g}\right) \\ q_w &= -q_w. \end{aligned}} \tag{3.184}$$

It is usually undesirable for the w coordinate of a point to be negated, so a typical implementation would leave it alone and calculate the negative value of $\mathbf{q}'_{xyz}$ to produce an equivalent point.

For the transformation of a line $\mathbf{k}$ into $\mathbf{k}' = -\mathbf{F} \vee \mathbf{k} \vee \underset{\sim}{\mathbf{F}}$, we derive formulas for the direction $\mathbf{k}'_\mathbf{v}$ and moment $\mathbf{k}'_\mathbf{m}$ separately. A direct calculation of $\mathbf{k}'_\mathbf{v}$ is given by

$$\mathbf{k}'_\mathbf{v} = 2\left[\mathbf{g}^2 \mathbf{k}_\mathbf{v} - \left(\mathbf{k}_\mathbf{v} \cdot \mathbf{g}\right)\mathbf{g} - F_{pw}\left(\mathbf{g} \times \mathbf{k}_\mathbf{v}\right)\right] - \mathbf{k}_\mathbf{v}, \tag{3.185}$$

We use the identity $\mathbf{g}^2 \mathbf{k}_\mathbf{v} - \left(\mathbf{k}_\mathbf{v} \cdot \mathbf{g}\right)\mathbf{g} = \left(\mathbf{g} \times \mathbf{k}_\mathbf{v}\right) \times \mathbf{g}$ to rewrite this as

Flector-line transformation (direction)

$$\boxed{\mathbf{k}'_\mathbf{v} = 2\left(\mathbf{a} \times \mathbf{g} - F_{pw}\mathbf{a}\right) - \mathbf{k}_\mathbf{v},} \tag{3.186}$$

where $\mathbf{a} = \mathbf{g} \times \mathbf{k}_\mathbf{v}$. A direct calculation of the moment $\mathbf{k}'_\mathbf{m}$ is given by

$$\begin{aligned} \mathbf{k}'_\mathbf{m} = 2\Big[& F_{gw}\left(\mathbf{g} \times \mathbf{k}_\mathbf{v}\right) + F_{pw}\left(\mathbf{p} \times \mathbf{k}_\mathbf{v} - \mathbf{g} \times \mathbf{k}_\mathbf{m}\right) + \mathbf{g}^2 \mathbf{k}_\mathbf{m} - \left(\mathbf{g} \cdot \mathbf{k}_\mathbf{m}\right)\mathbf{g} \\ & + \left(\mathbf{p} \cdot \mathbf{k}_\mathbf{v}\right)\mathbf{g} + \left(\mathbf{g} \cdot \mathbf{k}_\mathbf{v}\right)\mathbf{p} - 2\left(\mathbf{g} \cdot \mathbf{p}\right)\mathbf{k}_\mathbf{v}\Big] - \mathbf{k}_\mathbf{m}. \end{aligned} \tag{3.187}$$

As with motors, we apply the vector triple product identity three times and make the substitutions

$$\begin{aligned} \mathbf{g}^2 \mathbf{k}_\mathbf{m} - \left(\mathbf{g} \cdot \mathbf{k}_\mathbf{m}\right)\mathbf{g} &= \left(\mathbf{g} \times \mathbf{k}_\mathbf{m}\right) \times \mathbf{g} \\ \left(\mathbf{p} \cdot \mathbf{k}_\mathbf{v}\right)\mathbf{g} - \left(\mathbf{g} \cdot \mathbf{p}\right)\mathbf{k}_\mathbf{v} &= \mathbf{p} \times \left(\mathbf{g} \times \mathbf{k}_\mathbf{v}\right) \\ \left(\mathbf{g} \cdot \mathbf{k}_\mathbf{v}\right)\mathbf{p} - \left(\mathbf{g} \cdot \mathbf{p}\right)\mathbf{k}_\mathbf{v} &= \mathbf{g} \times \left(\mathbf{p} \times \mathbf{k}_\mathbf{v}\right). \end{aligned} \tag{3.188}$$

This lets us reuse the cross products $\mathbf{a} = \mathbf{g} \times \mathbf{k}_\mathbf{v}$, $\mathbf{b} = \mathbf{g} \times \mathbf{k}_\mathbf{m}$, and $\mathbf{c} = \mathbf{p} \times \mathbf{k}_\mathbf{v}$ so that we can write the transformed moment as

Flector-line transformation (moment)

$$\boxed{\mathbf{k}'_\mathbf{m} = 2\left[F_{gw}\mathbf{a} + F_{pw}\left(\mathbf{c} - \mathbf{b}\right) + \mathbf{g} \times \left(\mathbf{c} - \mathbf{b}\right) + \mathbf{p} \times \mathbf{a}\right] - \mathbf{k}_\mathbf{m}.} \tag{3.189}$$

Finally, for the transformation of a plane $\mathbf{h}$ into $\mathbf{h}' = -\mathbf{F} \vee \mathbf{h} \vee \underset{\sim}{\mathbf{F}}$, we derive formulas for the normal $\mathbf{h}'_{xyz}$ and position h'_w separately. A direct calculation of $\mathbf{h}'_{xyz}$ gives us

$$\mathbf{h}'_{xyz} = 2\left[\mathbf{g}^2 \mathbf{h}_{xyz} - \left(\mathbf{g} \cdot \mathbf{h}_{xyz}\right)\mathbf{g} - F_{pw}\left(\mathbf{g} \times \mathbf{h}_{xyz}\right)\right] - \mathbf{h}_{xyz}. \tag{3.190}$$

Using the identity $\mathbf{g}^2 \mathbf{h}_{xyz} - \left(\mathbf{g} \cdot \mathbf{h}_{xyz}\right)\mathbf{g} = \left(\mathbf{g} \times \mathbf{h}_{xyz}\right) \times \mathbf{g}$ lets us rewrite this as

Flector-plane transformation (normal)

$$\boxed{\mathbf{h}'_{xyz} = 2\left(\mathbf{a} \times \mathbf{g} - F_{pw}\mathbf{a}\right) - \mathbf{h}_{xyz},} \tag{3.191}$$

where $\mathbf{a} = \mathbf{g} \times \mathbf{h}_{xyz}$. No special simplifications are applied to a direct calculation of the w coordinate. After grouping terms, it is given by

Flector-plane transformation (position)

$$\boxed{h'_w = h_w + 2\left[\left(\mathbf{h}_{xyz} \times \mathbf{p} - F_{gw}\mathbf{h}_{xyz}\right) \cdot \mathbf{g} + F_{pw}\left(\mathbf{p} \cdot \mathbf{h}_{xyz}\right)\right].} \tag{3.192}$$

The numbers of multiply-add operations required to calculate points, lines, and planes transformed by a flector are the same as those required to transform by a motor. This again means that the same transformations can be performed with matrix multiplication using significantly fewer operations. If many objects are to be transformed by a flector, then it is much better for performance to convert the flector to a matrix first and use matrix multiplication.

Math Library Notes

- The `Flector3D` class stores the eight components of a 3D reflection operator as a `FlatPoint3D` object `p` and `Plane3D` object `g`.
- The * operator can multiply flectors by other flectors with the geometric antiproduct to produce a motor. Multiplication by a motor or quaternion produces another flector.
- The `Transform()` function takes a `FlatPoint3D`, `Point3D`, `Line3D`, or `Plane3D` object as its first parameter and a `Flector3D` object as its second parameter. It performs the flector transformation with the optimized formula and returns a result having the same type as the first parameter. In the case of a point $\mathbf{q}$ using either of the types `FlatPoint3D` or `Point3D`, the `Transform()` function calculates $\mathbf{F} \vee \mathbf{q} \vee \underset{\sim}{\mathbf{F}}$ without the minus sign in front avoid negating the w coordinate.
- The `Flector3D` class has a `GetTransformMatrix()` member function that calculates a 4×4 matrix with Equation (3.171) and returns it as a `Transform3D` object.
- The `Flector3D` class has a `SetTransformMatrix()` member function that accepts a `Transform3D` object as its parameter. It converts the matrix to a flector using the method described in this section. The code assumes that the upper-left 3×3 portion of the input is orthogonal and has a determinant of -1.

3.8 2D Rigid Transformations

In order to study Euclidean isometries in 2D space, we apply the geometric product and antiproduct to the 3D projective exterior algebra developed in Section 2.14. The geometric product on the same eight basis elements must respect the same metric in which $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = 1$, and $\mathbf{e}_3^2 = 0$, and the geometric antiproduct must obey the usual De Morgan law. Applying these requirements as we did in the 4D case at the beginning of this chapter leads to the multiplication tables shown in Table 3.11 for the 3D projective geometric algebra.

The reverse and antireverse operations continue to have the same meaning, and their effects on the eight basis elements are shown in Table 3.12. In the 3D projective algebra, it just happens to work out that the reverse and antireverse of any value are negatives of each other.

In any n-dimensional projective algebra representing $(n-1)$-dimensional Euclidean space, the set of multivectors with components having only even antigrade are motion operators, and the set of multivectors with components having only odd antigrade are reflection operators. The fixed geometry about which a rotation occurs is always represented by a quantity having an antigrade of two. In 3D space where the projective representation has four dimensions, the fixed geometry is a line that serves as the axis of rotation. In 2D space where the projective representation has three dimensions, the fixed geometry is a point that serves as the center of rotation. The fixed geometry in a reflection is always an antivector. In 3D space, we reflect across planes, and in 2D space, we reflect across lines.

Every proper isometry in 2D is a rotation about some point $\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$, and every motor $\mathbf{Q}$ has the form

$$\boxed{\mathbf{Q} = \mathbf{p} \sin \phi + \mathbb{1} \cos \phi.} \tag{3.193}$$

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$								
a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbb{1}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$-\mathbb{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	0	0	$-\mathbb{1}$	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbb{1}$	$-\mathbf{e}_3$	0	0	0	$\mathbf{e}_{31}$	0
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{23}$	0
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_3$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	0	0	0	$\mathbf{e}_3$	0

Geometric Antiproduct $\mathbf{a} ⟇ \mathbf{b}$								
a \ b	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\mathbf{1}$	0	0	0	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	0	$\mathbf{1}$
$\mathbf{e}_1$	0	0	0	$-\mathbf{e}_2$	$-\mathbf{1}$	$\mathbf{e}_{12}$	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	$\mathbf{e}_1$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	0	$\mathbf{e}_2$
$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbb{1}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_3$
$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbb{1}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	$-\mathbf{e}_2$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$\mathbb{1}$	$-\mathbf{e}_1$	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	$-\mathbf{1}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	$\mathbf{e}_{12}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$

Table 3.11. These are the multiplication tables for the geometric product and geometric antiproduct in the 3D projective algebra representing 2D Euclidean space.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$
$\underset{\sim}{\mathbf{u}}$	$-\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbb{1}$

Table 3.12. For each of the 8 basis elements $\mathbf{u}$ in the 3D projective exterior algebra, this table lists the reverse $\tilde{\mathbf{u}}$ and the antireverse $\underset{\sim}{\mathbf{u}}$.

The orientation $\mathbb{1} = \mathbf{e}_{321}$ was originally chosen for the 3D volume element in Section 2.14 so that this motor would be expressed as the sum of two terms containing a sine and cosine instead of the difference. When applied using the sandwich antiproduct $\mathbf{Q} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{Q}}$, the motor $\mathbf{Q}$ rotates the object $\mathbf{u}$ through the angle 2ϕ about the center given by the point $\mathbf{p}$. A 2D motor has four components with even antigrade, and it can be written generically as

Rigid motor (2D)

$$\mathbf{Q} = \underbrace{Q_x \mathbf{e}_1 + Q_y \mathbf{e}_2 + Q_z \mathbf{e}_3}_{\text{Fixed Point}} + \underbrace{Q_w \mathbb{1}}_{\text{Rotation}}, \tag{3.194}$$

which is the sum of a vector part and an antiscalar part. The absolute value of the angle of rotation can always be determined by looking at the antiscalar value. It is always true that $\mathbf{Q} ⟇ \underset{\sim}{\mathbf{Q}} = \mathbf{Q} \circ \mathbf{Q}$, so there is no internal constraint imposed on motors in two dimensions.

A 2D motor can be expressed in the exponential form

$$\mathbf{Q} = \exp_{\vee}(\phi \mathbf{p}), \tag{3.195}$$

where $\mathbf{p}$ is unitized so that $p_z = 1$. Extracting the parameters $\mathbf{p}$ and ϕ from the components of a motor $\mathbf{Q}$ is rather easy. We can just read off the center from the vector part and unitize it. Assuming the motor is unitized, the angle ϕ is given by

$$\phi = \tan^{-1} \frac{Q_z}{Q_w}. \tag{3.196}$$

We take the inverse tangent here because neither the inverse sine of Q_z nor the inverse cosine of Q_w is adequate. The inverse sine of Q_z by itself would erroneously give different angles for the motors $\mathbf{Q}$ and $-\mathbf{Q}$, but those are equivalent operators. The inverse cosine of Q_w by itself cannot distinguish between positive and negative angles.

The bulk and weight of a 2D motor $\mathbf{Q}$ are given by

$$\mathbf{Q}_\bullet = Q_x\mathbf{e}_1 + Q_y\mathbf{e}_2 \quad \text{and} \quad \mathbf{Q}_\circ = Q_z\mathbf{e}_3 + Q_w\mathbb{1}. \tag{3.197}$$

If the bulk is zero, then $\mathbf{Q}$ *is either the identity or a rotation about the origin* $\mathbf{e}_3$. The set of all 2D motors having a bulk of zero is equivalent to the set of complex numbers when we set $i = \mathbf{e}_3$. Motors of the form $a\mathbb{1} + b\mathbf{e}_3$ multiply under the geometric antiproduct just as complex numbers $a + bi$ do. When two such motors are multiplied together, their rotation angles accumulate, just as they do for the equivalent complex values.

If $Q_z = 0$, then the center of rotation is a point at infinity and the motor $\mathbf{Q}$ represents a translation operator $\mathbf{T}$ that we can write as

$$\mathbf{T} = -\tau_y\mathbf{e}_1 + \tau_x\mathbf{e}_2 + \mathbb{1}. \tag{3.198}$$

This operator translates an object $\mathbf{u}$ by twice the vector $\boldsymbol{\tau}$ with the sandwich antiproduct $\mathbf{T} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{T}}$. Notice that the translation vector does not appear directly in the $\mathbf{e}_1$ and $\mathbf{e}_2$ components of the operator, but is instead rotated 90 degrees. This happens because the vector $\boldsymbol{\tau}$ points toward the center of rotation in the horizon, and the direction of the motion going around that center must be perpendicular to the direction that points at it.

The 3×3 matrix $\mathbf{M}_\mathbf{Q}$ that is equivalent to a motor $\mathbf{Q}$ is given by

$$\mathbf{M}_\mathbf{Q} = \begin{bmatrix} 1 - 2Q_z^2 & -2Q_zQ_w & 2(Q_xQ_z + Q_yQ_w) \\ 2Q_zQ_w & 1 - 2Q_z^2 & 2(Q_yQ_z - Q_xQ_w) \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.199}$$

and its inverse is

$$\mathbf{M}_\mathbf{Q}^{-1} = \begin{bmatrix} 1 - 2Q_z^2 & 2Q_zQ_w & 2(Q_xQ_z - Q_yQ_w) \\ -2Q_zQ_w & 1 - 2Q_z^2 & 2(Q_yQ_z + Q_xQ_w) \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.200}$$

When applied to a point $\mathbf{p}$, these perform the transformations

$$\mathbf{M}_\mathbf{Q}\mathbf{p} = \mathbf{Q} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{Q}} \quad \text{and} \quad \mathbf{M}_\mathbf{Q}^{-1}\mathbf{p} = \underset{\sim}{\mathbf{Q}} \vee \mathbf{p} \vee \mathbf{Q}. \tag{3.201}$$

Every improper isometry in 2D is a transflection with respect to some line $\mathbf{g} = g_x\mathbf{e}_{23} + g_y\mathbf{e}_{31} + g_z\mathbf{e}_{12}$, and every flector $\mathbf{F}$ has the form

$$\boxed{\mathbf{F} = \mathbf{g} + \delta\mathbf{1}.} \tag{3.202}$$

When applied using the sandwich antiproduct $-\mathbf{F} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{F}}$, this flector reflects the object $\mathbf{u}$ across the line $\mathbf{g}$ and translates by the distance 2δ in the direction $(-g_y, g_x)$. The translation direction is parallel to the line and is equal to a 90-degree counterclockwise rotation of the line's normal vector. A 2D flector has four components with odd antigrade, and it can be written generically as

Rigid flector (2D)

$$\mathbf{F} = \underbrace{F_x\mathbf{e}_{23} + F_y\mathbf{e}_{31} + F_z\mathbf{e}_{12}}_{\text{Fixed Line}} + \underbrace{F_w\mathbf{1}}_{\text{Translation}}, \tag{3.203}$$

which is the sum of a bivector part and a scalar part. If the scalar part is zero, then the flector represents a reflection with no accompanying translation. In two dimensions, inversion through a point is not represented by a flector because the operation is equivalent to a 180-degree rotation, which is represented by a motor with zero antiscalar part.

The 3×3 matrix $\mathbf{M_F}$ that is equivalent to a flector $\mathbf{F}$ is given by

$$\mathbf{M_F} = \begin{bmatrix} 1-2F_x^2 & -2F_xF_y & -2\left(F_xF_z + F_yF_w\right) \\ -2F_xF_y & 1-2F_y^2 & -2\left(F_yF_z - F_xF_w\right) \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.204}$$

and its inverse is

$$\mathbf{M_Q^{-1}} = \begin{bmatrix} 1-2F_x^2 & -2F_xF_y & -2\left(F_xF_z - F_yF_w\right) \\ -2F_xF_y & 1-2F_y^2 & -2\left(F_yF_z + F_xF_w\right) \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.205}$$

When applied to a point $\mathbf{p}$, these perform the transformations

$$\mathbf{M_F p} = -\mathbf{F} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{F}} \quad \text{and} \quad \mathbf{M_F^{-1} p} = -\underset{\sim}{\mathbf{F}} \vee \mathbf{p} \vee \mathbf{F}. \tag{3.206}$$

The bulk norm and weight norm of a 2D motor $\mathbf{Q}$ are given by

$$\left\|\mathbf{Q}\right\|_\bullet = \mathbb{1}\sqrt{Q_x^2 + Q_y^2} \quad \text{and} \quad \left\|\mathbf{Q}\right\|_\circ = \mathbb{1}\sqrt{Q_z^2 + Q_w^2}. \tag{3.207}$$

For a 2D flector $\mathbf{F}$, the bulk norm and weight norm are

$$\left\|\mathbf{F}\right\|_\bullet = \mathbb{1}\sqrt{F_z^2 + F_w^2} \quad \text{and} \quad \left\|\mathbf{F}\right\|_\circ = \mathbb{1}\sqrt{F_x^2 + F_y^2}. \tag{3.208}$$

The geometric norm has the same interpretations in two dimensions as it does in three dimensions. In addition to giving the distance to the origin for points and lines, the geometric norm is always equal to half the distance that the origin is moved by any operator. This is summarized in Table 3.13 for all four types of objects that arise in the 3D projective geometric algebra.

Type	**Geometric Norm**	**Interpretation**
Point $\mathbf{p}$	$\widehat{\left\|\mathbf{p}\right\|} = \dfrac{\sqrt{p_x^2 + p_y^2}}{\lvert p_z \rvert}$	Distance from the origin to the point $\mathbf{p}$. Half the distance that the origin is moved by the motor $\mathbf{p}$.
Line $\mathbf{g}$	$\widehat{\left\|\mathbf{g}\right\|} = \dfrac{\lvert g_z \rvert}{\sqrt{g_x^2 + g_y^2}}$	Perpendicular distance from the origin to the line $\mathbf{g}$. Half the distance that the origin is moved by the flector $\mathbf{g}$.
Motor $\mathbf{Q}$	$\widehat{\left\|\mathbf{Q}\right\|} = \sqrt{\dfrac{Q_x^2 + Q_y^2}{Q_z^2 + Q_w^2}}$	Half the distance that the origin is moved by the motor $\mathbf{Q}$.
Flector $\mathbf{F}$	$\widehat{\left\|\mathbf{F}\right\|} = \sqrt{\dfrac{F_z^2 + F_w^2}{F_x^2 + F_y^2}}$	Half the distance that the origin is moved by the flector $\mathbf{F}$.

Table 3.13. The geometric norm in two dimensions can always be interpreted as half the distance that the origin $\mathbf{o}$ is moved by an object $\mathbf{X}$ applied as an operator with the sandwich antiproduct $\mathbf{X} \vee \mathbf{o} \vee \underset{\sim}{\mathbf{X}}$.

Math Library Notes

- The `Motor2D` class stores the four components of a 2D motion operator as floating-point values named `x`, `y`, `z`, and `w`.
- The `*` operator can multiply motors by other motors with the geometric antiproduct.
- The `Transform()` function takes a `FlatPoint2D`, `Point2D`, or `Line2D` object as its first parameter and a `Motor2D` object as its second parameter. It performs the motor transformation and returns a result having the same type as the first parameter.
- The `Motor2D` class has a `GetTransformMatrix()` member function that calculates a 3×3 matrix with Equation (3.199) and returns it as a `Transform2D` object.
- The `Motor2D` class has a `SetTransformMatrix()` member function that accepts a `Transform2D` object as its parameter. It converts the matrix to a motor under the assumption that the upper-left 2×2 portion of the input is a true rotation matrix, meaning that it is orthogonal and has a determinant of $+1$.
- The `Flector2D` class stores the four components of a 2D reflection operator as floating-point values named `x`, `y`, `z`, and `w`.
- The `*` operator can multiply flectors by other flectors with the geometric antiproduct to produce a motor. Multiplication by a motor produces another flector.
- The `Transform()` function takes a `FlatPoint2D`, `Point2D`, or `Line2D` object as its first parameter and a `Flector2D` object as its second parameter. It performs the flector transformation and returns a result having the same type as the first parameter. In the case of a point $\mathbf{q}$ using either of the types `FlatPoint2D` or `Point2D`, the `Transform()` function calculates $\mathbf{F} \vee \mathbf{q} \vee \underset{\sim}{\mathbf{F}}$ without the minus sign in front avoid negating the z coordinate.
- The `Flector3D` class has a `GetTransformMatrix()` member function that calculates a 3×3 matrix with Equation (3.204) and returns it as a `Transform2D` object.
- The `Flector3D` class has a `SetTransformMatrix()` member function that accepts a `Transform2D` object as its parameter. It converts the matrix to a flector under the assumption that the upper-left 2×2 portion of the input is orthogonal and has a determinant of -1.

3.9 Operator Duality

All of the Euclidean isometries discussed in this chapter have been performed by sandwiching an object $\mathbf{u}$ inside an operator $\mathbf{X}$ and multiplying with the geometric antiproduct. We had decided in Section 3.5 that it had to be the antiproduct $\vee$ that performed these rigid transformations, and not the product $\wedge$, because the antiproduct keeps the horizon fixed while allowing all points not in the horizon to move around. We also came to the conclusion that the product $\wedge$ does not perform rigid transformations because it always fixes the origin and is able to move the horizon to locations not infinitely far away.

As previously discussed in Section 2.6, every object in projective geometric algebra is really a representation of two complementary things at the same time because it has one interpretation in space and another interpretation in antispace. Every operation in projective geometric algebra really performs two complementary actions at the same time, where one happens in space and the other happens in antispace. These concepts emerge from the intrinsic duality in the algebra, and the case is no different when it comes to the geometric product and geometric antiproduct. When we transform an object with the antiproduct $\vee$, it performs a Euclidean isometry in the space of the object. *Simultaneously*, it performs a *complement* isometry in the *antispace* of the same object. If we were to replace the geometric antiproduct with the geometric product, then the operations performed in space and antispace trade places. The product $\wedge$ performs a complement isometry in space while simultaneously performing a Euclidean isometry in antispace.

3.9.1 Complement Isometries

So what exactly is a complement isometry? All we know at this point is that they have to keep the origin fixed, but we can discover much more by examining all the invariants of Euclidean transformations and their complements more closely. An *invariant* is any geometry that is fixed in place by a transformation, and knowing what that geometry is tells us a lot about the nature of the transformation itself. The Euclidean isometries performed by motors and flectors all have invariant geometries that are part of the operator's algebraic representation. In three dimensions, the invariant of a reflection is the fixed plane across which everything else is mirrored, the invariant of a rotation is the fixed line about which everything else orbits, and the invariant of an inversion is the fixed point through which everything else is turned inside out.

The important thing to understand about complement isometries is that their invariants are the complements of the invariants of their associated Euclidean isometries. For example, when a reflection fixes its mirroring plane $\mathbf{g}$ in regular space, the complement isometry happening at the same time in antispace must fix the the complement of $\mathbf{g}$, which is a point. Likewise, when a rotation fixes its axis $\boldsymbol{l}$ in regular space, its complement isometry must fix the complement of $\boldsymbol{l}$, which is a different line. This tells us part of the story, but the mirroring plane of a reflection and the axis of a rotation are not the only invariants fixed by those transformations.

We know that all motors and flectors fix the horizon as a whole, but they also fix specific lower-dimensional geometries that lie in the horizon. The complements of these fixed geometries must contain the origin, so they show up very prominently as invariants of the complement isometries. In general, we can demonstrate that for any geometry $\mathbf{u}$ that is fixed by a Euclidean isometry, the weight dual $\mathbf{u}^{☆}$ is also fixed by the same isometry. When we say that $\mathbf{u}$ is fixed by an operator $\mathbf{X}$ under the geometric antiproduct, we mean that

$$\mathbf{X} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{X}} = s\mathbf{u} \tag{3.209}$$

for some scalar value s that could be positive or negative. The algebraic representation of $\mathbf{u}$ may be scaled, but all nonzero multiples are equivalent geometries in the homogeneous model. Assuming that Equation (3.209) is true for some operator $\mathbf{X}$ and some geometry $\mathbf{u}$, we now consider the expression $\mathbf{X} \vee \mathbf{u}^{☆} \vee \underset{\sim}{\mathbf{X}}$. As stated by Equation (3.63), the weight dual $\mathbf{u}^{☆}$ is always equal to $\underset{\sim}{\mathbf{u}} \vee \mathbf{1}$, so we can write

$$\mathbf{X} \vee \mathbf{u}^{☆} \vee \underset{\sim}{\mathbf{X}} = \mathbf{X} \vee (\underset{\sim}{\mathbf{u}} \vee \mathbf{1}) \vee \underset{\sim}{\mathbf{X}}, \tag{3.210}$$

and this lets us reassociate the products. Since $\mathbf{X}$ has components that have either all even grades or all odd grades, the effect of swapping the factors $\mathbf{1}$ and $\underset{\sim}{\mathbf{X}}$ at the end of the product is at most a sign change. Also, since $\mathbf{u}$ represents a geometry, its components all have the same grade, so its antireverse is either $\mathbf{u}$ itself or its negation. In all cases, we have

$$\mathbf{X} \vee \mathbf{u}^{☆} \vee \underset{\sim}{\mathbf{X}} = \pm \mathbf{X} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{X}} \vee \mathbf{1}, \tag{3.211}$$

where the sign isn't known without specifying $\mathbf{X}$ and $\mathbf{u}$, but we don't care. The original transformation of $\mathbf{u}$ now appears, and we know it could only scale $\mathbf{u}$, so we reach the conclusion

$$\mathbf{X} \vee \mathbf{u}^{☆} \vee \underset{\sim}{\mathbf{X}} = \pm s\mathbf{u} \vee \mathbf{1} = \pm s\mathbf{u}^{☆}, \tag{3.212}$$

which shows that $\mathbf{u}^{☆}$ must be a fixed geometry. Note that this happens only for the weight dual, and the bulk dual $\mathbf{u}^{★}$ is not fixed in a similar manner.

The primary invariant of a motor $\mathbf{Q}$ in the form given by Equation (3.93) is the line $\boldsymbol{l}$ serving as the axis of rotation. The weight dual $\boldsymbol{l}^{☆} = -l_{vx}\,\mathbf{e}_{23} - l_{vy}\,\mathbf{e}_{31} - l_{vz}\,\mathbf{e}_{12}$ must also be invariant under the screw motion that $\mathbf{Q}$ represents. This secondary invariant is the line at infinity in directions perpendicular to the axis of rotation, essentially an infinitely large ring around the line $\boldsymbol{l}$. For a flector $\mathbf{F}$ in the form given by Equation (3.156), the primary invariant is the reflection plane $\mathbf{g}$. The weight dual $\mathbf{g}^{\star} = -g_x\mathbf{e}_1 - g_y\mathbf{e}_2 - g_z\mathbf{e}_3$ is the point at infinity in the direction perpendicular to the plane, and it must also be invariant.

When a geometry $\mathbf{u}$ and its weight dual $\mathbf{u}^{\star}$ are fixed in space by a Euclidean isometry, the complements of these two invariants must be fixed in antispace. This can be seen algebraically by taking the complement of Equations (3.209) and (3.212) to get

$$\overline{\mathbf{X} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{X}}} = \bar{\mathbf{X}} \mathbin{\dot{\wedge}} \bar{\mathbf{u}} \mathbin{\dot{\wedge}} \tilde{\bar{\bar{\mathbf{X}}}} = \pm s\bar{\mathbf{u}} \tag{3.213}$$

and

$$\overline{\mathbf{X} \mathbin{\dot{\vee}} \mathbf{u}^{\star} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{X}}} = \bar{\mathbf{X}} \mathbin{\dot{\wedge}} \overline{\mathbf{u}^{\star}} \mathbin{\dot{\wedge}} \tilde{\bar{\bar{\mathbf{X}}}} = \pm s\overline{\mathbf{u}^{\star}} = \pm s\bar{\mathbf{u}}^{\bigstar}, \tag{3.214}$$

where we have used Equation (3.52) to swap reverse operations, and we have applied the identity

$$\overline{\mathbf{u}^{\star}} = \overline{\overline{\mathbb{G}\mathbf{u}}} = \overline{\mathbf{G}\bar{\mathbf{u}}} = \bar{\mathbf{u}}^{\bigstar}. \tag{3.215}$$

For an operator $\mathbf{X}$ representing a Euclidean isometry, the sandwich product $\bar{\mathbf{X}} \mathbin{\dot{\wedge}} \mathbf{u} \mathbin{\dot{\wedge}} \tilde{\bar{\bar{\mathbf{X}}}}$ performs the associated complement isometry. If a geometry $\mathbf{u}$ is fixed by the complement isometry, then the bulk dual $\mathbf{u}^{\bigstar}$, not the weight dual, is also fixed by the same operation. Since the bulk dual always contains the origin, this means that in addition to fixing some geometry $\mathbf{u}$ that is the complement of a Euclidean isometry's invariant, a complement isometry fixes another geometry at a finite location that is parallel to $\bar{\mathbf{u}}$ and passes through the origin. This makes the transformations performed by complement isometries look very odd.

Since the set of Euclidean isometries is generated by composing reflections across planes with the geometric antiproduct, the set of complement isometries is generated by composing complement reflections across points with the geometric product. When a reflection operation fixes the plane $\mathbf{g}$, its associated complement reflection fixes the point $\bar{\mathbf{g}}$. This is illustrated in Figure 3.13 for the plane $\mathbf{g} = \mathbf{e}_{423} + \frac{1}{4}\mathbf{e}_{321}$. The left side of the figure shows the ordinary Euclidean reflection that fixes the plane $\mathbf{g}$ where $x = -\frac{1}{4}$, and the right side of the figure shows the associated complement

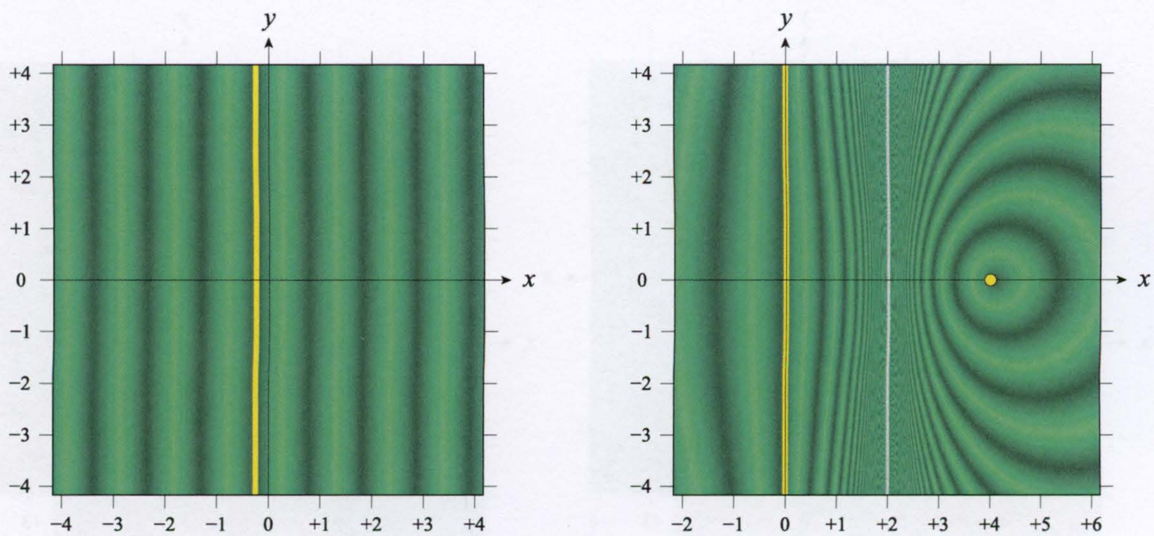


Figure 3.13. (Left) This is the Euclidean reflection across the plane $\mathbf{g} = \mathbf{e}_{423} + \frac{1}{4}\mathbf{e}_{321}$, which mirrors space across the yellow line where $x = -\frac{1}{4}$. (Right) This is the corresponding complement reflection with respect to the point $\bar{\mathbf{g}} = -\mathbf{e}_1 - \frac{1}{4}\mathbf{e}_4$, which combines a mirroring of space through the yellow point at $(4, 0, 0)$ and another mirroring of space across the plane $\bar{\mathbf{g}}^{\bigstar} = \mathbf{e}_{423}$ shown as the yellow line at $x = 0$. In both graphs, points are exchanged with other points having the same shade of green.

reflection that fixes the point $(4, 0, 0)$, which is the complement of $\mathbf{g}$. The fixed point $\mathbf{g}^{\smallstar} = \mathbf{e}_1$ in the horizon for the Euclidean reflection shows up as the fixed plane $\overline{\mathbf{g}}^{\bigstar} = \mathbf{e}_{423}$ containing the origin in the complement reflection. This plane is parallel to the reflection plane $\mathbf{g}$, but with the bulk removed. In the figure, points are exchanged under both operations with other points having the same shade of green. The Euclidean reflection of course shows a simple mirroring across the reflection plane, but the complement reflection is a complicated combination of two different kinds of mirroring. Space is reflected through the fixed point $\overline{\mathbf{g}}$ at $(4, 0, 0)$ in such a way that the infinitely wide region where $x > 4$ is exchanged with region between $x = 2$ and $x = 4$ of finite width. Space is also reflected across the plane $\overline{\mathbf{g}}^{\bigstar}$ at $x = 0$ in such a way that the infinitely wide region where $x < 0$ is exchanged with the region between $x = 0$ and $x = 2$ of finite width. Points lying on the plane $x = 2$ forming the boundary between these two reflections are exchanged with the horizon.

As the point $\overline{\mathbf{g}}$ moves farther and farther to the right in the graph of the complement reflection, the transformation looks more and more like an ordinary Euclidean reflection. When $\overline{\mathbf{g}}$ actually lies at infinity, then the transformation only mirrors across the plane $\overline{\mathbf{g}}^{\bigstar}$, but this is the same as the plane $\mathbf{g}$ in this case, so the complement reflection is exactly the same as a Euclidean reflection. If the plane $\mathbf{g}$ contains the origin, and thus the point $\overline{\mathbf{g}}$ is contained by the horizon, then the Euclidean reflection $\mathbf{g} \vee \mathbf{u} \vee \mathbf{g}$ and the complement reflection $\overline{\mathbf{g}} \wedge \mathbf{u} \wedge \tilde{\overline{\mathbf{g}}}$ perform the same exact operation on $\mathbf{u}$. The sets of Euclidean isometries and complement isometries intersect where both the origin and the horizon are fixed, and this is discussed further below.

When two Euclidean reflections across planes $\mathbf{g}$ and $\mathbf{h}$ are composed with the geometric antiproduct $\vee$, the invariant of the resulting rotation is the meet $\boldsymbol{l} = \mathbf{h} \vee \mathbf{g}$, which is the line contained by both planes. When two complement reflections with respect to points $\overline{\mathbf{g}}$ and $\overline{\mathbf{h}}$ are composed with the geometric product $\wedge$, the invariant of the resulting complement rotation is the join $\overline{\boldsymbol{l}} = \overline{\mathbf{h}} \wedge \overline{\mathbf{g}}$, which is the line containing both points. The bulk dual $\overline{\boldsymbol{l}}^{\bigstar}$, which is a line passing through the origin and running perpendicular to the moment of $\boldsymbol{l}$, is also fixed by the complement rotation. A comparison of the Euclidean rotation performed by $\mathbf{R} = \left(\mathbf{e}_{43} - \tfrac{1}{2}\mathbf{e}_{31}\right)\sin\phi + \mathbb{1}\cos\phi$ and the associated complement rotation $\overline{\mathbf{R}} = \left(\tfrac{1}{2}\mathbf{e}_{42} - \mathbf{e}_{12}\right)\sin\phi + \mathbf{1}\cos\phi$ is shown in Figure 3.14. The axis of the Euclidean rotation shown on the left side of the figure is the line parallel to the z axis that passes

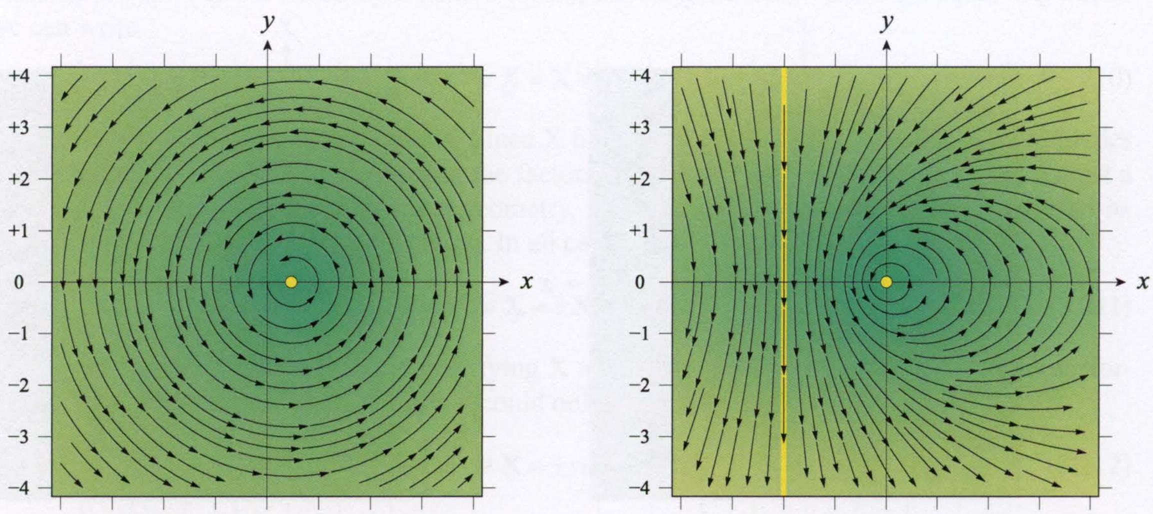


Figure 3.14. (Left) This is the flow field in the x-y plane for the rotation $\mathbf{R} = \left(\mathbf{e}_{43} - \tfrac{1}{2}\mathbf{e}_{31}\right)\sin\phi + \mathbb{1}\cos\phi$. The axis of rotation passes through the yellow point at $\left(\tfrac{1}{2}, 0, 0\right)$ and is perpendicular to the page. (Right) This is the flow field in the x-y plane for the complement rotation $\overline{\mathbf{R}} = \left(\tfrac{1}{2}\mathbf{e}_{42} - \mathbf{e}_{12}\right)\sin\phi + \mathbf{1}\cos\phi$. Points follow orbits of constant eccentricity with respect to the focus at the origin and the directrix indicated by the yellow line at $x = -2$.

through the x-y plane at the point $\left(\frac{1}{2}, 0, 0\right)$. In the complement rotation on the right side, the complement line $\overline{\boldsymbol{l}}$ acts as a directrix for conic section orbits, and the point where its bulk dual $\overline{\boldsymbol{l}}^{\star}$ intersects any plane containing $\overline{\boldsymbol{l}}$ acts as a focus. In the x-y plane, the origin is the focus, and a point $\mathbf{p}$ follows an orbit of constant eccentricity e with respect to the directrix $\overline{\boldsymbol{l}}$ under the transformation $\overline{\mathbf{R}} \wedge \mathbf{p} \wedge \tilde{\overline{\mathbf{R}}}$. (Eccentricity is defined as the distance to the focus divided by the perpendicular distance to the directrix.) The orbit is elliptical when $e < 1$, parabolic when $e = 1$, and hyperbolic when $e > 1$. In the hyperbolic case, a full orbit through a complement rotation of 2π radians has pieces on both sides of the directrix. As the directrix moves farther and farther away from the origin, the orbits become more and more circular, and they become exactly circular when the directrix lies in the horizon. Rotations about the origin are both Euclidean isometries and complement isometries.

A complement translation is interesting because it performs a transformation that is more practical than those performed by complement reflections and rotations. An ordinary Euclidean translation does not fix any point having a finite location, but it must fix the horizon as a whole because no point can be translated away from infinity. Unlike other Euclidean isometries, a translation actually fixes the horizon pointwise, which means every point and every line contained by the horizon is invariant. A complement translation must therefore fix not only the origin but every line and every plane that contains the origin as well. That being the case, any point undergoing a complement translation can move only along the straight line connecting it to the origin because points cannot leave a line containing the origin. This is visible in Figure 3.15, which compares a Euclidean translation in the negative z direction performed by the operator $\mathbf{T} = -\frac{1}{4}\mathbf{e}_{12} + \mathbb{1}$ to the associated complement translation performed by the operator $\overline{\mathbf{T}} = \frac{1}{4}\mathbf{e}_{43} + \mathbf{1}$.

To see what the overall effect of a complement translation is, we consider the operator

$$\overline{\mathbf{T}} = \frac{1}{2f}\mathbf{e}_{43} + \mathbf{1}. \tag{3.216}$$

When we transform a unitized point $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_4$ with this operator using the geometric product, we get

$$\overline{\mathbf{T}} \wedge \mathbf{p} \wedge \tilde{\overline{\mathbf{T}}} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \left(\frac{p_z}{f} + 1\right)\mathbf{e}_4. \tag{3.217}$$

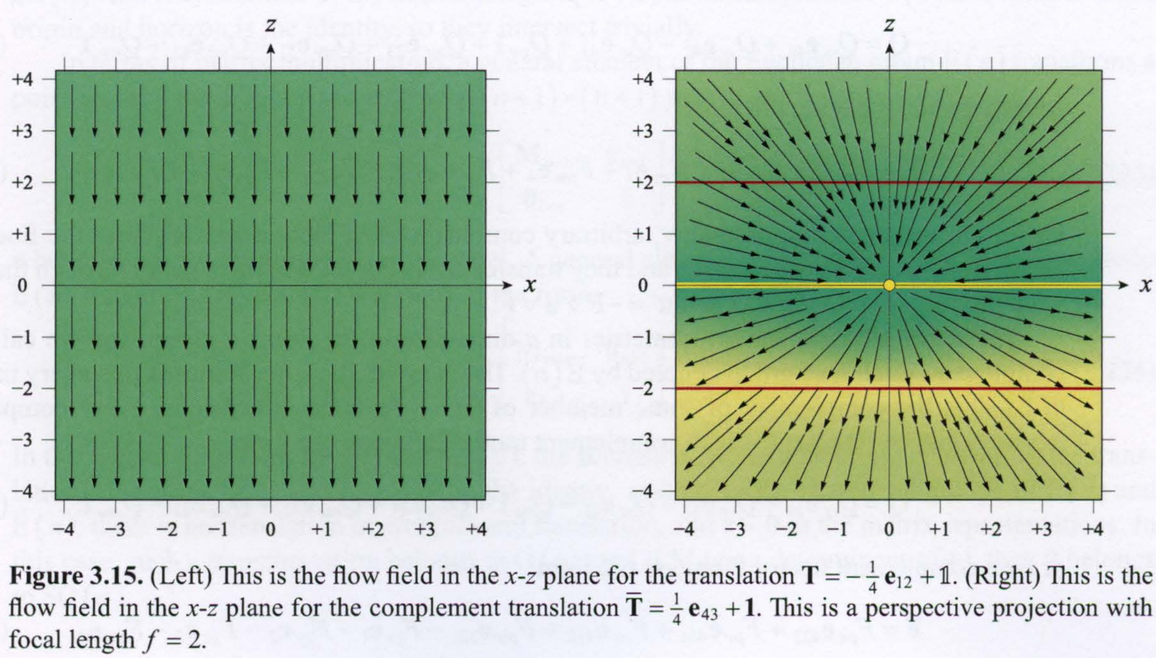


Figure 3.15. (Left) This is the flow field in the x-z plane for the translation $\mathbf{T} = -\frac{1}{4}\mathbf{e}_{12} + \mathbb{1}$. (Right) This is the flow field in the x-z plane for the complement translation $\overline{\mathbf{T}} = \frac{1}{4}\mathbf{e}_{43} + \mathbf{1}$. This is a perspective projection with focal length $f = 2$.

We can unitize this point by multiplying it by $f/(p_z + f)$, and this gives us the point

$$\frac{f}{p_z + f}(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3) + \mathbf{e}_4. \tag{3.218}$$

This acts like a perspective projection with focal length f in which the center of projection lies at $(0, 0, -f)$, and the viewing direction points along the positive z axis. The x and y coordinates are scaled precisely by the ratio of the focal length to the distance along the viewing direction between the center of projection and the point $\mathbf{p}$. The z coordinate is also scaled by the same factor, and this has the effect of squishing the range $[0, \infty]$ into the range $[0, f]$ in the z direction. It also causes the range $[-f, 0]$ to be expanded into the range $[-\infty, 0]$, and it moves everything in the range $[-\infty, -f]$ to the range $[f, \infty]$.

Geometric Product	**DUALITY**	**Geometric Antiproduct**
$\mathbf{Q} ⟑ \mathbf{u} ⟑ \tilde{\mathbf{Q}}$		$\mathbf{Q} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{Q}}$
▪ Fixes the origin $\mathbf{e}_n$.		▪ Fixes the horizon $\overline{\mathbf{e}}_n$.
▪ Performs complement isometries in regular space.		▪ Performs Euclidean isometries in regular space.
▪ Performs Euclidean isometries in antispace.		▪ Performs complement isometries in antispace.

3.9.2 Transformation Groups

The set of all Euclidean isometries in n dimensions forms an algebraic group called the Euclidean group and denoted by $\mathrm{E}(n)$. In three dimensions, every member of the Euclidean group can be represented by a motor $\mathbf{Q}$ having the form

$$\mathbf{Q} = Q_{vx}\mathbf{e}_{41} + Q_{vy}\mathbf{e}_{42} + Q_{vz}\mathbf{e}_{43} + Q_{vw}\mathbb{1} + Q_{mx}\mathbf{e}_{23} + Q_{my}\mathbf{e}_{31} + Q_{mz}\mathbf{e}_{12} + Q_{mw}\mathbf{1} \tag{3.219}$$

or by a flector $\mathbf{F}$ having the form

$$\mathbf{F} = F_{px}\mathbf{e}_1 + F_{py}\mathbf{e}_2 + F_{pz}\mathbf{e}_3 + F_{pw}\mathbf{e}_4 + F_{gx}\mathbf{e}_{423} + F_{gy}\mathbf{e}_{431} + F_{gz}\mathbf{e}_{412} + F_{gw}\mathbf{e}_{321}. \tag{3.220}$$

Under the geometric antiproduct $⟇$, arbitrary combinations of these operators form the Euclidean group $\mathrm{E}(3)$ with $\mathbb{1}$ as the identity, and they transform any object $\mathbf{u}$ in the algebra through the sandwich products $\mathbf{u}' = \mathbf{Q} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{Q}}$ and $\mathbf{u}' = -\mathbf{F} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{F}}$.

The set of all complement isometries in n dimensions also forms a group, and we call it the complement Euclidean group denoted by $\overline{\mathrm{E}}(n)$. The bar over the letter E means that every member of $\overline{\mathrm{E}}(n)$ is the complement of some member of $\mathrm{E}(n)$. In three dimensions, every complement isometry can be represented by a complement motor $\mathbf{Q}$ having the form

$$\mathbf{Q} = Q_{vx}\mathbf{e}_{23} + Q_{vy}\mathbf{e}_{31} + Q_{vz}\mathbf{e}_{12} - Q_{vw}\mathbb{1} + Q_{mx}\mathbf{e}_{41} + Q_{my}\mathbf{e}_{42} + Q_{mz}\mathbf{e}_{43} - Q_{mw}\mathbf{1} \tag{3.221}$$

or by a complement flector $\mathbf{F}$ having the form

$$\mathbf{F} = F_{px}\mathbf{e}_{423} + F_{py}\mathbf{e}_{431} + F_{pz}\mathbf{e}_{412} + F_{pw}\mathbf{e}_{321} - F_{gx}\mathbf{e}_1 - F_{gy}\mathbf{e}_2 - F_{gz}\mathbf{e}_3 - F_{gw}\mathbf{e}_4. \tag{3.222}$$

Under the geometric product $\wedge$, arbitrary combinations of these operators form the complement Euclidean group $\overline{\mathrm{E}}(3)$ with $\mathbf{1}$ as the identity, and they transform any object $\mathbf{u}$ in the algebra through the sandwich products $\mathbf{u}' = \mathbf{Q} \wedge \mathbf{u} \wedge \tilde{\mathbf{Q}}$ and $\mathbf{u}' = -\mathbf{F} \wedge \mathbf{u} \wedge \tilde{\mathbf{F}}$.

The geometric product corresponds to transform composition in the group $\overline{\mathrm{E}}(n)$, and the geometric antiproduct corresponds to transform composition in the group $\mathrm{E}(n)$. Reflections across planes are represented by antivectors (having antigrade one), and they meet at lower-dimensional invariants under the geometric antiproduct. Symmetrically, complement reflections across points are represented by vectors (having grade one), and they join at higher-dimensional invariants under the geometric product. A sandwich product $\mathbf{u}' = \mathbf{Q} \wedge \mathbf{u} \wedge \tilde{\mathbf{Q}}$ transforms the space of $\mathbf{u}$ with an element of $\overline{\mathrm{E}}(n)$, and it transforms the antispace of $\mathbf{u}$ with the complementary element of $\mathrm{E}(n)$. Symmetrically, a sandwich antiproduct $\mathbf{u}' = \mathbf{Q} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{Q}}$ transforms the space of $\mathbf{u}$ with an element of $\mathrm{E}(n)$, and it transforms the antispace of $\mathbf{u}$ with the complementary element of $\overline{\mathrm{E}}(n)$. Every element of $\overline{\mathrm{E}}(n)$ fixes the origin, and every element of $\mathrm{E}(n)$ fixes the horizon.

The groups $\mathrm{E}(n)$ and $\overline{\mathrm{E}}(n)$ are isomorphic due to the De Morgan laws relating the geometric product and antiproduct. They each have a number of important subgroups, some of which are shared between them, and their hierarchical relationships are shown in Figure 3.16. In particular, the Euclidean group $\mathrm{E}(n)$ contains the special Euclidean subgroup, denoted by $\mathrm{SE}(n)$, that includes all proper isometries performed by motors but excludes improper isometries performed by flectors. All complements of proper isometries belong to the complementary subgroup $\mathrm{S}\overline{\mathrm{E}}(n)$ of $\overline{\mathrm{E}}(n)$. The subgroups $\mathrm{SE}(n)$ and $\mathrm{S}\overline{\mathrm{E}}(n)$ further contain a translation subgroup $\mathrm{T}(n)$ and a complementary translation subgroup $\overline{\mathrm{T}}(n)$.

Any transformation that fixes both the origin and the horizon belongs to both the groups $\mathrm{E}(n)$ and $\overline{\mathrm{E}}(n)$. The Euclidean group and its complement intersect at the orthogonal group, which is denoted $\mathrm{O}(n)$, and it contains all rotations about the origin as well as all rotoreflections about the origin. Every member of $\mathrm{O}(n)$ has a representation with zero weight that transforms elements with the geometric product and a complementary representation with zero bulk that transforms elements with the geometric antiproduct. For example, conventional quaternions $\mathbf{q}$ have two representations, one that transforms any object $\mathbf{u}$ through the sandwich product $\mathbf{q} \wedge \mathbf{u} \wedge \tilde{\mathbf{q}}$ and another that transforms any object $\mathbf{u}$ through the sandwich antiproduct $\mathbf{q} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{q}}$, as discussed further below. The orthogonal group has a subgroup called the special orthogonal group, denoted $\mathrm{SO}(n)$, and it excludes improper isometries. The special orthogonal group is the intersection between $\mathrm{SE}(n)$ and $\mathrm{S}\overline{\mathrm{E}}(n)$. The only member of the translation group $\mathrm{T}(n)$ or its complement $\overline{\mathrm{T}}(n)$ that fixes both the origin and horizon is the identity, so they intersect trivially.

In terms of matrix multiplication, a general element of the Euclidean group $\mathrm{E}(n)$ transforms a point by multiplying it on the left by an $(n+1)\times(n+1)$ matrix of the form

$$\begin{bmatrix} \mathbf{M}_{n\times n} & \boldsymbol{\tau}_{n\times 1} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}, \tag{3.223}$$

where the $n \times n$ submatrix $\mathbf{M}$ is orthogonal. A general element of the complement Euclidean group $\overline{\mathrm{E}}(n)$ transforms points with matrices of the form

$$\begin{bmatrix} \mathbf{M}_{n\times n} & \mathbf{0}_{n\times 1} \\ \boldsymbol{\tau}_{1\times n} & 1 \end{bmatrix}. \tag{3.224}$$

In the special subgroups $\mathrm{SE}(n)$ and $\mathrm{S}\overline{\mathrm{E}}(n)$, the submatrix $\mathbf{M}$ has a determinant of $+1$. In the translation subgroups $\mathrm{T}(n)$ and $\overline{\mathrm{T}}(n)$, $\mathbf{M}$ is the identity matrix. In the shared subgroups of $\mathrm{E}(n)$ and $\overline{\mathrm{E}}(n)$, there is no translation or complement translation, and $\boldsymbol{\tau} = \mathbf{0}$ in the matrix representations. In this case, such a transformation belongs to $\mathrm{O}(n)$, and if $\mathbf{M}$ has a determinant of $+1$, then it belongs to $\mathrm{SO}(n)$.

The matrix representations of a transformation in $\mathrm{E}(n)$ and its complement in $\overline{\mathrm{E}}(n)$ are related by the inverse transpose operation. That is, if $\mathbf{H}$ is an $(n+1)\times(n+1)$ matrix representing an element of $\mathrm{E}(n)$, then the corresponding element of $\overline{\mathrm{E}}(n)$ is given by $\left(\mathbf{H}^{-1}\right)^{\mathrm{T}}$, and vice versa. This is what happens to the matrix representation when we take the complement of a geometric algebra operator and apply it with the opposite product.

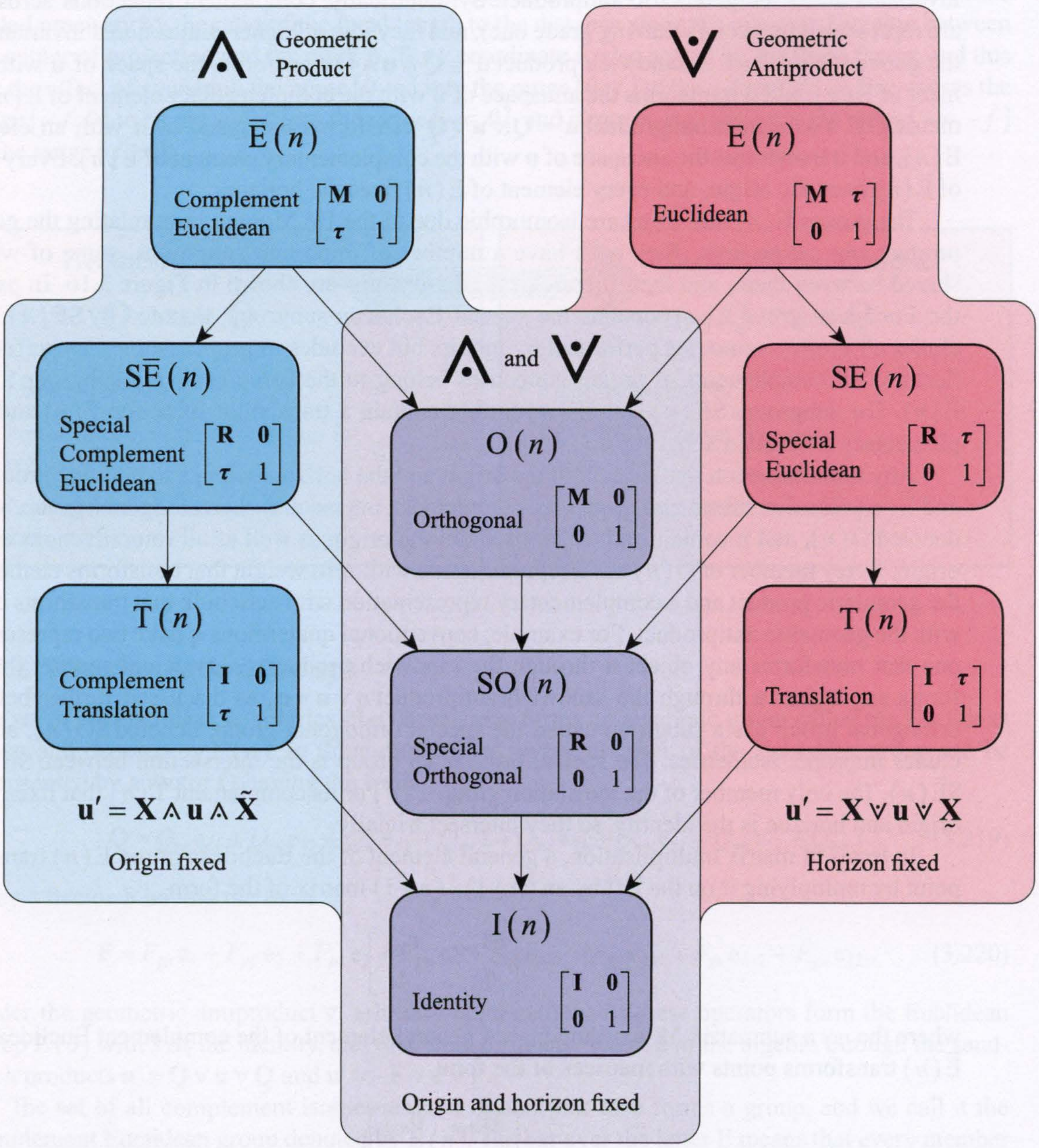


Figure 3.16. These are the hierarchical relationships among important transformation groups that arise in projective geometric algebras. Every transformation $\mathbf{X}$ belonging to a group in a blue box on the left operates on an object $\mathbf{u}$ through the sandwich product $\mathbf{X} \wedge \mathbf{u} \wedge \tilde{\mathbf{X}}$, and every transformation $\mathbf{X}$ belonging to a group in a red box on the right operates on an object $\mathbf{u}$ through the sandwich antiproduct $\mathbf{X} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{X}}$. Transformations belonging to groups in purple boxes in the center have two operator forms and can be used with either the product or antiproduct. In the matrix representations, $\mathbf{M}$ is an orthogonal $n\times n$ matrix with determinant ±1, $\mathbf{R}$ is an orthogonal $n\times n$ matrix with determinant $+1$, and $\mathbf{I}$ is the $n\times n$ identity matrix. The translation vector $\boldsymbol{\tau}$ is a $1\times n$ row in blue boxes and an $n\times 1$ column in red boxes.

3.9.3 Quaternions Revisited

Because quaternions keep both the origin and horizon fixed, they are members of the group $\mathrm{SO}(3)$ where the special Euclidean group $\mathrm{SE}(3)$ and complementary special Euclidean group $\overline{\mathrm{SE}}(3)$ intersect. Consequently, every quaternion has two different representations in the three-dimensional geometric algebra. First, the set of quaternions is exactly the subset of motors with bulk zero that perform pure rotations about the origin without any translation. In this case, the units i, j, and k are identified as

$$i = \mathbf{e}_{41},\ j = \mathbf{e}_{42},\ \text{and}\ k = \mathbf{e}_{43}. \tag{3.225}$$

A quaternion $\mathbf{q}$ can then be written as

$$\mathbf{q} = q_x \mathbf{e}_{41} + q_y \mathbf{e}_{42} + q_z \mathbf{e}_{43} + q_w \mathbb{1}, \tag{3.226}$$

and any point, line, or plane $\mathbf{u}$ is rotated about the origin through the sandwich antiproduct

$$\mathbf{u}' = \mathbf{q} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{q}}. \tag{3.227}$$

A unit quaternion can also be written as

$$\mathbf{q} = \mathbf{a} \sin \phi + \mathbb{1} \cos \phi, \tag{3.228}$$

where $\mathbf{a} = a_x \mathbf{e}_{41} + a_y \mathbf{e}_{42} + a_z \mathbf{e}_{43}$ is a unit bivector representing the axis of rotation, and ϕ is half the angle of rotation.

Second, the set of quaternions is exactly the subset of complement motors for which the directrix lies in the horizon. In this case, the weight is zero, and the units i, j, and k are identified as

$$i = -\mathbf{e}_{23},\ j = -\mathbf{e}_{31},\ \text{and}\ k = -\mathbf{e}_{12}. \tag{3.229}$$

A quaternion $\mathbf{q}$ can then be written as

$$\mathbf{q} = -q_x \mathbf{e}_{23} - q_y \mathbf{e}_{31} - q_z \mathbf{e}_{12} + q_w \mathbf{1}, \tag{3.230}$$

and any point, line, or plane $\mathbf{u}$ is rotated about the origin through the sandwich product

$$\mathbf{u}' = \mathbf{q} \wedge \mathbf{u} \wedge \tilde{\mathbf{q}}. \tag{3.231}$$

A unit quaternion can also be written as

$$\mathbf{q} = -\mathbf{a} \sin \phi + \mathbf{1} \cos \phi, \tag{3.232}$$

where $\mathbf{a} = a_x \mathbf{e}_{23} + a_y \mathbf{e}_{31} + a_z \mathbf{e}_{12}$ is a unit bivector representing the axis of rotation, and ϕ is half the angle of rotation. This form of a quaternion matches the operator that we derived in a nonprojective setting in Section 3.3. However, the first form of a quaternion above is the one typically used in the projective algebra because that one can be extended to include translations.

Historical Remarks

William Kindgon Clifford was an English mathematician who coined the term geometric algebra when he successfully combined the multiplication of Hamilton's quaternions with the building blocks of Grassmann's extension theory. Today, a broad class of mathematical structures known as Clifford algebras are named in his honor.

William Kingdon Clifford
(1845–1879)

Clifford wrote about Grassmann algebra and quaternions in a paper entitled "Applications of Grassmann's Extensive Algebra" [Cliff1878a] that was published in the *American Journal of Mathematics* in 1878. This paper discussed four-dimensional projective geometries very similar to those found in Chapter 2 of this book, but he didn't quite reach the point of using a degenerate metric. Importantly, his paper did include the realization that the i, j, and k components of a quaternion were not the same as the components of an ordinary vector, but were instead the combinations of pairs of vector basis elements, what we now call bivectors.

Also in 1878, Clifford published his book *Elements of Dynamic* [Cliff1878b] which contained a description of the two different types of multiplication between three-component quantities that arise when two quaternions are multiplied together. These were equivalent to the definitions of the dot product and cross product that would become widely known as part of the vector analysis framework established roughly two decades later.

Chapter 4

Round Projective Geometry

In the preceding chapters, the addition of one extra dimension to an n-dimensional Euclidean vector space allowed us to build a projective model that we refer to as the *rigid geometric algebra* (RGA) in order to distinguish it from the algebra introduced in this chapter. In $(n+1)$-dimensional rigid geometric algebra, flat geometries with homogeneous representations are combined in various ways using the exterior products, and they are rigidly transformed using the geometric products. We now add *two* extra dimensions to build a larger, doubly projective model called the *conformal geometric algebra* (CGA). This $(n+2)$-dimensional algebra includes everything that we have already seen in the rigid geometric algebra, but the geometric objects are generalized in such a way that they are all round and have specific radii. The flat points, lines, and planes that we are familiar with now become special cases in which the radii of the round objects are infinite. This chapter introduces round objects and explores the same kinds of geometric manipulation with the exterior products that was covered in Chapter 2, such as the join, meet, and projection operations. The topic of the next chapter is similar to that of Chapter 3, and it discusses the non-rigid "conformal" transformations that can be performed with the geometric products.

4.1 Construction

Conformal geometric algebra is constructed by adding two projective basis vectors called $\mathbf{e}_-$ and $\mathbf{e}_+$ to the set of ordinary basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of n-dimensional Euclidean space. The new vectors are named this way because their squares under the dot product are

$$\mathbf{e}_- \cdot \mathbf{e}_- = -1 \quad \text{and} \quad \mathbf{e}_+ \cdot \mathbf{e}_+ = +1. \tag{4.1}$$

This by itself is enough to build the entire algebra and observe all of its emergent properties. The projections that follow are part of the standard way to interpret what's going on when we talk about specific vectors, bivectors, trivectors, etc., and start multiplying them together with the exterior product and geometric product.

We proceed in the algebra representing three-dimensional Euclidean space because it has been the primary focus of this book and will continue to be the main area on which we concentrate in this chapter. We begin by considering a homogeneous point

$$\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_-, \tag{4.2}$$

where the basis vector $\mathbf{e}_-$ is temporarily filling the role that $\mathbf{e}_4$ previously played. As in the algebras we have already studied in the preceding chapters, any nonzero scalar multiple of this point belongs to the same equivalence class, so we can just assume $w = 1$. Now, we perform a stereographic projection of the point $\mathbf{p}$ onto the four-dimensional unit hypersphere that is centered at $\mathbf{e}_-$ and extends

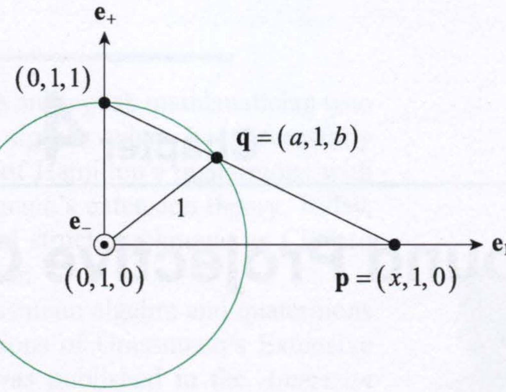


Figure 4.1. The stereographic projection of the point $\mathbf{p}$ is the point $\mathbf{q}$ where the line connecting $\mathbf{p}$ to the north pole at $\mathbf{e}_- + \mathbf{e}_+$ intersects the unit hypersphere centered at $\mathbf{e}_-$. The $\mathbf{e}_-$ axis points out of the page.

with a radius of one into the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_+\}$ subspace. Since this is extremely difficult to visualize, we drop the $\mathbf{e}_2$ and $\mathbf{e}_3$ dimensions so that we are left with only $\mathbf{e}_1$, $\mathbf{e}_-$, and $\mathbf{e}_+$, which we keep in that order. The point $\mathbf{p}$ then has the coordinates $(x, 1, 0)$, as shown in Figure 4.1.

The hypersphere is centered at the coordinates $(0, 1, 0)$, and we are projecting toward the north pole at the coordinates $(0, 1, 1)$. The projection of $\mathbf{p}$ is the point $\mathbf{q} = (a, 1, b)$ where the straight line connecting $\mathbf{p}$ to the north pole intersects the surface of the hypersphere. Everything is happening in the plane where the $\mathbf{e}_-$ coordinate is one, so we can ignore it and just worry about the $\mathbf{e}_1$ and $\mathbf{e}_+$ coordinates. Since $\mathbf{q}$ is on the surface of the circle where the hypersphere intersects this plane, we know that $a^2 + b^2 = 1$. If we subtract the north pole coordinates from both $\mathbf{p}$ and $\mathbf{q}$, then the resulting directions must be parallel, so we can also say

$$(a, 0, b-1) \times (x, 0, -1) = 0, \tag{4.3}$$

and that gives us the relationship $(1-b)\,x = a$. Squaring both sides and using $a^2 + b^2 = 1$ produces the equation

$$(1-b)^2\, x^2 = a^2 = 1 - b^2 = (1-b)(1+b), \tag{4.4}$$

from which we can determine that the value of b is given by

$$b = \frac{x^2 - 1}{x^2 + 1}. \tag{4.5}$$

Plugging this into $a^2 + b^2 = 1$ allows us to solve for a to get

$$a = \frac{2x}{x^2 + 1}. \tag{4.6}$$

The projected point $\mathbf{q}$ is thus

$$\mathbf{q} = \frac{2x}{x^2 + 1}\mathbf{e}_1 + \mathbf{e}_- + \frac{x^2 - 1}{x^2 + 1}\mathbf{e}_+. \tag{4.7}$$

Since we are using homogeneous coordinates, we can multiply $\mathbf{q}$ by any nonzero scalar value without changing its geometric meaning. To make the $\mathbf{e}_1$ coordinate the value x that we started with at the point $\mathbf{p}$, we choose to multiply by $\frac{1}{2}\left(x^2 + 1\right)$, and that produces

$$\mathbf{q} = x\mathbf{e}_1 + \tfrac{1}{2}\left(x^2 + 1\right)\mathbf{e}_- + \tfrac{1}{2}\left(x^2 - 1\right)\mathbf{e}_+. \tag{4.8}$$

Regrouping the coefficients of the $\mathbf{e}_-$ and $\mathbf{e}_+$ components lets us write

$$\mathbf{q} = x\mathbf{e}_1 + \tfrac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) + \tfrac{1}{2}x^2(\mathbf{e}_- + \mathbf{e}_+). \tag{4.9}$$

It is convenient to introduce two new vectors $\mathbf{e}_4$ and $\mathbf{e}_5$ defined as

$$\boxed{\begin{aligned} \mathbf{e}_4 &= \tfrac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) \\ \mathbf{e}_5 &= \mathbf{e}_- + \mathbf{e}_+, \end{aligned}} \tag{4.10}$$

where we have purposely included the factor of $\frac{1}{2}$ in one vector but not the other. Using $\mathbf{e}_4$ and $\mathbf{e}_5$, the projected point $\mathbf{q}$ is finally expressed as

$$\mathbf{q} = x\mathbf{e}_1 + \mathbf{e}_4 + \tfrac{1}{2}x^2\mathbf{e}_5. \tag{4.11}$$

When we add the $\mathbf{e}_2$ and $\mathbf{e}_3$ dimensions back in, this becomes

$$\mathbf{q} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + \mathbf{e}_4 + \tfrac{1}{2}\left(x^2 + y^2 + z^2\right)\mathbf{e}_5. \tag{4.12}$$

In conformal geometric algebra, every unitized point $\mathbf{q}$ has the form given by Equation (4.12). The first four components are the same as they were in rigid geometric algebra, but now there is a fifth component that holds half the squared distance to the origin. We will expand upon the meaning of this component in the next section, but first, it's important to establish a meaning for the vectors $\mathbf{e}_4$ and $\mathbf{e}_5$ by themselves. When we consider the case in which $(x, y, z) = (0, 0, 0)$, then we can immediately identify the vector $\mathbf{e}_4$ as the origin, which is nicely consistent with the meaning it has in rigid geometric algebra. Now assuming that $(x, y, z) \neq (0, 0, 0)$, we can homogeneously scale $\mathbf{q}$ by the value $2/(x^2 + y^2 + z^2)$, and we have

$$\mathbf{q} = \frac{2x}{x^2 + y^2 + z^2}\mathbf{e}_1 + \frac{2y}{x^2 + y^2 + z^2}\mathbf{e}_2 + \frac{2z}{x^2 + y^2 + z^2}\mathbf{e}_3 + \frac{2}{x^2 + y^2 + z^2}\mathbf{e}_4 + \mathbf{e}_5. \tag{4.13}$$

As the magnitude of (x, y, z) is allowed to become arbitrarily large, the first four components all approach zero, leaving only a constant $\mathbf{e}_5$. This makes it apparent that $\mathbf{e}_5$ corresponds to any point that is infinitely far away from the origin. Since it is defined as $\mathbf{e}_- + \mathbf{e}_+$, the vector $\mathbf{e}_5$ coincides with the north pole of the hypersphere onto which we projected, which is known as the *point at infinity*. This is a different kind of point at infinity compared to the points in the horizon that we have in rigid geometric algebra because there is no directional information. All points infinitely far from the origin stereographically project onto the same point at the north pole. Points in the horizon still exist in conformal geometric algebra, but not as the kind of point in Equation (4.12).

Figure 4.2 provides a visualization of the double projection happening in conformal geometric algebra. We are working in five-dimensional space, but that obviously can't be drawn on a two-dimensional page, so the y and z axes corresponding to the $\mathbf{e}_2$ and $\mathbf{e}_3$ basis vectors are omitted. This leaves the x axis corresponding to the basis vector $\mathbf{e}_1$ and the two directions corresponding to the $\mathbf{e}_-$ and $\mathbf{e}_+$ basis vectors. The yellow plane is really a three-dimensional subspace that lies one unit away from the origin $\mathbf{o}$ in the $\mathbf{e}_4$ direction, and it represents the homogeneous projection. A point $\mathbf{q}$ of the form given by Equation (4.12) can be homogeneously scaled by any value, but when it has an $\mathbf{e}_4$ coordinate of one, it lies in the yellow plane. The green surface is really a four-dimensional subspace called the *null cone*, and it represents the stereographic projection. The circle centered at $\mathbf{e}_-$ inside the cone is the same circle that appears in Figure 4.1, and the point $\mathbf{e}_5$ is the north pole. The null cone gets its name from the fact that any point lying on its surface is a *null vector*, which means it squares to zero under the dot product. Using the dot products defined for $\mathbf{e}_-$ and $\mathbf{e}_+$ in Equation (4.1) together with the definitions of $\mathbf{e}_4$ and $\mathbf{e}_5$ in Equation (4.10), we can work out the dot products

$$\begin{aligned}\mathbf{e}_4 \cdot \mathbf{e}_4 &= \mathbf{e}_5 \cdot \mathbf{e}_5 = 0 \\ \mathbf{e}_4 \cdot \mathbf{e}_5 &= \mathbf{e}_5 \cdot \mathbf{e}_4 = -1.\end{aligned} \tag{4.14}$$

When we apply these to any point $\mathbf{q}$ of the form shown in Equation (4.12) or any scalar multiple of such a point, we find it's always the case that $\mathbf{q} \bullet \mathbf{q} = 0$. Every point $\mathbf{q}$ falls on the green cone, and when it is unitized so the $\mathbf{e}_4$ coordinate is one, it also lies in the yellow plane. This intersection between the null cone and the subspace lying one unit into the $\mathbf{e}_4$ direction is called the *horosphere*. Although drawn as a purple parabola with only one dimension in the figure, the horosphere is really a three-dimensional paraboloidal surface. This is where the geometric objects we will be studying in this chapter exist when they have a unit weight.

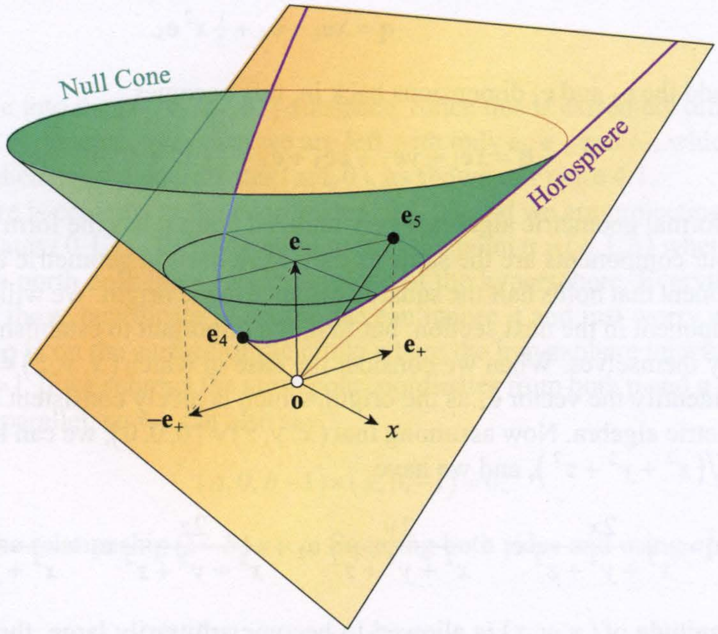


Figure 4.2. The yellow plane corresponds to the subspace that is one unit from the origin $\mathbf{o}$ in the $\mathbf{e}_4$ direction, and it represents the homogeneous projection. The green cone corresponds to the surface where every point $\mathbf{q}$ of the form given by Equation (4.12) lies, and it represents the stereographic projection. The intersection of these two subspaces is the horosphere, shown here as the purple parabola.

There is somewhat of an arbitrary choice in the definitions of $\mathbf{e}_4$ and $\mathbf{e}_5$ appearing in Equation (4.10). Several alternate definitions are possible in which $\mathbf{e}_4 = s(\mathbf{e}_- - \mathbf{e}_+)$ and $\mathbf{e}_5 = t(\mathbf{e}_- + \mathbf{e}_+)$ for various constant values of s and t, though it is common to require $st = 1/2$ because that makes the bivectors $\mathbf{e}_- \wedge \mathbf{e}_+$ and $\mathbf{e}_4 \wedge \mathbf{e}_5$ equal. Definitions that set $s = 1/2$ and $t = 1$ tend to produce the cleanest formulation of the entire algebra. The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ and $\mathbf{e}_5$ constitute an alternate vector basis that we use instead of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_-$, and $\mathbf{e}_+$ because doing so allows us to interpret the meanings of geometric objects and conformal transformations in a more intuitive way. We can easily transform from one basis to the other with Equation (4.10) and the inverse relationships

$$\boxed{\begin{aligned}\mathbf{e}_- &= \tfrac{1}{2}\mathbf{e}_5 + \mathbf{e}_4 \\ \mathbf{e}_+ &= \tfrac{1}{2}\mathbf{e}_5 - \mathbf{e}_4.\end{aligned}} \tag{4.15}$$

However, we will be working with the $\mathbf{e}_4$-$\mathbf{e}_5$ basis virtually all the time, and there will rarely be any need to consider the $\mathbf{e}_-$-$\mathbf{e}_+$ basis.[1]

All 32 basis elements covering the six grades in the 5D conformal exterior algebra are listed in Table 4.1. Half of the basis elements, those missing the factor $\mathbf{e}_5$, are exactly the same as the basis elements shown in Table 2.3 for the projective exterior algebra. By convention, the basis elements without a factor $\mathbf{e}_5$ are listed first for each grade, and they are kept in the same order as they were listed in the 4D algebra. The basis elements that do have a factor of $\mathbf{e}_5$ follow, and the factor of $\mathbf{e}_5$ is always last in a product of basis vectors. The preceding factors always occur in the same order as they do in a basis element having the same factors without $\mathbf{e}_5$ at the end. So there are two copies of the 16 basis elements from the 4D projective exterior algebra, one copy that is exactly the same, and a second copy for which each basis element has been multiplied by $\mathbf{e}_5$ on the right side. In addition to the scalar basis element $\mathbf{1}$ and five basis vectors, there are ten bivectors, ten trivectors, five quadrivectors, and the volume element $\mathbb{1}$, which is equal to $\mathbf{e}_{12345}$ here.

The wedge product works in conformal geometric algebra in exactly the same way that it does in rigid geometric algebra. The full multiplication table in the 5D algebra contains 1024 entries, and it is displayed across two pages in Appendix A. We define complements with respect to the $\mathbf{e}_4$-$\mathbf{e}_5$ basis and the volume element $\mathbf{e}_{12345}$. The complements of all 32 basis elements in the 5D algebra are shown in Table 4.2. Since there are an odd number of dimensions, right complements and left complements are the same, and we always use the overbar notation $\overline{\mathbf{u}}$. With complements defined, we can construct the antiwedge product by using the De Morgan law. The full multiplication table for the antiwedge product in the 5D algebra is also shown in Appendix A.

Type	Grade	Basis Elements
Scalar	0	$\mathbf{1}$
Vectors	1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$
Bivectors	2	$\mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{15}, \mathbf{e}_{25}, \mathbf{e}_{35}, \mathbf{e}_{45}$
Trivectors	3	$\mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}, \mathbf{e}_{321}, \mathbf{e}_{415}, \mathbf{e}_{425}, \mathbf{e}_{435}, \mathbf{e}_{235}, \mathbf{e}_{315}, \mathbf{e}_{125}$
Quadrivectors	4	$\mathbf{e}_{1234}, \mathbf{e}_{4235}, \mathbf{e}_{4315}, \mathbf{e}_{4125}, \mathbf{e}_{3215}$
Antiscalar	5	$\mathbb{1} = \mathbf{e}_{12345}$

Table 4.1. These are the 32 basis elements of the 5D conformal exterior algebra.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\overline{\mathbf{u}}$	$\mathbb{1}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbf{e}_{1234}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	$-\mathbf{e}_{415}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{435}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$

$\mathbf{u}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
$\overline{\mathbf{u}}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{45}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_5$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{1}$

Table 4.2. For each of the 32 basis elements $\mathbf{u}$ in the 5D conformal exterior algebra, this table lists the complement $\overline{\mathbf{u}}$. (Right and left complements are equivalent in odd numbers of dimensions.)

[1] In addition to different choices for the constants s and t, the literature contains several different notations for the basis vectors that we call $\mathbf{e}_4$ and $\mathbf{e}_5$, and this includes symbols like o and ∞ that reflect their interpretations as the origin and the point at infinity. We prefer $\mathbf{e}_4$ and $\mathbf{e}_5$ because the subscripts match the storage order of coordinates in the math library (following $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$), and they correspond nicely to the rows and columns of the matrices appearing in Chapter 5.

In conformal geometric algebra, the metric tensor $\mathbf{g}_\pm$ with respect to the set of basis vectors $B_\pm = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_-, \mathbf{e}_+\}$ is the 5×5 diagonal matrix

$$\mathbf{g}_\pm = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.16}$$

Since we will be using the set of basis vectors $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$, we need to transform $\mathbf{g}_\pm$ into the equivalent metric tensor $\mathbf{g}$ that operates on vectors over the basis B rather than the basis $B_\pm$. We accomplish this by examining the dot product between two vectors $\mathbf{u}_\pm$ and $\mathbf{v}_\pm$ defined on the basis $B_\pm$, which is given by

$$\mathbf{u}_\pm \bullet \mathbf{v}_\pm = \mathbf{u}_\pm^{\mathrm{T}} \mathbf{g}_\pm \mathbf{v}_\pm. \tag{4.17}$$

To take the dot product between vectors $\mathbf{u}$ and $\mathbf{v}$ defined on the basis B, we need to first change the basis to $B_\pm$ by multiplying by the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}, \tag{4.18}$$

which implements Equation (4.10) to transform $\mathbf{e}_4$ and $\mathbf{e}_5$. The dot product between $\mathbf{u}$ and $\mathbf{v}$ is then given by

$$\mathbf{u} \bullet \mathbf{v} = (\mathbf{M}\mathbf{u})^{\mathrm{T}} \mathbf{g}_\pm (\mathbf{M}\mathbf{v}) = \mathbf{u}^{\mathrm{T}} \left(\mathbf{M}^{\mathrm{T}} \mathbf{g}_\pm \mathbf{M} \right) \mathbf{v}, \tag{4.19}$$

and this means that the metric tensor $\mathbf{g}$ for the basis B must be equal to the matrix product $\mathbf{M}^{\mathrm{T}} \mathbf{g}_\pm \mathbf{M}$. Showing just the lower-right 2×2 portion of this product, we have

$$\mathbf{g} = \mathbf{M}^{\mathrm{T}} \mathbf{g}_\pm \mathbf{M} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{4.20}$$

We can now write the full 5×5 metric tensor $\mathbf{g}$ as

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{4.21}$$

This is the metric tensor that we use throughout our study of conformal geometric algebra. Notice that it matches the dot products between $\mathbf{e}_4$ and $\mathbf{e}_5$ that we already figured out in Equation (4.14). Because they both represent valid points, the origin and the point at infinity, $\mathbf{e}_4$ and $\mathbf{e}_5$ are both null vectors, and they indeed do square to zero under the dot product. However, we now have a new type of interaction between basis vectors that doesn't show up in rigid geometric algebra. The dot products between $\mathbf{e}_4$ and $\mathbf{e}_5$ are both -1. This feature of the metric tensor effects how duals are calculated in this chapter and how the geometric product is calculated in the Chapter 5.

The metric tensor $\mathbf{g}$ shown in Equation (4.21) is extended to the full metric exomorphism $\mathbf{G}$ laid out in Figure 4.3. Interestingly, both the metric and antimetric for the 4D rigid algebra appear inside the full metric for the 5D conformal algebra. The entries of the 4D metric given by Equation (2.61) are enclosed in boxes with solid outlines, and the negative entries of the 4D antimetric given by Equation (2.64) are enclosed in boxes with dotted outlines. That both of these are part of the 5D metric reflects the fact that there are two copies of the rigid geometric algebra inside the conformal geometric algebra, one that works with products and one that works with antiproducts.

Since $\det(\mathbf{g}) = -1$, the metric and antimetric must satisfy $\mathbf{G}\mathbb{G} = -\mathbf{I}$. Since $\mathbf{g}$ is its own inverse, so is $\mathbf{G}$, and we have the simple relationship

$$\mathbb{G} = -\mathbf{G} \tag{4.22}$$

in the conformal geometric algebra for any number of dimensions. Because the dot product and antidot product are defined with the metric and antimetric by Equations (2.74) and (2.75), they also

$$\mathbf{G} = \left[\begin{array}{c|ccccc|cccccccccc|cccccccccc|ccccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
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\end{array}\right]$$

Figure 4.3. This is the metric exomorphism $\mathbf{G}$ for the 5D conformal exterior algebra, where rows and columns correspond to the basis elements in the order shown in Table 4.1. The metric antiexomorphism $\mathbb{G}$ is simply equal to $-\mathbf{G}$. The entries of the metric and the negative entries of the antimetric in the 4D rigid exterior algebra are enclosed in boxes with solid outlines and dotted outlines, respectively.

have a simple relationship that we can express as

$$\mathbf{a} \circ \mathbf{b} = -\mathbf{a} \bullet \mathbf{b}. \tag{4.23}$$

That is, the antidot product as an antiscalar is just the negative complement of the dot product as a scalar. Similarly, because the dual and antidual are defined with the metric and antimetric by Equations (2.100) and (2.101), they are related by

$$\mathbf{u}^{\star} = -\mathbf{u}^{\bigstar}. \tag{4.24}$$

The metric duals and antiduals of all 32 basis elements in the 5D conformal geometric algebra are listed in Table 4.3.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$-\mathbf{e}_{1234}$	$-\mathbf{e}_{3215}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{415}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{435}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{321}$
$\mathbf{u}^{\star}$	$-\mathbb{1}$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{4315}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{3215}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{321}$

$\mathbf{u}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_4$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_5$	$-\mathbf{1}$
$\mathbf{u}^{\star}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{45}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_4$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_5$	$\mathbf{1}$

Table 4.3. For each of the 32 basis elements $\mathbf{u}$ in the 5D conformal exterior algebra, this table lists the metric dual $\mathbf{u}^{\bigstar}$ and antidual $\mathbf{u}^{\star}$. (Right and left duals are equivalent in odd numbers of dimensions.)

4.2 3D Round Geometry

In Section 2.4, flat geometric objects were constructed by joining points together with the wedge product, and k-vectors represented $(k-1)$-dimensional flat subspaces. A vector was a 0D point, a bivector was a 1D line, and a trivector was a 2D plane. The process is exactly the same in conformal geometric algebra, except now the results are generalized to round geometric objects. A k-vector in CGA corresponds to the $(k-2)$-dimensional spherical surface of a $(k-1)$-dimensional solid ball. A bivector is a 0-sphere that we call a *dipole* because it consists of the two points on either end of some line segment. A trivector is a circle corresponding to the boundary of a disk aligned to some plane. Finally, a quadrivector is a sphere. Applying the same pattern to a point bizarrely implies that a vector is the surface of a zero-dimensional ball, which has dimension -1. One way to interpret this is that a point in CGA is a kind of inverted sphere that's hollow on the inside and solid throughout all space on the outside. Indeed, the algebra allows us to assign a radius to a point just like we can for dipoles, circles, and spheres.

For all four types of geometry, the radius shows up in the algebraic representation as a squared quantity r^2, and it can be positive, negative, or zero. We refer to any geometry for which $r^2 > 0$ as a *real* object, whether it be a point, dipole, circle, or sphere. A geometry for which $r^2 < 0$ is called an *imaginary* object because the radius r must be an imaginary number if its square is negative. A geometry for which $r^2 = 0$ is called a *null* object.

The flat geometries from rigid geometric algebra are included in conformal geometric algebra, and they show up when one of the points participating in the wedge product is the point at infinity $\mathbf{e}_5$. The points that are multiplied together to construct a higher-dimensional object are contained by that object. If one of the points is infinitely far away, then the object containing that point and the others must have an infinite radius. Everything in the rigid algebra transfers over to the conformal algebra through multiplication by $\mathbf{e}_5$, and this includes not only geometric objects but also motors, flectors, and even homogeneous magnitudes.

A flat point $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$ gets multiplied by $\mathbf{e}_5$ in the conformal algebra and becomes

Flat point (3D)

$$\boxed{\mathbf{p} = p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + p_w\mathbf{e}_{45}.} \tag{4.25}$$

This bivector is really a dipole with one end at $\mathbf{p}$ and the other end at the point at infinity. We use the specific term *flat point* to distinguish it from the vector $\mathbf{q}$ shown in Equation (4.12), which we will call a *round point* from now on. A round point can have a finite radius that is real, imaginary, or null, but a flat point always has an infinite radius.

A flat line $\boldsymbol{l} = l_{vx}\mathbf{e}_{41} + l_{vy}\mathbf{e}_{42} + l_{vz}\mathbf{e}_{43} + l_{mx}\mathbf{e}_{23} + l_{my}\mathbf{e}_{31} + l_{mz}\mathbf{e}_{12}$ gets multiplied by $\mathbf{e}_5$ in the conformal algebra and becomes

Line (3D)

$$\boxed{\boldsymbol{l} = l_{vx}\mathbf{e}_{415} + l_{vy}\mathbf{e}_{425} + l_{vz}\mathbf{e}_{435} + l_{mx}\mathbf{e}_{235} + l_{my}\mathbf{e}_{315} + l_{mz}\mathbf{e}_{125}.} \tag{4.26}$$

This trivector is a circle that passes through the point at infinity, and it thus has an infinite radius. We don't need to include the word "flat" when describing a line because there is no round geometry that it can be confused with. The remaining flat geometry, a plane $\mathbf{g} = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412} + g_w\mathbf{e}_{321}$, gets multiplied by $\mathbf{e}_5$ in the conformal algebra and becomes

Plane (3D)

$$\boxed{\mathbf{g} = g_x\mathbf{e}_{4235} + g_y\mathbf{e}_{4315} + g_z\mathbf{e}_{4125} + g_w\mathbf{e}_{3215}.} \tag{4.27}$$

This quadrivector is a sphere that contains the point at infinity on its surface. A plane is a sphere with an infinite radius.

4.2.1 Representations

Vectors, bivectors, trivectors, and quadrivectors correspond to round points, dipoles, circles, and spheres in CGA. Round points and spheres each have five components because they are the vectors and antivectors in the 5D algebra. Dipoles and circles each have ten components, but there is some redundant information stored in them, and we will see in Section 4.10 that they really have only six degrees of freedom. Here, we break down the components and assign some geometric meaning to all of them. A flat geometric object appears prominently in each representation of a round object, and it corresponds to the round object's *carrier*, which is the smallest flat subspace that contains the round object. The diagrams of a dipole and circle appearing below also show a *container*, which is the smallest sphere that contains a round object. Carriers and cocarriers are discussed further in Section 4.2.3, and containers are discussed further in Section 4.2.5.

In the geometric interpretations that we give below, the components of each type of object are divided into two groups that we call the round part and the flat part. The *round part* of an algebraic representation is the collection of components that do not contain a factor of the basis vector $\mathbf{e}_5$. The *flat part* is everything else that does contain a factor of $\mathbf{e}_5$. The round part is highlighted green in the equations below that define component naming that we'll be using, and the flat part is highlighted purple. The components of the round part are called the round part because they exist only for round objects. Flat points, lines, and planes do not have a round part, only a flat part. The round part always contains a round object's carrier geometry, and a round object is considered unitized when its carrier is unitized.

We begin with the round point having the form that we already derived in Equation (4.12), but now we make a small tweak by adding a radius. A round point $\mathbf{a}$ at the position $\mathbf{p} = (p_x, p_y, p_z)$ with radius r is given by

Round point (3D)

$$\boxed{\mathbf{a} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_4 + \frac{\mathbf{p}^2 + r^2}{2}\mathbf{e}_5.} \tag{4.28}$$

As is the case for all round objects, the value of r^2 can be positive, negative, or zero. If $r^2 > 0$, then $\mathbf{a}$ is a real round point, and if $r^2 < 0$, then $\mathbf{a}$ is an imaginary round point. If $r^2 = 0$, then $\mathbf{a}$ is a null round point, and we recover Equation (4.12).

We mentioned earlier that all points having the form shown in Equation (4.12) are null vectors that lie on the null cone, but if we endow a point with a radius, that is no longer true. A point $\mathbf{a}$ with a nonzero radius is not a null vector and does not satisfy $\mathbf{a} \bullet \mathbf{a} = 0$. A point with a real radius lies inside the horosphere shown in Figure 4.2, and a point with an imaginary radius lies outside the horosphere. As we'll see below in Section 4.3, this dot product is actually measuring the radius.

A round point $\mathbf{a}$ is written in terms of generic coordinates as

$$\mathbf{a} = \underbrace{a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3 + a_w\mathbf{e}_4}_{\text{Carrier Point}} + \underbrace{a_u\mathbf{e}_5}_{\substack{\text{Infinity} \\ (\text{when } a_x = a_y = a_z = a_w = 0)}}, \tag{4.29}$$

where we always use the subscripts x, y, z, w, and u. The carrier point stored in the round part is the flat point that coincides with the round point but has no radius. If a round point has no round part, leaving only the $\mathbf{e}_5$ component, then it is the point at infinity.

Let $\mathbf{a}$ and $\mathbf{b}$ be round points of radius zero that are both the same distance r from a center point $\mathbf{p} = (p_x, p_y, p_z)$ in opposite directions along the unit-length vector $\mathbf{n} = (n_x, n_y, n_z)$, as shown in Figure 4.4. If we multiply them together with the wedge product to calculate $\mathbf{a} \wedge \mathbf{b}$ and unitize the result, then we have constructed a dipole $\mathbf{d}$ of the form

Dipole (3D)

$$\boxed{\begin{aligned}\mathbf{d} = {} & n_x\mathbf{e}_{41} + n_y\mathbf{e}_{42} + n_z\mathbf{e}_{43} + (p_y n_z - p_z n_y)\mathbf{e}_{23} + (p_z n_x - p_x n_z)\mathbf{e}_{31} + (p_x n_y - p_y n_x)\mathbf{e}_{12} \\ & + (\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + \mathbf{e}_{45}) - \frac{\mathbf{p}^2 + r^2}{2}(n_x\mathbf{e}_{15} + n_y\mathbf{e}_{25} + n_z\mathbf{e}_{35}).\end{aligned}} \tag{4.30}$$

The six components of the round part appearing first in this formula are exactly the components of a line containing the point $\mathbf{p}$ and running in the direction $\mathbf{n}$. This is the carrier of a dipole, which is shown as the black line in the figure. The cocarrier is the subspace containing the center $\mathbf{p}$ but perpendicular to the carrier, and it is shown as the yellow plane.

A dipole $\mathbf{d}$ is written in terms of generic coordinates as

$$\mathbf{d} = \underbrace{\overbrace{d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43}}^{\text{Cocarrier Normal}} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12}}_{\text{Carrier Line}} + \underbrace{d_{px}\mathbf{e}_{15} + d_{py}\mathbf{e}_{25} + d_{pz}\mathbf{e}_{35} + \overbrace{d_{pw}\mathbf{e}_{45}}^{\text{Cocarrier Position}}}_{\substack{\text{Flat Point} \\ (\text{when } d_{vz} = d_{vy} = d_{vz} = d_{mx} = d_{my} = d_{mz} = 0)}}. \tag{4.31}$$

The subscripts vx, vy, vz, mx, my, and mz identifying the components of the carrier have the same meanings as they do for a flat line. The four remaining components have the subscripts px, py, pz, and pw because they constitute a flat point $\mathbf{p}$ when the round part is zero. This happens when one of the two points $\mathbf{a}$ and $\mathbf{b}$ that were multiplied together to construct the dipole is the point at infinity.

As we did for the two parts of a line $\boldsymbol{l}_\mathbf{v}$ and $\boldsymbol{l}_\mathbf{m}$, we sometimes use a shorthand notation for various three-component parts of a dipole. The notation $\mathbf{d}_\mathbf{v}$ means the vector (d_{vx}, d_{vy}, d_{vz}), and the notation $\mathbf{d}_\mathbf{m}$ means the vector (d_{mx}, d_{my}, d_{mz}). For the vector (d_{px}, d_{py}, d_{pz}), we use the longer notation $\mathbf{d}_{pxyz}$ to make it clear that it's a three-dimensional vector that excludes the pw component.

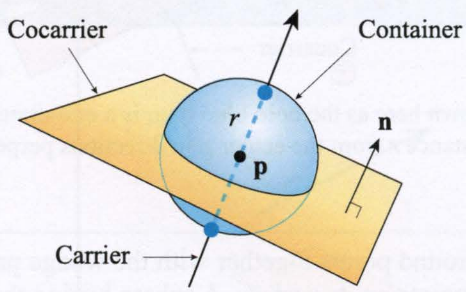


Figure 4.4. A dipole, shown here as the pair of blue points connected by a dashed line, is a zero-dimensional sphere. The points of radius zero on its surface lie at the distance r from the center $\mathbf{p}$ along the direction $\mathbf{n}$.

When we multiply three round points together with the wedge product, the result is the unique circle $\mathbf{c}$ that contains all three of them on its surface. As shown in Figure 4.5, this circle has a center $\mathbf{p}$ and a radius r, and it lies in a plane with the unit-length normal vector $\mathbf{n}$ determined by the three points. Using the parameters $\mathbf{p}$, $\mathbf{n}$, and r, a circle can be written as the trivector

Circle (3D)

$$\mathbf{c} = n_x\mathbf{e}_{423} + n_y\mathbf{e}_{431} + n_z\mathbf{e}_{412} + (p_y n_z - p_z n_y)\mathbf{e}_{415} + (p_z n_x - p_x n_z)\mathbf{e}_{425} + (p_x n_y - p_y n_x)\mathbf{e}_{435}$$
$$+ (\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{235} + p_y\mathbf{e}_{315} + p_z\mathbf{e}_{125} - \mathbf{e}_{321}) - \frac{\mathbf{p}^2 - r^2}{2}(n_x\mathbf{e}_{235} + n_y\mathbf{e}_{315} + n_z\mathbf{e}_{125}). \tag{4.32}$$

The normal of the carrier plane can be seen in the first three terms, which make use of the same components $\mathbf{e}_{423}$, $\mathbf{e}_{431}$, and $\mathbf{e}_{412}$ that a flat plane has in the rigid algebra. The position of the carrier plane in the $\mathbf{e}_{321}$ component has the value $-\mathbf{p}\cdot\mathbf{n}$ as it must for a plane with normal $\mathbf{n}$ that contains the point $\mathbf{p}$. The carrier is shown as the yellow plane in the figure, and the cocarrier is shown as the black line that is perpendicular to it and contains the center $\mathbf{p}$.

A circle $\mathbf{c}$ is written in terms of generic coordinates as

$$\mathbf{c} = \underbrace{\overbrace{c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}}^{\text{Cocarrier Direction}} + c_{gw}\mathbf{e}_{321}}_{\text{Carrier Plane}} + \underbrace{\overbrace{c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435}}^{\text{Cocarrier Moment}} + c_{mx}\mathbf{e}_{235} + c_{my}\mathbf{e}_{315} + c_{mz}\mathbf{e}_{125}}_{\substack{\text{Flat Line} \\ (\text{when } c_{gx} = c_{gy} = c_{gz} = c_{gw} = 0)}}. \tag{4.33}$$

The subscripts gx, gy, gz, and gw correspond to the carrier and have the same meaning as they do for a flat plane. The six subscripts vx, vy, vz, mx, my, and mz constitute the flat line that $\mathbf{c}$ becomes when the round part is zero. A straight line is the special case of a circle that contains the point at infinity $\mathbf{e}_5$. Again, we have a shorthand notation for the various three-component parts of a circle. The notation $\mathbf{c}_\mathbf{v}$ means the vector (c_{vx}, c_{vy}, c_{vz}), and the notation $\mathbf{c}_\mathbf{m}$ means the vector (c_{mx}, c_{my}, c_{mz}). Since the carrier has four components, we use the notation $\mathbf{c}_{gxyz}$ for the vector (c_{gx}, c_{gy}, c_{gz}) to exclude the gw coordinate.

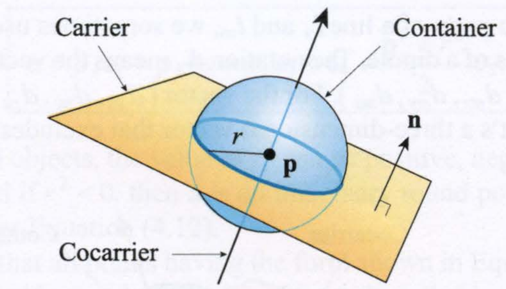


Figure 4.5. A circle, shown here as the bold blue ring, is a one-dimensional sphere. The points of radius zero on its surface lie at a distance r from the center $\mathbf{p}$ in directions perpendicular to the normal vector $\mathbf{n}$.

Multiplying four round points together with the wedge product constructs the unique sphere $\mathbf{s}$ that contains all four points on its surface. A sphere having the center $\mathbf{p}$ and radius r is represented by the quadrivector

Sphere (3D)

$$\boxed{\mathbf{s} = p_x \mathbf{e}_{4235} + p_y \mathbf{e}_{4315} + p_z \mathbf{e}_{4125} - \mathbf{e}_{1234} - \frac{\mathbf{p}^2 - r^2}{2} \mathbf{e}_{3215}.} \tag{4.34}$$

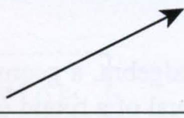
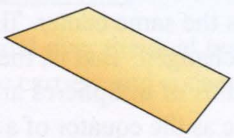
The $\mathbf{e}_{1234}$ component holds the weight of a sphere, and it is intentionally negative one here because that makes it possible to read the coordinates of the center directly from the first three terms. This is similar to how the center of a round point can be read directly when its weight component $\mathbf{e}_4$ is positive one. Round points and spheres are duals of each other, and as can be seen in Table 4.3, either the three components containing p_x, p_y, and p_z or the one weight component has to be negated no matter which dual is applied to convert between them.

A sphere $\mathbf{s}$ is written in terms of generic coordinates as

$$\mathbf{s} = \underbrace{s_u \mathbf{e}_{1234}}_{\text{Carrier Space}} + \underbrace{s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125} + s_w \mathbf{e}_{3215}}_{\substack{\text{Flat Plane} \\ (\text{when } s_u = 0)}}, \tag{4.35}$$

where we always use the subscripts x, y, z, w, and u just as we do for a round point. Our convention is to write the round part before the flat part, so the u coordinate appears first for a sphere and last for a round point. The carrier for a sphere is the entire 3D space, which is represented by the 4D volume element $\mathbf{e}_{1234}$ from the rigid algebra. When the $\mathbf{e}_{1234}$ component is zero, a sphere becomes a flat plane that contains the point at infinity $\mathbf{e}_5$.

In the numerous diagrams that appear throughout this chapter, we draw the various types of flat and round geometric objects with a consistent visual appearance for easy identification. Examples are shown in Table 4.4. Flat points are always enclosed by a small orange disk so they are not confused with round points. Lines are always drawn as a bold black segment with an arrowhead at one end, and planes are always filled with a yellow-orange gradient. Each of the four types of round object can appear in two different colors that are used to distinguish between real and imaginary radii. For dipoles, circles, and spheres, the color blue indicates that the object has a real radius, and the color red indicates that it has an imaginary radius. The colors are different for round points because they otherwise look exactly like spheres, and we need a way to tell the difference. For round points, the color green indicates a real radius, and the color purple indicates an imaginary radius. Aside from color, spheres and round points are drawn as balls with black points at their centers. The centers of round points are a little bit smaller than the centers of spheres. Dipoles are drawn as a pair of points connected by a dashed line, where the dashing is meant to convey that the segment between the two points is not part of the dipole's surface. Finally, circles are drawn as bold rings with black points at their centers.

Flat Object	**Appearance**
Flat point **p**	
Line $\boldsymbol{l}$	
Plane **g**	

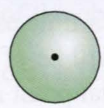
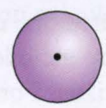
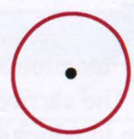
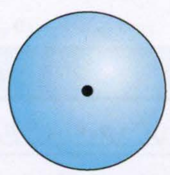
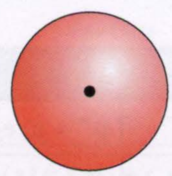
Round Object	**Real**	**Imaginary**
Round point **a**		
Dipole **d**		
Circle **c**		
Sphere **s**		

Table 4.4. The various types of geometries in conformal geometry algebra are consistently drawn as shown in this table. The color blue is generally used to indicate that a round geometry is real, and the color red is used to indicate that it is imaginary. The exception is round points which use green and purple for real and imaginary to distinguish them from spheres.

Math Library Notes

- The `RoundPoint3D` class stores the five coordinates of a 5D vector representing a round point, and they are named `x`, `y`, `z`, `w`, and `u`.
- The `Dipole3D` class stores the ten components of a 5D bivector representing a dipole. These components are divided into three parts named `v`, `m`, and `p`, which are the direction of the carrier line as a `Vector3D`, the moment of the carrier line as a `Bivector3D`, and the flat point as a `FlatPoint3D`.

- The `Circle3D` class stores the ten components of a 5D trivector representing a circle. These components are divided into three parts named `g`, `v`, and `m`, which are the carrier plane as a `Plane3D`, the direction of the flat line as a `Vector3D`, and the moment of the flat line as a `Bivector3D`.
- The `Sphere3D` class stores the five coordinates of a 5D quadrivector representing a sphere, and they are named `u`, `x`, `y`, `z`, and `w`.
- The `FlatPoint3D`, `Line3D`, and `Plane3D` classes are used for flat geometries in conformal geometric algebra as well as rigid geometric algebra.

4.2.2 Duals

In conformal geometric algebra, a geometric object $\mathbf{u}$ and its dual $\mathbf{u}^\bigstar$ have symmetric properties. When we calculate the dual of a round object, what we get is another round object of the complementary grade that has the same center. The magnitude of the radius is also the same, but real and imaginary radii are exchanged. That is, the squared radius of $\mathbf{u}^\bigstar$ has the same size but the opposite sign of the squared radius of $\mathbf{u}$. Spheres are dual to round points, and circles are dual to dipoles. If we think of a real circle as the equator of a globe, then the dual of that circle is the imaginary dipole whose surface coincides with the north and south poles of the same globe. The opposite is also true. The dual of a real dipole coinciding with the poles is the imaginary circle at the equator.

The duals of all seven types of geometric objects in 3D space are listed in Table 4.5, and they are easily calculated one component at a time using the information in Table 4.3. The duals of the flat geometries are included, but they will be of little interest except to note that they each have a carrier in the horizon. For example, a flat point is a dipole with one pole at infinity, so the equator in its dual is a circle whose carrier plane is the whole horizon. Because the metric is not degenerate, no information is lost when we calculate a dual, which is unlike the rigid algebra. Taking the dual of a dual in CGA returns to the original object, but it's always negated in the 5D case.

4.2.3 Carriers

We have already mentioned the *carrier* of a round object. It is the lowest dimensional flat object that contains it. The carrier of a round point is the flat point at the same location, the carrier of a dipole is the line in which it lies, the carrier of a circle is the plane in which it lies, and the carrier

Type	Dual
Flat point $\mathbf{p}$	$\mathbf{p}^\bigstar = p_w\mathbf{e}_{321} - p_x\mathbf{e}_{235} - p_y\mathbf{e}_{315} - p_z\mathbf{e}_{125}$
Line $\boldsymbol{l}$	$\boldsymbol{l}^\bigstar = l_{vx}\mathbf{e}_{23} + l_{vy}\mathbf{e}_{31} + l_{vz}\mathbf{e}_{12} + l_{mx}\mathbf{e}_{15} + l_{my}\mathbf{e}_{25} + l_{mz}\mathbf{e}_{35}$
Plane $\mathbf{g}$	$\mathbf{g}^\bigstar = -g_x\mathbf{e}_1 - g_y\mathbf{e}_2 - g_z\mathbf{e}_3 + g_w\mathbf{e}_5$
Round point $\mathbf{a}$	$\mathbf{a}^\bigstar = -a_w\mathbf{e}_{1234} + a_x\mathbf{e}_{4235} + a_y\mathbf{e}_{4315} + a_z\mathbf{e}_{4125} - a_u\mathbf{e}_{3215}$
Dipole $\mathbf{d}$	$\mathbf{d}^\bigstar = -d_{vx}\mathbf{e}_{423} - d_{vy}\mathbf{e}_{431} - d_{vz}\mathbf{e}_{412} + d_{pw}\mathbf{e}_{321}$ $- d_{mx}\mathbf{e}_{415} - d_{my}\mathbf{e}_{425} - d_{mz}\mathbf{e}_{435} - d_{px}\mathbf{e}_{235} - d_{py}\mathbf{e}_{315} - d_{pz}\mathbf{e}_{125}$
Circle $\mathbf{c}$	$\mathbf{c}^\bigstar = c_{gx}\mathbf{e}_{41} + c_{gy}\mathbf{e}_{42} + c_{gz}\mathbf{e}_{43}$ $+ c_{vx}\mathbf{e}_{23} + c_{vy}\mathbf{e}_{31} + c_{vz}\mathbf{e}_{12} + c_{mx}\mathbf{e}_{15} + c_{my}\mathbf{e}_{25} + c_{mz}\mathbf{e}_{35} - c_{gw}\mathbf{e}_{45}$
Sphere $\mathbf{s}$	$\mathbf{s}^\bigstar = -s_x\mathbf{e}_1 - s_y\mathbf{e}_2 - s_z\mathbf{e}_3 + s_u\mathbf{e}_4 + s_w\mathbf{e}_5$

Table 4.5. These are the duals of the geometric objects arising in the conformal geometric algebra over 3D space. The antidual $\mathbf{u}^\star$ is always the negation of the dual $\mathbf{u}^\bigstar$ in this algebra.

of a sphere is all of 3D space. The carrier of any object can be calculated by taking the wedge product with the point at infinity $\mathbf{e}_5$. Denoting the carrier of an object $\mathbf{u}$ by $\operatorname{car}(\mathbf{u})$, this gives us the definition

Carrier

$$\boxed{\operatorname{car}(\mathbf{u}) = \mathbf{u} \wedge \mathbf{e}_5.} \tag{4.36}$$

This product has the effect of annihilating the flat part of the object (because $\mathbf{e}_5 \wedge \mathbf{e}_5 = 0$) and appending a factor of $\mathbf{e}_5$ to each component of the round part to make it a flat geometry.

The *cocarrier* of a round object $\mathbf{u}$, which we denote by $\operatorname{ccr}(\mathbf{u})$, is defined to be the carrier of the object's antidual $\mathbf{u}^{\star}$, which we can write as

Cocarrier

$$\boxed{\operatorname{ccr}(\mathbf{u}) = \mathbf{u}^{\star} \wedge \mathbf{e}_5.} \tag{4.37}$$

There is no geometric reason to choose the antidual over the dual here, and we do so only because it produces results with more favorable signs. The carriers and cocarriers of the round objects in 3D space are listed in Table 4.6.

For a round object $\mathbf{u}$ of grade k, the carrier of $\mathbf{u}$ has grade $k+1$, and the cocarrier of $\mathbf{u}$ has grade $n-k+1$, where n is the dimension of the algebra. The carrier and cocarrier both contain the center of the object, and their antiwedge product has grade two. Thus, the meet of the carrier and cocarrier is a flat point $\mathbf{p}$ coinciding with the center of the object that we can calculate with the formula

$$\mathbf{p} = \operatorname{ccr}(\mathbf{u}) \vee \operatorname{car}(\mathbf{u}). \tag{4.38}$$

This is not the formula we will typically use for the center of an object, however. Instead, we replace $\operatorname{car}(\mathbf{u})$ with $\mathbf{u}$ itself below because it provides a way to incorporate the radius into the center and express it as a round point.

Type	Carrier	Cocarrier
Round point $\mathbf{a}$	$\operatorname{car}(\mathbf{a}) = a_x\mathbf{e}_{15} + a_y\mathbf{e}_{25} + a_z\mathbf{e}_{35} + a_w\mathbf{e}_{45}$	$\operatorname{ccr}(\mathbf{a}) = a_w\mathbb{1}$
Dipole $\mathbf{d}$	$\operatorname{car}(\mathbf{d}) = d_{vx}\mathbf{e}_{415} + d_{vy}\mathbf{e}_{425} + d_{vz}\mathbf{e}_{435} + d_{mx}\mathbf{e}_{235} + d_{my}\mathbf{e}_{315} + d_{mz}\mathbf{e}_{125}$	$\operatorname{ccr}(\mathbf{d}) = d_{vx}\mathbf{e}_{4235} + d_{vy}\mathbf{e}_{4315} + d_{vz}\mathbf{e}_{4125} - d_{pw}\mathbf{e}_{3215}$
Circle $\mathbf{c}$	$\operatorname{car}(\mathbf{c}) = c_{gx}\mathbf{e}_{4235} + c_{gy}\mathbf{e}_{4315} + c_{gz}\mathbf{e}_{4125} + c_{gw}\mathbf{e}_{3215}$	$\operatorname{ccr}(\mathbf{c}) = -c_{gx}\mathbf{e}_{415} - c_{gy}\mathbf{e}_{425} - c_{gz}\mathbf{e}_{435} - c_{vx}\mathbf{e}_{235} - c_{vy}\mathbf{e}_{315} - c_{vz}\mathbf{e}_{125}$
Sphere $\mathbf{s}$	$\operatorname{car}(\mathbf{s}) = s_u\mathbb{1}$	$\operatorname{ccr}(\mathbf{s}) = s_x\mathbf{e}_{15} + s_y\mathbf{e}_{25} + s_z\mathbf{e}_{35} - s_u\mathbf{e}_{45}$

Table 4.6. These are the carriers and cocarriers of the round geometric objects arising in the conformal geometric algebra over 3D space, as defined by Equations (4.36) and (4.37).

4.2.4 Centers

The *center* of a round object is the round point having the same position and radius. The center of an object $\mathbf{u}$ is denoted by $\operatorname{cen}(\mathbf{u})$, and it is given by

Center

$$\boxed{\operatorname{cen}(\mathbf{u}) = \operatorname{ccr}(\mathbf{u}) \vee \mathbf{u},} \tag{4.39}$$

which is the meet of $\mathbf{u}$ and its own cocarrier. We will sometimes call this the *round center* to distinguish it from the flat center given by Equation (4.38). The squared radius of an object's round center has the same sign as the squared radius of the object itself. That is, the center of a real object is real, and the center of an imaginary object is imaginary. It's useful to express the center of a round object as a round point because it can serve as the anchor point for a parametric description of the object. The centers of the round objects in 3D space are listed in Table 4.7. As shown in the table, applying the function in Equation (4.39) to a round point returns the same geometry, as would be expected, but with some homogeneous scaling based on the point's weight.

Type	Center
Round point **a**	$\operatorname{cen}(\mathbf{a}) = a_x a_w \mathbf{e}_1 + a_y a_w \mathbf{e}_2 + a_z a_w \mathbf{e}_3 + a_w^2 \mathbf{e}_4 + a_w a_u \mathbf{e}_5$
Dipole **d**	$\operatorname{cen}(\mathbf{d}) = (d_{vy} d_{mz} - d_{vz} d_{my} + d_{vx} d_{pw}) \mathbf{e}_1 + (d_{vz} d_{mx} - d_{vx} d_{mz} + d_{vy} d_{pw}) \mathbf{e}_2 + (d_{vx} d_{my} - d_{vy} d_{mx} + d_{vz} d_{pw}) \mathbf{e}_3 + (d_{vx}^2 + d_{vy}^2 + d_{vz}^2) \mathbf{e}_4 + (d_{pw}^2 - d_{vx} d_{px} - d_{vy} d_{py} - d_{vz} d_{pz}) \mathbf{e}_5$
Circle **c**	$\operatorname{cen}(\mathbf{c}) = (c_{gy} c_{vz} - c_{gz} c_{vy} - c_{gx} c_{gw}) \mathbf{e}_1 + (c_{gz} c_{vx} - c_{gx} c_{vz} - c_{gy} c_{gw}) \mathbf{e}_2 + (c_{gx} c_{vy} - c_{gy} c_{vx} - c_{gz} c_{gw}) \mathbf{e}_3 + (c_{gx}^2 + c_{gy}^2 + c_{gz}^2) \mathbf{e}_4 + (c_{vx}^2 + c_{vy}^2 + c_{vz}^2 + c_{gx} c_{mx} + c_{gy} c_{my} + c_{gz} c_{mz}) \mathbf{e}_5$
Sphere **s**	$\operatorname{cen}(\mathbf{s}) = -s_x s_u \mathbf{e}_1 - s_y s_u \mathbf{e}_2 - s_z s_u \mathbf{e}_3 + s_u^2 \mathbf{e}_4 + (s_x^2 + s_y^2 + s_z^2 - s_w s_u) \mathbf{e}_5$

Table 4.7. These are the centers of the round geometric objects arising in the conformal geometric algebra over 3D space, as round points defined by Equation (4.39).

4.2.5 Containers

The *container* of a round object is the smallest sphere that contains it. The container of an object $\mathbf{u}$ is denoted by $\operatorname{con}(\mathbf{u})$, and it is given by

Container

$$\boxed{\operatorname{con}(\mathbf{u}) = \mathbf{u} \wedge \operatorname{car}(\mathbf{u})^{\star},} \tag{4.40}$$

which is an expansion of $\mathbf{u}$ onto its own carrier using the antidual operation. (The dual would also produce the same container, but we use the antidual to be consistent with the expansions appearing later in Section 4.8.) As with centers, the squared radius of an object's container has the same sign as the squared radius of the object itself. That is, a real object has a real container, and an imaginary object has an imaginary container. The containers of the round objects in 3D space are listed in Table 4.8. As shown in the table, applying the function in Equation (4.40) to a sphere returns the same geometry, as would be expected, but with some homogeneous scaling based on the sphere's weight.

4.2.6 Partners

The *partner* of a round object is the object of the same type having the same position, same carrier, and same absolute size, but having a squared radius of the opposite sign. The partner of an object $\mathbf{u}$ is denoted by $\operatorname{par}(\mathbf{u})$, and it is given by

Partner

$$\boxed{\operatorname{par}(\mathbf{u}) = (-1)^{\operatorname{gr}(\mathbf{u})+1} \operatorname{con}(\mathbf{u}^{\star}) \vee \operatorname{car}(\mathbf{u}),} \tag{4.41}$$

which is the meet of the container of $\mathbf{u}^{\star}$ and the carrier of $\mathbf{u}$. This formula works because taking the dual of $\mathbf{u}$ exchanges real and imaginary radii, and the container function keeps it that way. The power of -1 in front isn't technically necessary, but it conveniently preserves the orientation of the geometry $\mathbf{u}$. The partners of the round objects in 3D space are listed in Table 4.9, which shows that some of the calculations can be surprisingly complicated for such a simple change to the geometry.

Type	Container
Round point $\mathbf{a}$	$\operatorname{con}(\mathbf{a}) = -a_w^2\mathbf{e}_{1234} + a_x a_w \mathbf{e}_{4235} + a_y a_w \mathbf{e}_{4315} + a_z a_w \mathbf{e}_{4125} + \left(a_w a_u - a_x^2 - a_y^2 - a_z^2\right)\mathbf{e}_{3215}$
Dipole $\mathbf{d}$	$\operatorname{con}(\mathbf{d}) = \left(d_{vx}^2 + d_{vy}^2 + d_{vz}^2\right)\mathbf{e}_{1234}$ $+ (d_{vz}d_{my} - d_{vy}d_{mz} - d_{vx}d_{pw})\mathbf{e}_{4235}$ $+ (d_{vx}d_{mz} - d_{vz}d_{mx} - d_{vy}d_{pw})\mathbf{e}_{4315}$ $+ (d_{vy}d_{mx} - d_{vx}d_{my} - d_{vz}d_{pw})\mathbf{e}_{4125}$ $+ \left(d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{vx}d_{px} + d_{vy}d_{py} + d_{vz}d_{pz}\right)\mathbf{e}_{3215}$
Circle $\mathbf{c}$	$\operatorname{con}(\mathbf{c}) = -\left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2\right)\mathbf{e}_{1234}$ $+ (c_{gy}c_{vz} - c_{gz}c_{vy} - c_{gx}c_{gw})\mathbf{e}_{4235}$ $+ (c_{gz}c_{vx} - c_{gx}c_{vz} - c_{gy}c_{gw})\mathbf{e}_{4315}$ $+ (c_{gx}c_{vy} - c_{gy}c_{vx} - c_{gz}c_{gw})\mathbf{e}_{4125}$ $+ \left(c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz} - c_{gw}^2\right)\mathbf{e}_{3215}$
Sphere $\mathbf{s}$	$\operatorname{con}(\mathbf{s}) = s_u^2\mathbf{e}_{1234} + s_x s_u \mathbf{e}_{4235} + s_y s_u \mathbf{e}_{4315} + s_z s_u \mathbf{e}_{4125} + s_w s_u \mathbf{e}_{3215}$

Table 4.8. These are the containers of the round geometric objects arising in the conformal geometric algebra over 3D space, as spheres defined by Equation (4.40).

Type	Partner
Round point $\mathbf{a}$	$\operatorname{par}(\mathbf{a}) = a_x a_w^2\mathbf{e}_1 + a_y a_w^2\mathbf{e}_2 + a_z a_w^2\mathbf{e}_3 + a_w^3\mathbf{e}_4 + \left(a_x^2 + a_y^2 + a_z^2 - a_w a_u\right)a_w\mathbf{e}_5$
Dipole $\mathbf{d}$	$\operatorname{par}(\mathbf{d}) = \left(d_{vx}^2 + d_{vy}^2 + d_{vz}^2\right)(d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12} + d_{pw}\mathbf{e}_{45})$ $+ \left(d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - d_{vx}d_{px} - d_{vy}d_{py} - d_{vz}d_{pz}\right)(d_{vx}\mathbf{e}_{15} + d_{vy}\mathbf{e}_{25} + d_{vz}\mathbf{e}_{35})$ $+ (d_{mz}d_{vy} - d_{my}d_{vz})d_{pw}\mathbf{e}_{15} + (d_{mx}d_{vz} - d_{mz}d_{vx})d_{pw}\mathbf{e}_{25} + (d_{my}d_{vx} - d_{mx}d_{vy})d_{pw}\mathbf{e}_{35}$
Circle $\mathbf{c}$	$\operatorname{par}(\mathbf{c}) = \left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2\right)(c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412} + c_{gw}\mathbf{e}_{321} + c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435})$ $+ \left(c_{gw}^2 - c_{vx}^2 - c_{vy}^2 - c_{vz}^2 - c_{gx}c_{mx} - c_{gy}c_{my} - c_{gz}c_{mz}\right)(c_{gx}\mathbf{e}_{235} + c_{gy}\mathbf{e}_{315} + c_{gz}\mathbf{e}_{125})$ $+ (c_{vy}c_{gz} - c_{vz}c_{gy})c_{gw}\mathbf{e}_{235} + (c_{vz}c_{gx} - c_{vx}c_{gz})c_{gw}\mathbf{e}_{315} + (c_{vx}c_{gy} - c_{vy}c_{gx})c_{gw}\mathbf{e}_{125}$
Sphere $\mathbf{s}$	$\operatorname{par}(\mathbf{s}) = s_u^3\mathbf{e}_{1234} + s_x s_u^2\mathbf{e}_{4235} + s_y s_u^2\mathbf{e}_{4315} + s_z s_u^2\mathbf{e}_{4125} + \left(s_x^2 + s_y^2 + s_z^2 - s_w s_u\right)s_u\mathbf{e}_{3215}$

Table 4.9. These are the partners of the round geometric objects arising in the conformal geometric algebra over 3D space, as defined by Equation (4.41).

4.2.7 Attitude

The attitude function that we first defined in Section 2.8.4 continues to apply in conformal geometric algebra. In 3D space, we extract the attitude of an object with the formula

Attitude

$$\boxed{\operatorname{att}(\mathbf{u}) = \mathbf{u} \vee \overline{\mathbf{e}}_4,} \tag{4.42}$$

which removes any component containing a factor of $\mathbf{e}_4$. The attitudes of all flat objects and round objects in 3D space are listed in Table 4.10. For flat objects, the attitudes listed here match the attitudes shown in Table 2.9, but now each one has an additional factor of $\mathbf{e}_5$. For round objects, we can see that the attitude of the carrier is extracted, but there is some additional information as well, and it has to do with the radii of surface points. This will enable parametric formulations of the surfaces of round objects later in Section 4.6.

Type	Attitude
Flat point $\mathbf{p}$	$\operatorname{att}(\mathbf{p}) = p_w \mathbf{e}_5$
Line $\boldsymbol{l}$	$\operatorname{att}(\boldsymbol{l}) = l_{vx}\mathbf{e}_{15} + l_{vy}\mathbf{e}_{25} + l_{vz}\mathbf{e}_{35}$
Plane $\mathbf{g}$	$\operatorname{att}(\mathbf{g}) = g_x\mathbf{e}_{235} + g_y\mathbf{e}_{315} + g_z\mathbf{e}_{125}$
Round point $\mathbf{a}$	$\operatorname{att}(\mathbf{a}) = a_w\mathbf{1}$
Dipole $\mathbf{d}$	$\operatorname{att}(\mathbf{d}) = d_{vx}\mathbf{e}_1 + d_{vy}\mathbf{e}_2 + d_{vz}\mathbf{e}_3 + d_{pw}\mathbf{e}_5$
Circle $\mathbf{c}$	$\operatorname{att}(\mathbf{c}) = c_{gx}\mathbf{e}_{23} + c_{gy}\mathbf{e}_{31} + c_{gz}\mathbf{e}_{12} + c_{vx}\mathbf{e}_{15} + c_{vy}\mathbf{e}_{25} + c_{vz}\mathbf{e}_{35}$
Sphere $\mathbf{s}$	$\operatorname{att}(\mathbf{s}) = s_u\mathbf{e}_{321} + s_x\mathbf{e}_{235} + s_y\mathbf{e}_{315} + s_z\mathbf{e}_{125}$

Table 4.10. These are the attitudes of the geometric objects arising in the conformal geometric algebra over 3D space, as defined by Equation (4.42).

Math Library Notes

- The dual and antidual operations are implemented by the `Dual()` and `Antidual()` functions for round objects only.
- The carriers and cocarriers of round objects are returned by the `Carrier()` and `Cocarrier()` functions.
- The `Center()` function calculates the center of a `RoundPoint3D`, `Dipole3D`, `Circle3D`, or `Sphere3D` object. The return value always has type `RoundPoint3D`.
- The `Container()` function calculates the container of a `RoundPoint3D`, `Dipole3D`, `Circle3D`, or `Sphere3D` object. The return value always has type `Sphere3D`.
- The `Partner()` function calculates the partner of a `RoundPoint3D`, `Dipole3D`, `Circle3D`, or `Sphere3D` object. The return value always has the same type as its input.
- The `Attitude()` function returns the attitude of a `RoundPoint3D`, `Dipole3D`, `Circle3D`, or `Sphere3D` object. The return type is a floating-point value, a `RoundPoint3D` object, a `Dipole3D` object, and a `Circle3D` object, respectively.

4.3 Norms

In Section 4.2.1, we divided the components of a geometric object into a round part and a flat part, which are those components that do not have a factor of $\mathbf{e}_5$ and those that do have a factor of $\mathbf{e}_5$, respectively. Each of those parts can be further divided into a bulk and weight depending on whether a factor of $\mathbf{e}_4$ is present. This lets us classify all the components of an object as belonging to one of four parts as follows.

- The *round bulk* of $\mathbf{u}$, denoted by $\mathbf{u}_\bullet$ with a solid black circle written as a subscript, consists of all components of $\mathbf{u}$ that do not have a factor of either $\mathbf{e}_5$ or $\mathbf{e}_4$. The round bulk has the same meaning as in the rigid algebra, and it is equivalent to the bulk of the carrier geometry for round objects. The round bulk for flat objects is always zero.
- The *round weight* of $\mathbf{u}$, denoted by $\mathbf{u}_\circ$ with an empty white circle written as a subscript, consists of all components of $\mathbf{u}$ that do not have a factor of $\mathbf{e}_5$ but do have a factor of $\mathbf{e}_4$. The round weight has the same meaning as in the rigid algebra, and it is equivalent to the weight of the carrier geometry for round objects. The round weight for flat objects is always zero.
- The *flat bulk* of $\mathbf{u}$, denoted by $\mathbf{u}_\blacksquare$ with a solid black square written as a subscript, consists of all components of $\mathbf{u}$ that have a factor of $\mathbf{e}_5$ but do not have a factor of $\mathbf{e}_4$. For flat objects, the flat bulk has the same meaning as bulk does in the rigid algebra.
- The *flat weight* of $\mathbf{u}$, denoted by $\mathbf{u}_\square$ with an empty white square written as a subscript, consists of all components of $\mathbf{u}$ that have a factor of both $\mathbf{e}_5$ and $\mathbf{e}_4$. For flat objects, the flat weight has the same meaning as weight does in the rigid algebra.

The bulk and weight of the round and flat parts of each type of geometry arising in conformal geometric algebra are listed in Table 4.11.

By measuring the size of the components belonging to each of the four separate parts of an object $\mathbf{u}$, we can define four separate norms. Naturally, we call these the round bulk norm $\|\mathbf{u}\|_\bullet$, the round weight norm $\|\mathbf{u}\|_\circ$, the flat bulk norm $\|\mathbf{u}\|_\blacksquare$, and the flat weight norm $\|\mathbf{u}\|_\square$, where we are using the same kind of notation as we did for the bulk and weight norms in the rigid algebra. All four norms are listed for each type of geometry in Table 4.12. As with the separate treatment of the bulk and weight norms in the rigid algebra, we are using the term "norm" loosely here. True norms need to measure a distance that satisfies the requirements that we previously listed in Table 2.14.

For flat objects, the ratio of the flat bulk norm to the flat weight norm is equivalent to the geometric norm in the rigid algebra, and its value is the distance from the origin to the object. For round objects, the ratio of the round bulk norm to the round weight norm is the distance from the object to the carrier geometry. The round weight always contains the homogeneous weight of a round object, so unitizing a round object amounts to making the round weight norm have unit magnitude. To remove homogeneous scaling, all other norms must be divided by the round weight norm.

The flat part of a round object contains information about the center and radius. In particular, the flat weight norm is the weighted distance between the support of the carrier and the center of an object. This means that the round bulk norm and flat weight norm form the sides of a right triangle in which the hypotenuse connects the origin to the center, as shown in Figure 4.6. That being the case, we define the *center norm* of a round object $\mathbf{u}$ as

Center norm

$$\boxed{\|\mathbf{u}\|_\odot = \sqrt{\|\mathbf{u}\|_\bullet^2 + \|\mathbf{u}\|_\square^2}.} \tag{4.43}$$

The center norm is denoted by double vertical bars with a circle containing a dot at its center written as a subscript. The center norm is weighted, so we need to divide by the round weight norm $\|\mathbf{u}\|_\circ$ to get the actual distance between the center and the origin. The center norms for the four types of round object in 3D space are listed in Table 4.13.

Whenever an inner product is defined, it induces a norm. In conformal geometric algebra, we have the dot product and antidot product, and their magnitudes are negatives of each other due to the the metric and antimetric having the relationship $\mathbb{G} = -\mathbf{G}$. When we take the dot product of a round object $\mathbf{u}$ with itself using any of the representations given in Section 4.2.1, we find it is always the case that

$$\mathbf{u} \bullet \mathbf{u} = -r^2, \tag{4.44}$$

where r is the radius of the object. If we take a square root of this value, then we'll get an imaginary number when $\mathbf{u}$ has a real radius and a real number when $\mathbf{u}$ has an imaginary radius. To avoid this reversal of real and imaginary values, we instead calculate the radius of an object with the square root of the antidot product because $\mathbf{u} \circ \mathbf{u} = r^2 \mathbb{1}$. The *radius norm* of a round object $\mathbf{u}$ is denoted by $\|\mathbf{u}\|_{\oslash}$ with a circle containing a radial line written as a subscript, and it is defined by

Radius norm

$$\boxed{\|\mathbf{u}\|_{\oslash} = \sqrt{\mathbf{u} \circ \mathbf{u}}.} \tag{4.45}$$

This norm produces real values for real objects and imaginary values for imaginary objects. The radius norm is weighted just like all the other norms, so it needs to be divided by the round weight norm $\|\mathbf{u}\|_{\bigcirc}$ in order to calculate the actual radius of $\mathbf{u}$. The radius norm for the four types of round object in 3D space are listed in Table 4.13.

Type	Round bulk and weight	Flat bulk and weight
Flat point $\mathbf{p}$	$\mathbf{p}_\bullet = 0$ $\mathbf{p}_\circ = 0$	$\mathbf{p}_\blacksquare = p_x \mathbf{e}_{15} + p_y \mathbf{e}_{25} + p_z \mathbf{e}_{35}$ $\mathbf{p}_\square = p_w \mathbf{e}_{45}$
Line $\boldsymbol{l}$	$\boldsymbol{l}_\bullet = 0$ $\boldsymbol{l}_\circ = 0$	$\boldsymbol{l}_\blacksquare = l_{mx} \mathbf{e}_{235} + l_{my} \mathbf{e}_{315} + l_{mz} \mathbf{e}_{125}$ $\boldsymbol{l}_\square = l_{vx} \mathbf{e}_{415} + l_{vy} \mathbf{e}_{425} + l_{vz} \mathbf{e}_{435}$
Plane $\mathbf{g}$	$\mathbf{g}_\bullet = 0$ $\mathbf{g}_\circ = 0$	$\mathbf{g}_\blacksquare = g_w \mathbf{e}_{3215}$ $\mathbf{g}_\square = g_x \mathbf{e}_{4235} + g_y \mathbf{e}_{4315} + g_z \mathbf{e}_{4125}$
Round point $\mathbf{a}$	$\mathbf{a}_\bullet = a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3$ $\mathbf{a}_\circ = a_w \mathbf{e}_4$	$\mathbf{a}_\blacksquare = a_u \mathbf{e}_5$ $\mathbf{a}_\square = 0$
Dipole $\mathbf{d}$	$\mathbf{d}_\bullet = d_{mx} \mathbf{e}_{23} + d_{my} \mathbf{e}_{31} + d_{mz} \mathbf{e}_{12}$ $\mathbf{d}_\circ = d_{vx} \mathbf{e}_{41} + d_{vy} \mathbf{e}_{42} + d_{vz} \mathbf{e}_{43}$	$\mathbf{d}_\blacksquare = d_{px} \mathbf{e}_{15} + d_{py} \mathbf{e}_{25} + d_{pz} \mathbf{e}_{35}$ $\mathbf{d}_\square = d_{pw} \mathbf{e}_{45}$
Circle $\mathbf{c}$	$\mathbf{c}_\bullet = c_{gw} \mathbf{e}_{321}$ $\mathbf{c}_\circ = c_{gx} \mathbf{e}_{423} + c_{gy} \mathbf{e}_{431} + c_{gz} \mathbf{e}_{412}$	$\mathbf{c}_\blacksquare = c_{mx} \mathbf{e}_{235} + c_{my} \mathbf{e}_{315} + c_{mz} \mathbf{e}_{125}$ $\mathbf{c}_\square = c_{vx} \mathbf{e}_{415} + c_{vy} \mathbf{e}_{425} + c_{vz} \mathbf{e}_{435}$
Sphere $\mathbf{s}$	$\mathbf{s}_\bullet = 0$ $\mathbf{s}_\circ = s_u \mathbf{e}_{1234}$	$\mathbf{s}_\blacksquare = s_w \mathbf{e}_{3215}$ $\mathbf{s}_\square = s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125}$

Table 4.11. For each type of geometric object $\mathbf{u}$ in the conformal geometric algebra over 3D space, this table lists the round bulk $\mathbf{u}_\bullet$, the round weight $\mathbf{u}_\circ$, the flat bulk $\mathbf{u}_\blacksquare$, and the flat weight $\mathbf{u}_\square$.

Type	Round bulk norm	Round weight norm	Flat bulk norm	Flat weight norm
Flat point $\mathbf{p}$	$\lVert\mathbf{p}\rVert_\bullet = 0$	$\lVert\mathbf{p}\rVert_\circ = 0$	$\lVert\mathbf{p}\rVert_\blacksquare = \sqrt{p_x^2 + p_y^2 + p_z^2}$	$\lVert\mathbf{p}\rVert_\square = \lvert p_w \rvert$
Line $\boldsymbol{l}$	$\lVert\boldsymbol{l}\rVert_\bullet = 0$	$\lVert\boldsymbol{l}\rVert_\circ = 0$	$\lVert\boldsymbol{l}\rVert_\blacksquare = \sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$	$\lVert\boldsymbol{l}\rVert_\square = \sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Plane $\mathbf{g}$	$\lVert\mathbf{g}\rVert_\bullet = 0$	$\lVert\mathbf{g}\rVert_\circ = 0$	$\lVert\mathbf{g}\rVert_\blacksquare = \lvert g_w \rvert$	$\lVert\mathbf{g}\rVert_\square = \sqrt{g_x^2 + g_y^2 + g_z^2}$
Round point $\mathbf{a}$	$\lVert\mathbf{a}\rVert_\bullet = \sqrt{a_x^2 + a_y^2 + a_z^2}$	$\lVert\mathbf{a}\rVert_\circ = \lvert a_w \rvert$	$\lVert\mathbf{a}\rVert_\blacksquare = \lvert a_u \rvert$	$\lVert\mathbf{a}\rVert_\square = 0$
Dipole $\mathbf{d}$	$\lVert\mathbf{d}\rVert_\bullet = \sqrt{d_{mx}^2 + d_{my}^2 + d_{mz}^2}$	$\lVert\mathbf{d}\rVert_\circ = \sqrt{d_{vx}^2 + d_{vy}^2 + d_{vz}^2}$	$\lVert\mathbf{d}\rVert_\blacksquare = \sqrt{d_{px}^2 + d_{py}^2 + d_{pz}^2}$	$\lVert\mathbf{d}\rVert_\square = \lvert d_{pw} \rvert$
Circle $\mathbf{c}$	$\lVert\mathbf{c}\rVert_\bullet = \lvert c_{gw} \rvert$	$\lVert\mathbf{c}\rVert_\circ = \sqrt{c_{gx}^2 + c_{gy}^2 + c_{gz}^2}$	$\lVert\mathbf{c}\rVert_\blacksquare = \sqrt{c_{mx}^2 + c_{my}^2 + c_{mz}^2}$	$\lVert\mathbf{c}\rVert_\square = \sqrt{c_{vx}^2 + c_{vy}^2 + c_{vz}^2}$
Sphere $\mathbf{s}$	$\lVert\mathbf{s}\rVert_\bullet = 0$	$\lVert\mathbf{s}\rVert_\circ = \lvert s_u \rvert$	$\lVert\mathbf{s}\rVert_\blacksquare = \lvert s_w \rvert$	$\lVert\mathbf{s}\rVert_\square = \sqrt{s_x^2 + s_y^2 + s_z^2}$

Table 4.12. These are the round bulk norms, round weight norms, flat bulk norms, and flat weight norms of geometric objects in the conformal geometric algebra over 3D space.

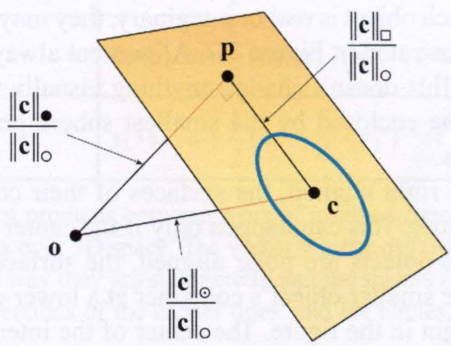


Figure 4.6. The circle $\mathbf{c}$ lies in a carrier plane with support $\mathbf{p}$. The round bulk norm $\lVert\mathbf{c}\rVert_\bullet$ corresponds to the perpendicular distance between the origin $\mathbf{o}$ and the plane, which is the length of the line segment connecting the origin and the support $\mathbf{p}$. The flat weight norm $\lVert\mathbf{c}\rVert_\square$ corresponds to the distance between the support $\mathbf{p}$ and the center of the circle $\mathbf{c}$ in the carrier plane. These form the sides of a right triangle for which the hypotenuse corresponds to the distance between the origin and the center of the circle, given by the center norm $\lVert\mathbf{c}\rVert_\odot$. All of the norms are homogeneously scaled and must be divided by the round weight norm $\lVert\mathbf{c}\rVert_\circ$ if the circle is not unitized.

Type	Center Norm	Radius Norm
Round point $\mathbf{a}$	$\lVert\mathbf{a}\rVert_\odot = \sqrt{a_x^2 + a_y^2 + a_z^2}$	$\lVert\mathbf{a}\rVert_\oslash = \sqrt{2a_w a_u - a_x^2 - a_y^2 - a_z^2}$
Dipole $\mathbf{d}$	$\lVert\mathbf{d}\rVert_\odot = \sqrt{d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{pw}^2}$	$\lVert\mathbf{d}\rVert_\oslash = \sqrt{d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - 2(d_{px}d_{vx} + d_{py}d_{vy} + d_{pz}d_{vz})}$
Circle $\mathbf{c}$	$\lVert\mathbf{c}\rVert_\odot = \sqrt{c_{gw}^2 + c_{vx}^2 + c_{vy}^2 + c_{vz}^2}$	$\lVert\mathbf{c}\rVert_\oslash = \sqrt{c_{vx}^2 + c_{vy}^2 + c_{vz}^2 - c_{gw}^2 + 2(c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz})}$
Sphere $\mathbf{s}$	$\lVert\mathbf{s}\rVert_\odot = \sqrt{s_x^2 + s_y^2 + s_z^2}$	$\lVert\mathbf{s}\rVert_\oslash = \sqrt{s_x^2 + s_y^2 + s_z^2 - 2s_w s_u}$

Table 4.13. This table lists the center norm $\lVert\mathbf{u}\rVert_\odot$ and radius norm $\lVert\mathbf{u}\rVert_\oslash$ for the round objects in the conformal geometric algebra over 3D space.

Math Library Notes

- The `SquaredCenterNorm()` and `SquaredRadiusNorm()` functions return the squares of the center norm and radius norm. Squares are returned so the quotient can be taken before a single square root is applied.
- The `SquaredBulkNorm()` and `SquaredWeightNorm()` functions return the squares of the round bulk norm and round weight norm. Their names do not contain the word `Round` so the same functions can be used with flat objects and round objects.
- The `SquaredFlatBulkNorm()` and `SquaredFlatWeightNorm()` functions return the squares of the flat bulk norm and flat weight norm.
- Round points, dipoles, circles, and spheres can be scaled to have unit weight by calling the `Unitize()` function.

4.4 Alignment

There are two types of alignment between round objects that appear everywhere in conformal geometric algebra. Two objects are "aligned" when they are algebraically orthogonal, as described in Section 4.5, or when one of the objects contains the other, as described in Section 4.6. Depending on whether the radius of each object is real or imaginary, they may be *right* aligned or *polar* aligned, and these two types are illustrated in Figure 4.7. Alignment always applies to the containers of the objects that are involved. This doesn't change anything visually for spheres and round points, but dipoles and circles must be enclosed by the smallest sphere possible in order to visualize their alignments to other objects.

When two objects are right aligned, the surfaces of their containers meet at right angles, as shown on the left in the figure. This can happen only if the center of each container lies outside the other container. When two objects are polar aligned, the surface of the larger object's container intersects the surface of the smaller object's container at a lower-dimensional surface of maximum radius, as shown on the right in the figure. The center of the intersection coincides with the center of the smaller object, and this center must lie inside the container of the larger object.

It is also possible for a round object to be aligned to a flat object. In the case, the flat object simply passes through the center of the round object. Such a geometric configuration satisfies the definitions of both right alignment and polar alignment.

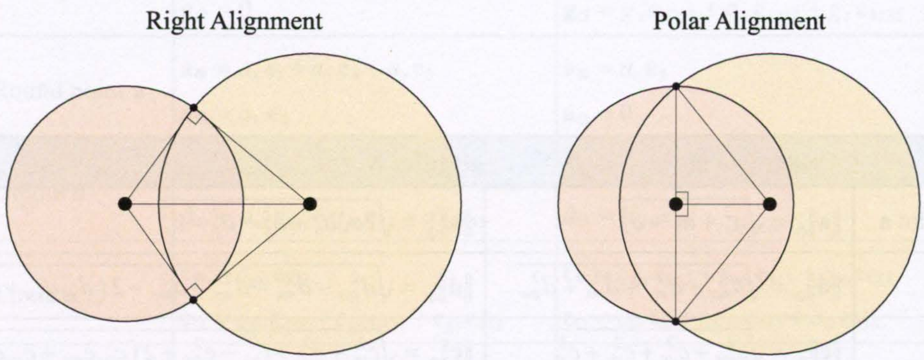


Figure 4.7. When two round objects are aligned because they are orthogonal or one contains the other, there are two possible configurations. (Left) Two objects are right aligned when the surfaces of their containers meet at right angles. (Right) Two objects are polar aligned when the surface of the larger object's container intersects the surface of the smaller object's container at a surface having a center that coincides with the the smaller object's center.

4.5 Dot Products

When we take the dot product between two unitized round objects $\mathbf{u}_1$ and $\mathbf{u}_2$ of the same type, we obtain a real number that depends on the difference $\mathbf{v}$ between their center positions and their radii r_1 and r_2. In the case of dipoles and circles, the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ corresponding to either the directions of the carrier lines or the normals of the carrier planes are also involved. The absolute positions of the objects do not matter, and in this setting, the dot product has the unusual property that it is not affected by a translation of the coordinate system. Formulas for the dot products between pairs of round objects in 3D space are listed in Table 4.14. The results produced for dipoles and circles are rather involved, but it's not too difficult to interpret the dot products of round points and spheres geometrically.

Type	Dot Product
Round points $\mathbf{a}_1$ and $\mathbf{a}_2$	$\mathbf{a}_1 \bullet \mathbf{a}_2 = -\frac{1}{2}\left(\mathbf{v}^2 + r_1^2 + r_2^2\right)$
Dipoles $\mathbf{d}_1$ and $\mathbf{d}_2$	$\mathbf{d}_1 \bullet \mathbf{d}_2 = -\frac{1}{2}(\mathbf{n}_1 \cdot \mathbf{n}_2)\left(\mathbf{v}^2 + r_1^2 + r_2^2\right) + (\mathbf{n}_1 \cdot \mathbf{v})(\mathbf{n}_2 \cdot \mathbf{v})$
Circles $\mathbf{c}_1$ and $\mathbf{c}_2$	$\mathbf{c}_1 \bullet \mathbf{c}_2 = +\frac{1}{2}(\mathbf{n}_1 \cdot \mathbf{n}_2)\left(\mathbf{v}^2 - r_1^2 - r_2^2\right) - (\mathbf{n}_1 \cdot \mathbf{v})(\mathbf{n}_2 \cdot \mathbf{v})$
Spheres $\mathbf{s}_1$ and $\mathbf{s}_2$	$\mathbf{s}_1 \bullet \mathbf{s}_2 = +\frac{1}{2}\left(\mathbf{v}^2 - r_1^2 - r_2^2\right)$

Table 4.14. These are the dot products between pairs of unitized round objects having the same type in the conformal geometric algebra over 3D space. The vector $\mathbf{v}$ is the difference between the centers of the objects (and it doesn't matter which way they are subtracted), and the scalars r_1 and r_2 are their radii. For dipoles, the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are the directions of the carrier lines, and for circles, the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normals of the carrier planes.

4.5.1 Round Points

Figure 4.8 shows the geometric relationship between two round points $\mathbf{a}_1$ and $\mathbf{a}_2$ as it pertains to the dot product $\mathbf{a}_1 \bullet \mathbf{a}_2$. We consider the three different cases that are possible, that both points are real, that one point is real and the other is imaginary, and that both points are imaginary. In all cases, the dot product gives us $\mathbf{v}^2 + r_1^2 + r_2^2$ multiplied by $-\frac{1}{2}$, but the geometric meaning of this number changes a little depending on whether r_1^2 and r_2^2 are positive or negative. In particular, what it means for two points to have a dot product of zero, and thus be considered orthogonal, is different among the three possible cases.

In the case that both round points $\mathbf{a}_1$ and $\mathbf{a}_2$ are real, we know that $r_1^2 > 0$ and $r_2^2 > 0$. As shown in the first row of Figure 4.8, the value $\mathbf{v}^2 + r_1^2 + r_2^2$ can be interpreted as the squared length of the hypotenuse of a right triangle in which the square of one leg is r_2^2 and the square of the other leg is $\mathbf{v}^2 + r_1^2$. The value $\mathbf{v}^2 + r_1^2$ is itself the squared length of the hypotenuse of another right triangle in which the two legs have squared lengths r_1^2 and $\mathbf{v}^2$. Since all of these values are positive, there are no nonzero real radii for which $\mathbf{a}_1 \bullet \mathbf{a}_2 = 0$, and consequently, no pair of real round points can ever be orthogonal. If both radii are reduced to zero, making both $\mathbf{a}_1$ and $\mathbf{a}_2$ null points, then the dot product between them becomes

$$\mathbf{a}_1 \bullet \mathbf{a}_2 = -\frac{1}{2}\mathbf{v}^2, \tag{4.46}$$

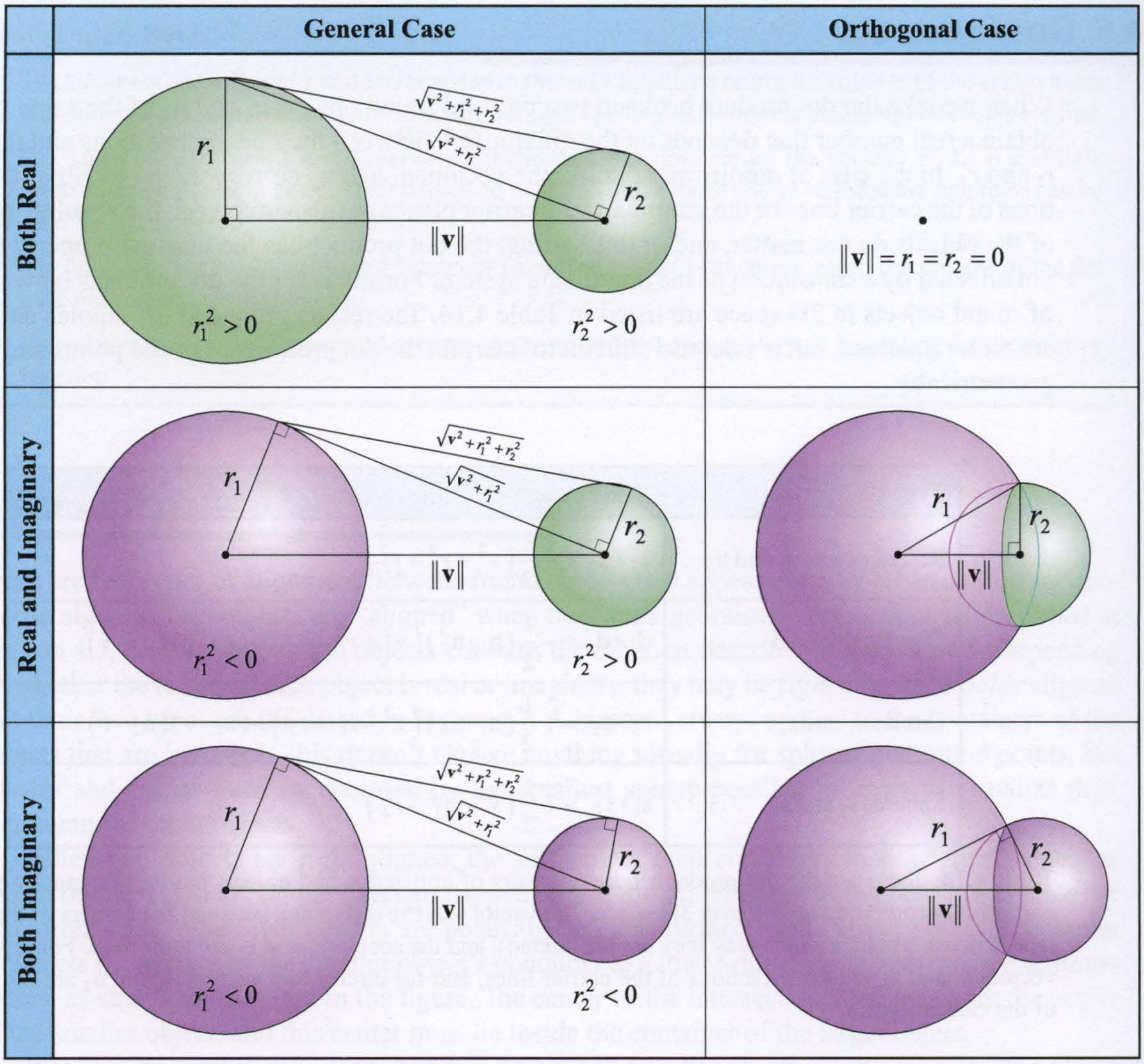


Figure 4.8. The dot product between two round points with radii r_1 and r_2 can be interpreted in three distinct ways depending on whether each is real (green) or imaginary (purple). The vector $\mathbf{v}$ represents the difference between the positions of the points. The right column illustrates the alignment when the points are considered orthogonal because the dot product between them is zero.

which is the squared distance between the two points multiplied by $-\frac{1}{2}$. (Remember that the points are unitized here, and the right side would be multiplied by their weights otherwise.) If $\mathbf{a}_1$ and $\mathbf{a}_2$ are the same point, but with a nonzero real radius r, then the distance between them is zero, and the dot product simplifies to Equation (4.44), which measures the squared radius. The only way to obtain a dot product of zero without using imaginary radii is to take the dot product of a round point having radius zero with itself.

In the case that $\mathbf{a}_1$ is imaginary and $\mathbf{a}_2$ is real, we have $r_1^2 < 0$ and $r_2^2 > 0$, and things get a little strange in the second row of Figure 4.8 because the diagrams include some imaginary lengths. Nevertheless, the use of lengths that square to negative numbers allows us to derive some geometric intuition from the algebraic relationship between the two points. The value $\mathbf{v}^2 + r_1^2 + r_2^2$ is still interpreted as the squared length of the hypotenuse of a right triangle, but it can become negative when the two points are close enough together. The squares of the legs are still r_2^2 and $\mathbf{v}^2 + r_1^2$, but in this case, $\mathbf{v}^2 + r_1^2$ must be less than $\mathbf{v}^2$ because $r_1^2 < 0$. This time, $\mathbf{v}^2$ is the squared length of the hypotenuse of the second triangle, and the squared lengths of the triangle's legs are r_1^2 and $\mathbf{v}^2 + r_1^2$.

Since one of the points is imaginary, it is possible to obtain a dot product of zero between them even if they both have nonzero radii and different center positions. An imaginary point $\mathbf{a}_1$ and a real point $\mathbf{a}_2$ are orthogonal precisely when

$$\mathbf{v}^2 + r_2^2 = |r_1|^2. \tag{4.47}$$

This configuration is illustrated in the second row of the figure, and it shows that $\|\mathbf{v}\|$ and r_2 are the legs of a right triangle for which the hypotenuse has the imaginary length r_1. The absolute value of r_1 must be larger than r_2 for this to be possible when r_2 is real. As demonstrated in the figure, when an imaginary point and real point are orthogonal, they are polar aligned.

In the final case that both round points $\mathbf{a}_1$ and $\mathbf{a}_2$ are imaginary, we have $r_1^2 < 0$ and $r_2^2 < 0$. We assume that the points have been numbered such that $|r_1| \geq |r_2|$ so that $\mathbf{a}_1$ is larger than $\mathbf{a}_2$ (or the same size) as drawn in the third row of Figure 4.8. Since the radii square to negative numbers, the value $\mathbf{v}^2 + r_1^2 + r_2^2$ is guaranteed to be smaller than $\mathbf{v}^2 + r_1^2$, and both of these values can be negative when the two points are close enough together. The value $\mathbf{v}^2 + r_1^2$ must also be smaller than $\mathbf{v}^2$. As a result of these relationships, $\mathbf{v}^2 + r_1^2 + r_2^2$ is the squared length of one leg of a right triangle for which $\mathbf{v}^2 + r_1^2$ is the squared length of the hypotenuse, and that becomes one leg of another right triangle for which $\mathbf{v}^2$ is the squared length of the hypotenuse. Two imaginary points are orthogonal precisely when

$$|r_1|^2 + |r_2|^2 = \mathbf{v}^2, \tag{4.48}$$

and this special case is shown in the third row of the figure. Here, r_1 and r_2 are the imaginary lengths of the legs of a right triangle for which the hypotenuse has the real length $\|\mathbf{v}\|$. When two imaginary points are orthogonal, they are right aligned.

4.5.2 Spheres

Since round points and spheres are duals of each other, they share similar properties. The dot product between two spheres $\mathbf{s}_1$ and $\mathbf{s}_2$ shown in Table 4.14 has the same form as the dot product between two points with the only difference being some sign changes that must exist due to the relationship $\mathbb{G} = -\mathbf{G}$ between the metric and antimetric. The squares of the radii are both negated because the dual of a real point is an imaginary sphere, and the dual of an imaginary point is a real sphere. Thus, as shown in the first row of Figure 4.9, the dot product between two real spheres behaves just like the dot product between two imaginary points except that it has the opposite sign. Two real spheres with centers separated by the vector $\mathbf{v}$ are orthogonal when $r_1^2 + r_2^2 = \mathbf{v}^2$, and they are right aligned just like two imaginary points. A real sphere $\mathbf{s}_1$ and an imaginary sphere $\mathbf{s}_2$ are orthogonal when $\mathbf{v}^2 + r_2^2 = |r_1|^2$, and they are polar aligned just like mixed points. Finally, since two real points can never be orthogonal, it must also be true that two imaginary spheres can never be orthogonal. The only way that the dot product between two non-real spheres can be zero is if they are both null spheres with the same center position.

4.5.3 Partners

If we consider the case of a round object $\mathbf{u}$ and its partner, then we find that they are always orthogonal. Recall that $\operatorname{par}(\mathbf{u})$ has the same center and absolute size, but its radius is changed from real to imaginary or vice versa. Thus, in each of the formulas listed in Table 4.14, the difference $\mathbf{v}$ between the centers is zero, and the radii must be related by $r_1^2 = -r_2^2$. For dipoles and circles, the attitudes of the carriers also match, so we have $\mathbf{n}_1 = \mathbf{n}_2$. Under these conditions, all four dot products in the table become zero, and we can state that in general,

$$\boxed{\mathbf{u} \bullet \operatorname{par}(\mathbf{u}) = 0.} \tag{4.49}$$

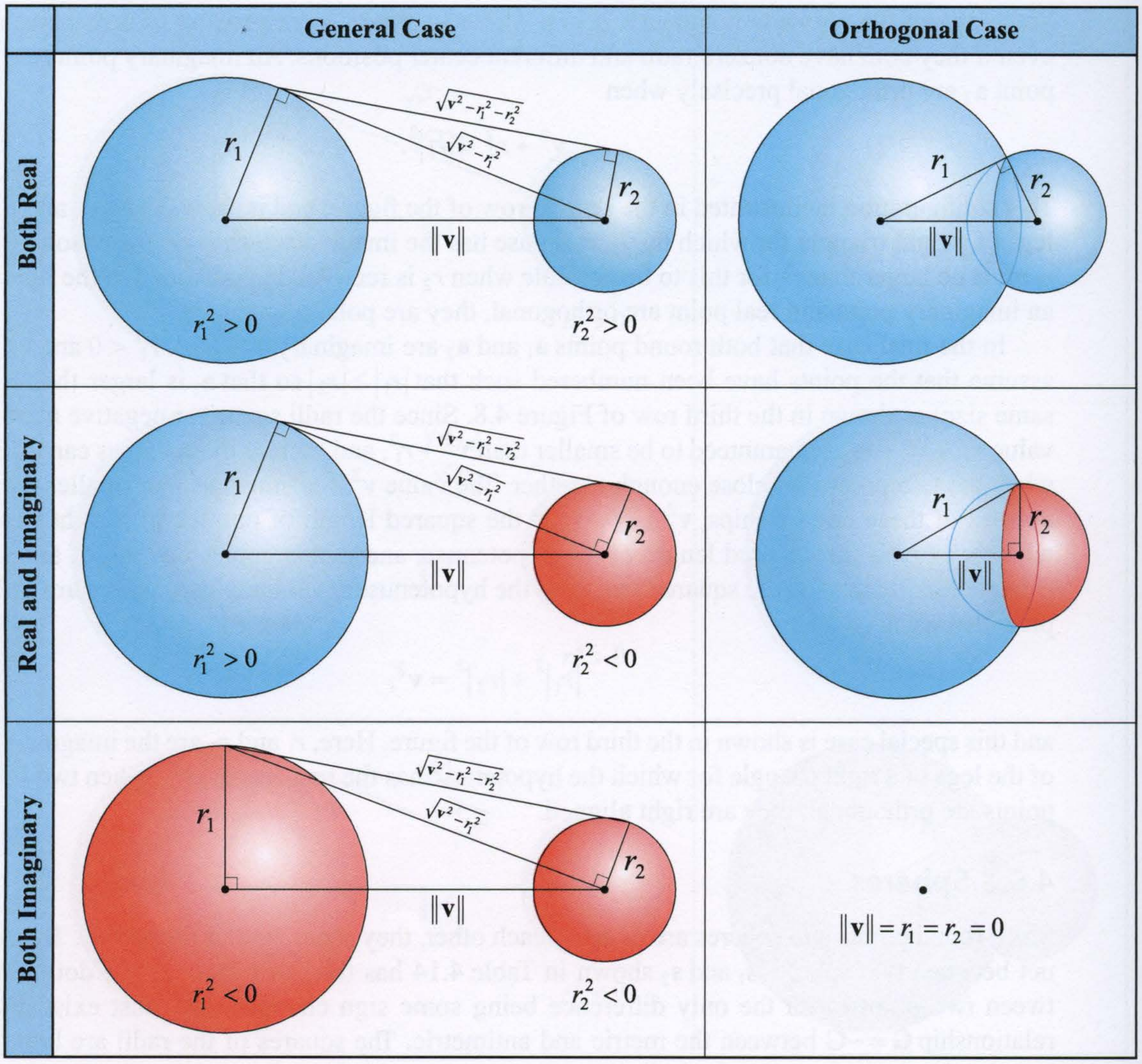


Figure 4.9. The dot product between two spheres with radii r_1 and r_2 can be interpreted in three distinct ways depending on whether each is real (blue) or imaginary (red). The vector $\mathbf{v}$ represents the difference between the centers of the spheres. The right column illustrates the alignment when spheres are considered orthogonal because the dot product between them is zero.

4.5.4 Conjugates

For every round object $\mathbf{u}$, there is another round object of the same type that we call the *conformal conjugate*, and it has a special relationship with $\mathbf{u}$. The conformal conjugate is calculated by negating the flat part of $\mathbf{u}$ while leaving the round part of $\mathbf{u}$ unchanged. Put another way, every component of $\mathbf{u}$ containing a factor of $\mathbf{e}_5$ is negated. Using the dagger notation $\mathbf{u}^\dagger$ to represent the conformal conjugate of $\mathbf{u}$, we can write this definition as

Conformal conjugate

$$\boxed{\mathbf{u}^\dagger = \mathbf{u}_\bullet + \mathbf{u}_\circ - \mathbf{u}_\blacksquare - \mathbf{u}_\square.} \tag{4.50}$$

Negating the flat part of $\mathbf{u}$ has the effect of changing the center position and radius in such a way that the dot product $\mathbf{u} \bullet \mathbf{u}^\dagger$ is the squared distance to the origin. This property lets us formulate an alternate definition for the center norm $\|\mathbf{u}\|_\odot$ that we previously defined in Equation (4.43). We can now express the distance between the origin and the center of an object as

Center norm (alternate)

$$\boxed{\|\mathbf{u}\|_{\odot} = \sqrt{\mathbf{u} \bullet \mathbf{u}^{\dagger}}.} \tag{4.51}$$

As with the original center norm, this alternate center norm is weighted and needs to be divided by the round weight norm $\|\mathbf{u}\|_{\bigcirc}$ if $\mathbf{u}$ is not unitized.

The conformal conjugate operation is an involution, and the conjugate of $\mathbf{u}^{\dagger}$ is just $\mathbf{u}$ itself. This means that $\|\mathbf{u}^{\dagger}\|_{\odot} = \|\mathbf{u}\|_{\odot}$, which says the conjugate of $\mathbf{u}$ must have a center that's the same distance from the origin as the center of $\mathbf{u}$. In the case of a unitized round point $\mathbf{a}$, only the a_u coordinate is negated by the conjugate operation, so $\mathbf{a}^{\dagger}$ has the same center as $\mathbf{a}$. However, by comparing the squared radius r_1^2 of $\mathbf{a}$ and the squared radius r_2^2 of $\mathbf{a}^{\dagger}$ using the radius norm listed in Table 4.13, we see that

$$r_1^2 = \|\mathbf{a}\|_{\oslash}^2 = 2a_u - a_x^2 - a_y^2 - a_z^2 \tag{4.52}$$

and

$$r_2^2 = \|\mathbf{a}^{\dagger}\|_{\oslash}^2 = -2a_u - a_x^2 - a_y^2 - a_z^2 = -2\left(a_x^2 + a_y^2 + a_z^2\right) - r_1^2, \tag{4.53}$$

where we assume $a_w = 1$. Plugging these into the dot product $\mathbf{a} \bullet \mathbf{a}^{\dagger}$ gives us $a_x^2 + a_y^2 + a_z^2$, which is indeed the squared distance to the origin.

In the case of a unitized sphere $\mathbf{s}$, the s_x, s_y, s_z, and s_w coordinates are all negated by the conjugate operation, but the s_u coordinate is left alone. The center of the sphere gets reflected through the origin and lies on the opposite side at the same distance away. The squared distance $\mathbf{v}^2$ between the centers of $\mathbf{s}$ and $\mathbf{s}^{\dagger}$ is thus equal to $4\left(s_x^2 + s_y^2 + s_z^2\right)$. The squared radius r_1^2 of $\mathbf{s}$ and the squared radius r_2^2 of $\mathbf{s}^{\dagger}$ are given by

$$r_1^2 = \|\mathbf{s}\|_{\oslash}^2 = s_x^2 + s_y^2 + s_z^2 + 2s_w \tag{4.54}$$

and

$$r_2^2 = \|\mathbf{s}^{\dagger}\|_{\oslash}^2 = s_x^2 + s_y^2 + s_z^2 - 2s_w = 2\left(s_x^2 + s_y^2 + s_z^2\right) - r_1^2, \tag{4.55}$$

where we assume $s_u = -1$. Plugging $\mathbf{v}^2$, r_1^2, and r_2^2 into the dot product $\mathbf{s} \bullet \mathbf{s}^{\dagger}$ gives us $s_x^2 + s_y^2 + s_z^2$, which is the distance that we need.

In all cases, the components of the flat weight are negated by the conformal conjugate operation, but they still have the same magnitude. The flat weight norm $\|\mathbf{u}\|_{\square}$ corresponds to the distance between the center of $\mathbf{u}$ and the support of its carrier. For dipoles and circles, this means the center of the conjugate is reflected through the support to a position lying in the same carrier on the opposite side of the support at the same distance away. This center of the conjugate $\mathbf{u}^{\dagger}$ also lies at the same distance from the origin as the center of $\mathbf{u}$, as required, and its radius is adjusted so that $\mathbf{u} \bullet \mathbf{u}^{\dagger}$ is the squared distance to the origin.

4.6 Containment

When we talk about join, meet, and expansion operations below, it will be important to understand what it means for one object $\mathbf{a}$ to be contained in another object $\mathbf{b}$ from a geometric perspective. Algebraically speaking, we define containment between $\mathbf{a}$ and $\mathbf{b}$ to be true precisely under the condition that $\mathbf{a} \wedge \mathbf{b} = 0$. When this condition is satisfied, the objects $\mathbf{a}$ and $\mathbf{b}$ share one of the two types of alignment defined in Section 4.4.

Figure 4.10 illustrates the three possible ways in which a round point $\mathbf{a}$ of radius r can be contained by a sphere $\mathbf{s}$ of radius R. The surface of the sphere itself is the set of null points having radius zero with positions that are exactly the distance R from the sphere's center. However, these null points are not the only points for which $\mathbf{a} \wedge \mathbf{s} = 0$. There are round points with nonzero radii both inside and outside the sphere that are algebraically still contained by the sphere because their wedge product with the sphere is zero.

Using the generic definitions for a round point and a sphere as given by Equations (4.29) and (4.35), the wedge product $\mathbf{a} \wedge \mathbf{s}$ is the antiscalar value

$$\mathbf{a} \wedge \mathbf{s} = \left(a_x s_x + a_y s_y + a_z s_z + a_w s_w + a_u s_u\right) \mathbb{1}. \tag{4.56}$$

This is just a 5D dot product if we treat both $\mathbf{a}$ and $\mathbf{s}$ as ordinary vectors and match the subscripts. It demonstrates that $\mathbf{a} \wedge \mathbf{s}$ is easy to calculate, but it doesn't tell us much about the geometric relationship between $\mathbf{a}$ and $\mathbf{s}$. If we instead write $\mathbf{a}$ and $\mathbf{s}$ in the forms given by Equations (4.28) and (4.34), which are always unitized such that $a_w = 1$ and $s_u = -1$, then we get a much more informative result. The wedge product $\mathbf{a} \wedge \mathbf{s}$ is now given by

$$\mathbf{a} \wedge \mathbf{s} = -\frac{1}{2}\left(\mathbf{v}^2 + r^2 - R^2\right), \tag{4.57}$$

where the vector $\mathbf{v}$ is the difference between the centers of $\mathbf{a}$ and $\mathbf{s}$. This looks an awful lot like the dot product between two points $\mathbf{a}_1$ and $\mathbf{a}_2$ listed in Table 4.14, and that is not a coincidence. The similarity is due to the bulk expansion property $\mathbf{a} \wedge \mathbf{b}^{\star} = (\mathbf{a} \bullet \mathbf{b})\,\mathbb{1}$ stated by Equation (2.156) back in Chapter 2. The sphere $\mathbf{s}$ having the squared radius R^2 is the dual of a round point $\mathbf{b}$ with the same coordinates, but since real and imaginary radii are exchanged for duals, $\mathbf{b}$ would have a squared radius of $-R^2$. The outcome of this relationship is that containment of a point by a sphere looks just like a pair of orthogonal points in which a real sphere is replaced by an imaginary point of the same size, and an imaginary sphere is replaced by a real point of the same size.

We consider the four separate cases in which a real sphere or an imaginary sphere contains a real round point or an imaginary round point. First, suppose that both the point $\mathbf{a}$ and the sphere $\mathbf{s}$ are real, so $r^2 > 0$ and $R^2 > 0$. We continue to use the vector $\mathbf{v}$ for the difference between the centers of $\mathbf{a}$ and $\mathbf{s}$. As shown in the upper-left part of Figure 4.10, the wedge product $\mathbf{a} \wedge \mathbf{s}$ given by Equation (4.57) is zero when $\mathbf{v}^2 + r^2 = R^2$, and this happens when the lengths $\|\mathbf{v}\|$ and r form the legs of a right triangle in which the hypotenuse has length R. The point is polar aligned with the sphere in this case. When $\|\mathbf{v}\|$ approaches the sphere's radius R, the point's radius r must approach zero, and when $\|\mathbf{v}\|$ approaches zero, the point's radius r must approach the sphere's radius R. Null points (with zero radius) contained by a sphere must have centers on the surface of the sphere, and real points contained by a sphere must have centers inside the sphere. The maximum radius of such a real point is R, the radius of the sphere, and this is exactly the round center of the sphere as defined in Section 4.2.4.

Next, suppose that the point $\mathbf{a}$ is imaginary and the sphere $\mathbf{s}$ is still real, so $r^2 < 0$ and $R^2 > 0$ as shown in the upper-right part of Figure 4.10. The Pythagorean relationship $\mathbf{v}^2 + r^2 = R^2$ that causes the wedge product $\mathbf{a} \wedge \mathbf{s}$ to be zero still holds, but we now rewrite it as $R^2 - r^2 = \mathbf{v}^2$ to highlight the fact that $\|\mathbf{v}\|$ takes over the role of the hypotenuse in a right triangle in which R and r correspond to the two legs. In this case, the point is right aligned with the sphere, and that means the point's center must lie outside the sphere. When $\|\mathbf{v}\|$ approaches the sphere's radius R, the point's radius r must approach zero, but there is no upper limit to the size of r.

Finally, we consider the case in which the sphere $\mathbf{s}$ is imaginary such that $R^2 < 0$. In this case, there is no real radius r that can satisfy the equation $\mathbf{v}^2 + r^2 - R^2 = 0$, so it is not possible for an imaginary sphere to contain any real round points at all. However, an imaginary sphere is still able to contain imaginary points as shown in the lower-right part of Figure 4.10. This time, the point is polar aligned with the sphere, and it is always at least as large as the sphere. We rewrite the Pythagorean relationship as $\mathbf{v}^2 - R^2 = -r^2$ to highlight that r is now the length of the hypotenuse. When $\|\mathbf{v}\|$ approaches zero, the point's radius r must approach the sphere's radius R. The two radii are

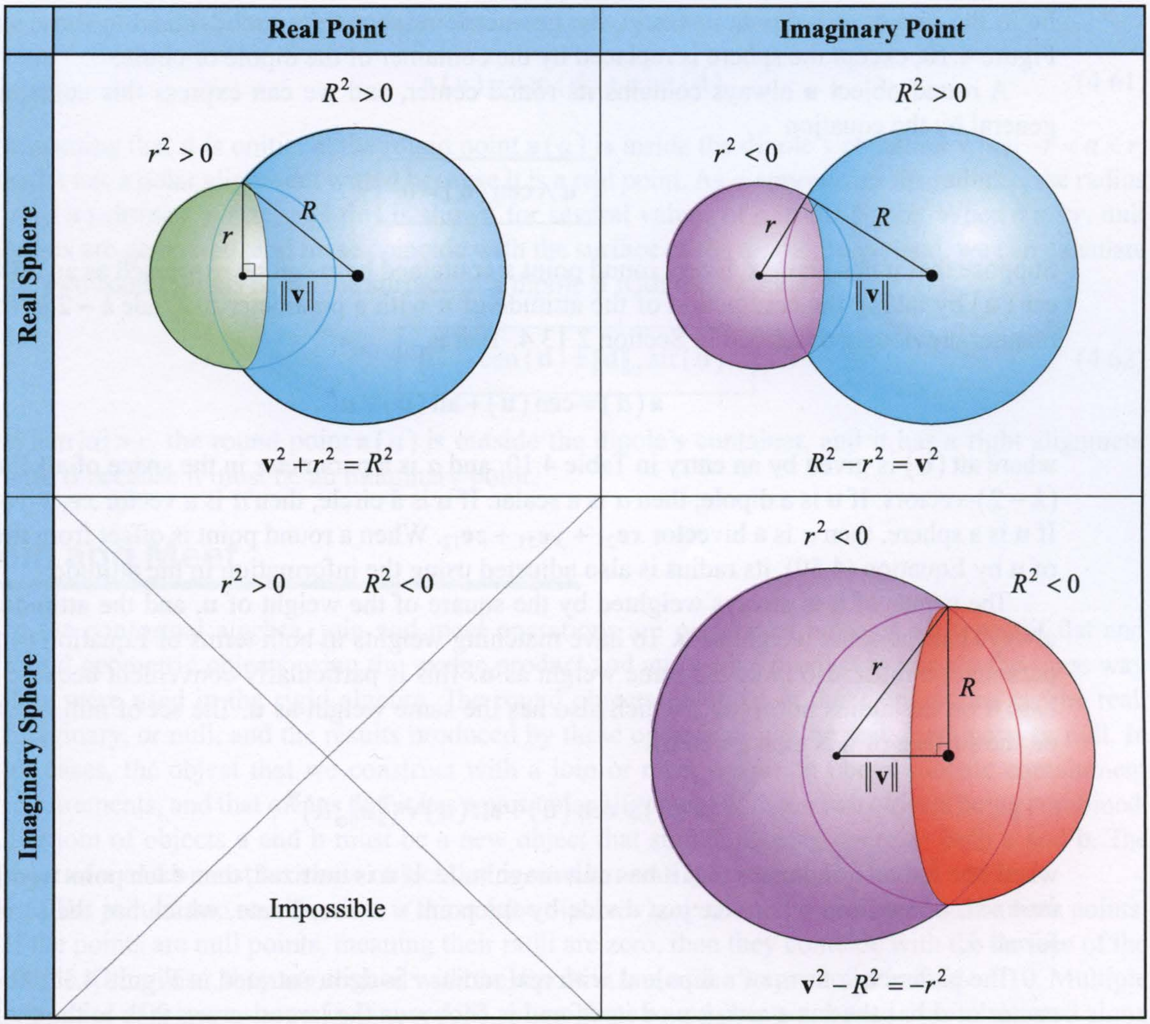


Figure 4.10. These are the cases in which a real or imaginary sphere of radius R contains a real or imaginary round point of radius r. The vector $\mathbf{v}$ is the difference between the center of the sphere and the center of the point. Real spheres contain real points centered inside the sphere and imaginary points centered outside. Imaginary spheres contain imaginary points both inside and outside, but they contain no real points at all.

equal when the point is the round center of the sphere, and this is the imaginary point of minimum size that can be contained by an imaginary sphere. There is no limit to how large r can be as the distance $\|\mathbf{v}\|$ from the center of the sphere grows.

Since a real sphere can only contain real points that have centers inside the sphere and imaginary points that have centers outside the sphere, it can be thought of as a solid ball of real space surrounded by an infinite expanse of empty imaginary space. When we consider the dual of a real sphere, it's interesting to note that an imaginary round point is only contained by imaginary spheres that have centers inside the point and by real spheres that have centers outside the point. An imaginary round point can thus be thought of as an empty ball of imaginary space surrounded by an infinite expanse of solid real space. This is consistent with our characterization of round points as inverted spheres at the beginning of Section 4.2.

Round points contained by dipoles and circles behave just as they do for spheres but under the restriction that their centers must lie in the carrier. The wedge product between a dipole and a point not centered on its carrier line always constructs a circle, and the wedge product between a circle and a point not centered on its carrier plane always constructs a sphere. When a round point does lie in the carrier of a dipole or circle, the geometric relationships are identical to those shown in Figure 4.10, except the sphere is replaced by the container of the dipole or circle.

A round object $\mathbf{u}$ always contains its round center, and we can express this containment in general by the equation

$$\boxed{\mathbf{u} \wedge \operatorname{cen}(\mathbf{u}) = 0.} \tag{4.58}$$

Suppose that $\mathbf{u}$ has grade k. Every round point $\mathbf{a}$ contained by $\mathbf{u}$ can be expressed as an offset from $\operatorname{cen}(\mathbf{u})$ by taking the contraction of the attitude of $\mathbf{u}$ with a parameter of grade $k-2$ in the same manner previously described in Section 2.13.4. That is,

$$\mathbf{a}(\alpha) = \operatorname{cen}(\mathbf{u}) + \operatorname{att}(\mathbf{u}) \vee \alpha^{\star}, \tag{4.59}$$

where $\operatorname{att}(\mathbf{u})$ is given by an entry in Table 4.10, and α is a parameter in the space of all Euclidean $(k-2)$-vectors. If $\mathbf{u}$ is a dipole, then α is a scalar. If $\mathbf{u}$ is a circle, then α is a vector $x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. If $\mathbf{u}$ is a sphere, then α is a bivector $x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12}$. When a round point is offset from the center of $\mathbf{u}$ by Equation (4.59), its radius is also adjusted using the information in the attitude.

The center of $\mathbf{u}$ is always weighted by the square of the weight of $\mathbf{u}$, and the attitude of $\mathbf{u}$ is always has the same weight as $\mathbf{u}$. To have matching weights in both terms of Equation (4.59), the parameter α must also have the same weight as $\mathbf{u}$. This is particularly convenient because we can base α on the radius norm $\|\mathbf{u}\|_{\circ}$, which also has the same weight as $\mathbf{u}$. The set of null points lying on the surface of $\mathbf{u}$ is then given by

$$\mathbf{a}(\hat{\alpha}) = \operatorname{cen}(\mathbf{u}) + \operatorname{att}(\mathbf{u}) \vee \left(\|\mathbf{u}\|_{\circ}\, \hat{\alpha}\right)^{\star}, \tag{4.60}$$

where the hat on $\hat{\alpha}$ indicates that it has unit magnitude. If $\mathbf{u}$ is unitized, then each point $\mathbf{a}(\hat{\alpha})$ is also unitized, but we can otherwise just divide by the point's w coordinate, which has the same value for all $\hat{\alpha}$.

The parametric form of a dipole $\mathbf{d}$ with real radius r is demonstrated in Figure 4.11. The round center of $\mathbf{d}$ has the same radius as $\mathbf{d}$ itself and is shown as the largest green disk in the figure. The

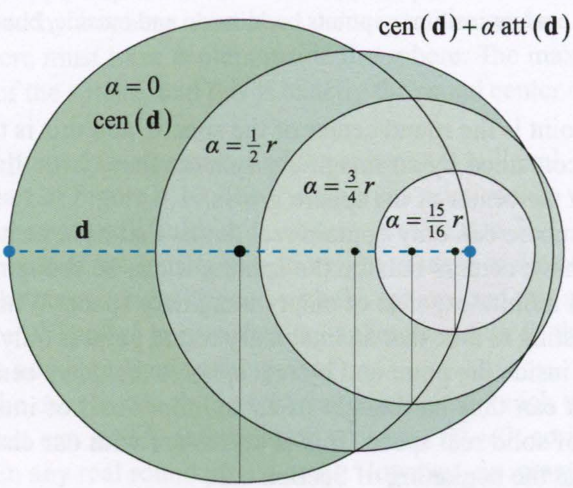


Figure 4.11. The unitized dipole $\mathbf{d}$ has a real radius r, and the two blue points represent its surface. The largest green disk is the round center of $\mathbf{d}$. Other points contained by $\mathbf{d}$ are generated parametrically by adding $\alpha \operatorname{att}(\mathbf{d})$ to the center position, where α is a scalar value. When $\alpha = \pm r$, the result is the pair of null points on the surface of the dipole.

parameter α is a scalar in this case, so we can express points contained by $\mathbf{d}$ as

$$\mathbf{a}(\alpha) = \operatorname{cen}(\mathbf{d}) + \alpha \operatorname{att}(\mathbf{d}). \tag{4.61}$$

Assuming that $\mathbf{d}$ is unitized, the round point $\mathbf{a}(\alpha)$ is inside the dipole's container when $-r < \alpha < r$, and it has a polar alignment with $\mathbf{d}$ because it is a real point. As α approaches the radius r, the radius of $\mathbf{a}(\alpha)$ shrinks in size, and this is shown for several values of α in the figure. When $\alpha = \pm r$, null points are generated, and these coincide with the surface of the dipole. In general, we can calculate the two round points $\mathbf{p}_\pm$ on the surface of a dipole $\mathbf{d}$ with the formula

Dipole surface points

$$\boxed{\mathbf{p}_\pm = \operatorname{cen}(\mathbf{d}) \pm \|\mathbf{d}\|_\circ \operatorname{att}(\mathbf{d}).} \tag{4.62}$$

When $|\alpha| > r$, the round point $\mathbf{a}(\alpha)$ is outside the dipole's container, and it has a right alignment with $\mathbf{d}$ because it must be an imaginary point.

4.7 Join and Meet

In the conformal algebra, join and meet operations are performed between all types of flat and round geometric objects using the wedge product and antiwedge product in exactly the same way they were used in the rigid algebra. The round objects involved in these operations can be real, imaginary, or null, and the results produced by these operations can be real, imaginary, or null. In all cases, the object that we construct with a join or meet operation obeys specific containment requirements, and that means that it has a particular alignment with the two objects being combined. The join of objects $\mathbf{a}$ and $\mathbf{b}$ must be a new object that simultaneously *contains* both $\mathbf{a}$ and $\mathbf{b}$. The meet of $\mathbf{a}$ and $\mathbf{b}$ must be a new object that is simultaneously *contained by* both $\mathbf{a}$ and $\mathbf{b}$.

The join of two round points is the unique dipole whose container is aligned to the both points. If the points are null points, meaning their radii are zero, then they coincide with the surface of the dipole. Otherwise, they are aligned with the dipole in one of the ways shown in Table 4.10. Multiple examples of this are shown in Figure 4.12, where three round points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are illustrated along with their wedge products. The join of all three points is the unique circle, also shown in the figure, that contains all three of them at the same time. This circle contains the surfaces of all three dipoles $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{b} \wedge \mathbf{c}$, and $\mathbf{c} \wedge \mathbf{a}$ as well, and we can think of it as the join of any one of the dipoles with the third point. If we were to join the circle with a fourth point not already contained by the circle, then we would get the unique sphere containing all four points.

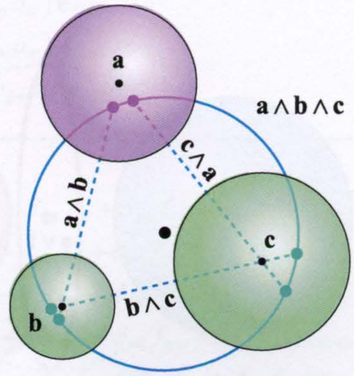


Figure 4.12. The three round points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (where $\mathbf{a}$ is imaginary, $\mathbf{b}$ is real, and $\mathbf{c}$ is real) are joined with the wedge product to construct three dipoles $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{b} \wedge \mathbf{c}$, and $\mathbf{c} \wedge \mathbf{a}$. The triple wedge product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ constructs the circle containing all three points and the surfaces of all three dipoles.

Flat geometries are created when we join a round geometry with the point at infinity $\mathbf{e}_5$, and as we have already seen, this extracts the carrier of the round geometry. The join of a flat geometry with a round geometry always yields another flat geometry because the factor of $\mathbf{e}_5$ can't be eliminated by the wedge product. The join of two flat geometries is always zero because they both already contain a factor of $\mathbf{e}_5$. Consequently, we can't join two flat geometries in the conformal algebra in the same way that we could in the rigid algebra. Instead, we can create a line by joining a round point with a flat point, and we can create a plane by either joining a round point with a line or joining a flat point with a dipole.

In total, there are seven different ways in which various objects can be joined in three dimensions. The exact calculations involved in the wedge product for each case are listed in Table 4.15 along with illustrations. All of the examples in the table depict configurations in which the result is either a flat object or a real round object. Imaginary round objects can also be constructed with the join operation, but it requires that both objects $\mathbf{a}$ and $\mathbf{b}$ participating in the wedge product be imaginary and that they are too close together for the container of their product $\mathbf{a} \wedge \mathbf{b}$ to be right aligned with $\mathbf{a}$ and $\mathbf{b}$.

The meet operation provides an extremely efficient way to both express and calculate the intersection of two geometric objects in the conformal algebra. In three dimensions, there are 12 possible ways to perform a meet between two objects of various types, and the per-component calculations of the corresponding antiwedge products are listed in Tables 4.16 and 4.17 with illustrations. With the exception of the last case, each of the examples given in the tables depicts a configuration in which two real or flat geometries actually intersect to produce a real result. However, it is much easier to produce imaginary results with the meet operation than it is with the join operation. Whenever two objects are not actually intersecting, their meet is still a meaningful geometry of the same type that we would get if there was an intersection, but it is an imaginary object.

Figure 4.13 shows an example in which a real sphere $\mathbf{s}$ and imaginary sphere $\mathbf{t}$ do not intersect. The meet $\mathbf{s} \vee \mathbf{t}$ is an imaginary circle between the two spheres lying in a plane perpendicular to the line segment connecting the centers of the spheres. The container of the circle is right aligned to the real sphere and polar aligned to the imaginary sphere. If both spheres were real but still not intersecting, then their meet would still be an imaginary circle, but it would be adjusted so that it is right aligned with both spheres, and that would cause its position and radius to be a little different. As the two spheres move closer together, the circle shrinks until it has radius zero when the spheres are tangent to each other. In general, the meet of tangent objects produces a null round geometry. These null objects still have carriers with a meaningful attitude, so it's possible to extract directional information from a circle or dipole where two tangent objects meet even though it has a radius of

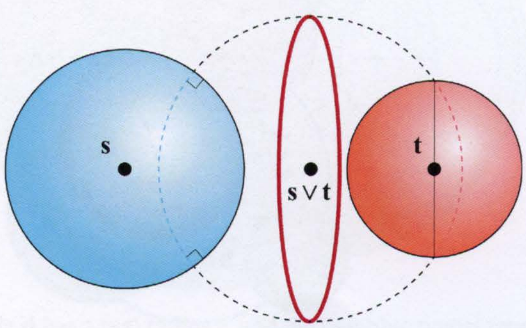


Figure 4.13. The meet of two spheres $\mathbf{s}$ and $\mathbf{t}$ that don't actually intersect is an imaginary circle. Here, $\mathbf{s}$ is a real sphere, and $\mathbf{t}$ is an imaginary sphere. The circle $\mathbf{s} \vee \mathbf{t}$ is right aligned to the real sphere $\mathbf{s}$ and polar aligned to the imaginary sphere $\mathbf{t}$.

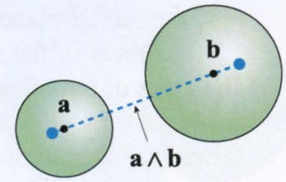
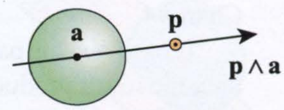
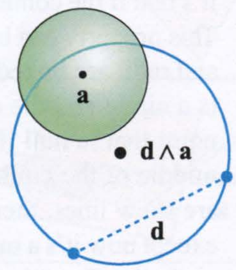
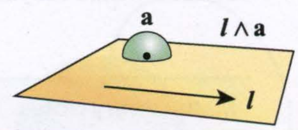
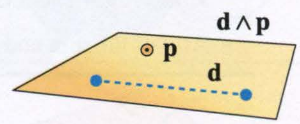
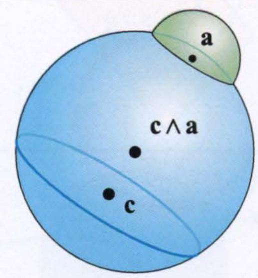
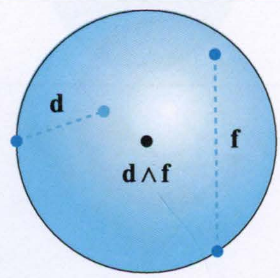
Join Operation	Illustration
Dipole containing round points $\mathbf{a}$ and $\mathbf{b}$. $\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= (a_w b_x - a_x b_w)\,\mathbf{e}_{41} + (a_w b_y - a_y b_w)\,\mathbf{e}_{42} + (a_w b_z - a_z b_w)\,\mathbf{e}_{43} \\ &+ (a_y b_z - a_z b_y)\,\mathbf{e}_{23} + (a_z b_x - a_x b_z)\,\mathbf{e}_{31} + (a_x b_y - a_y b_x)\,\mathbf{e}_{12} \\ &+ (a_x b_u - a_u b_x)\,\mathbf{e}_{15} + (a_y b_u - a_u b_y)\,\mathbf{e}_{25} \\ &+ (a_z b_u - a_u b_z)\,\mathbf{e}_{35} + (a_w b_u - a_u b_w)\,\mathbf{e}_{45}\end{aligned}$	
Line containing flat point $\mathbf{p}$ and round point $\mathbf{a}$. $\begin{aligned}\mathbf{p}\wedge\mathbf{a} &= (p_x a_w - p_w a_x)\,\mathbf{e}_{415} + (p_z a_y - p_y a_z)\,\mathbf{e}_{235} \\ &+ (p_y a_w - p_w a_y)\,\mathbf{e}_{425} + (p_x a_z - p_z a_x)\,\mathbf{e}_{315} \\ &+ (p_z a_w - p_w a_z)\,\mathbf{e}_{435} + (p_y a_x - p_x a_y)\,\mathbf{e}_{125}\end{aligned}$	
Circle containing dipole $\mathbf{d}$ and round point $\mathbf{a}$. $\begin{aligned}\mathbf{d}\wedge\mathbf{a} &= (d_{vy} a_z - d_{vz} a_y + d_{mx} a_w)\,\mathbf{e}_{423} + (d_{vz} a_x - d_{vx} a_z + d_{my} a_w)\,\mathbf{e}_{431} \\ &+ (d_{vx} a_y - d_{vy} a_x + d_{mz} a_w)\,\mathbf{e}_{412} - (d_{mx} a_x + d_{my} a_y + d_{mz} a_z)\,\mathbf{e}_{321} \\ &+ (d_{px} a_w - d_{pw} a_x + d_{vx} a_u)\,\mathbf{e}_{415} + (d_{pz} a_y - d_{py} a_z + d_{mx} a_u)\,\mathbf{e}_{235} \\ &+ (d_{py} a_w - d_{pw} a_y + d_{vy} a_u)\,\mathbf{e}_{425} + (d_{px} a_z - d_{pz} a_x + d_{my} a_u)\,\mathbf{e}_{315} \\ &+ (d_{pz} a_w - d_{pw} a_z + d_{vz} a_u)\,\mathbf{e}_{435} + (d_{py} a_x - d_{px} a_y + d_{mz} a_u)\,\mathbf{e}_{125}\end{aligned}$	
Plane containing line $\boldsymbol{l}$ and round point $\mathbf{a}$. $\begin{aligned}\boldsymbol{l}\wedge\mathbf{a} &= (l_{vz} a_y - l_{vy} a_z - l_{mx} a_w)\,\mathbf{e}_{4235} + (l_{vx} a_z - l_{vz} a_x - l_{my} a_w)\,\mathbf{e}_{4315} \\ &+ (l_{vy} a_x - l_{vx} a_y - l_{mz} a_w)\,\mathbf{e}_{4125} + (l_{mx} a_x + l_{my} a_y + l_{mz} a_z)\,\mathbf{e}_{3215}\end{aligned}$	
Plane containing dipole $\mathbf{d}$ and flat point $\mathbf{p}$. $\begin{aligned}\mathbf{d}\wedge\mathbf{p} &= (d_{vy} p_z - d_{vz} p_y + d_{mx} p_w)\,\mathbf{e}_{4235} \\ &+ (d_{vz} p_x - d_{vx} p_z + d_{my} p_w)\,\mathbf{e}_{4315} \\ &+ (d_{vx} p_y - d_{vy} p_x + d_{mz} p_w)\,\mathbf{e}_{4125} \\ &- (d_{mx} p_x + d_{my} p_y + d_{mz} p_z)\,\mathbf{e}_{3215}\end{aligned}$	
Sphere containing circle $\mathbf{c}$ and round point $\mathbf{a}$. $\begin{aligned}\mathbf{c}\wedge\mathbf{a} &= -(c_{gx} a_x + c_{gy} a_y + c_{gz} a_z + c_{gw} a_w)\,\mathbf{e}_{1234} \\ &+ (c_{vz} a_y - c_{vy} a_z + c_{gx} a_u - c_{mx} a_w)\,\mathbf{e}_{4235} \\ &+ (c_{vx} a_z - c_{vz} a_x + c_{gy} a_u - c_{my} a_w)\,\mathbf{e}_{4315} \\ &+ (c_{vy} a_x - c_{vx} a_y + c_{gz} a_u - c_{mz} a_w)\,\mathbf{e}_{4125} \\ &+ (c_{mx} a_x + c_{my} a_y + c_{mz} a_z + c_{gw} a_u)\,\mathbf{e}_{3215}\end{aligned}$	
Sphere containing dipoles $\mathbf{d}$ and $\mathbf{f}$. $\begin{aligned}\mathbf{d}\wedge\mathbf{f} &= -(d_{vx} f_{mx} + d_{vy} f_{my} + d_{vz} f_{mz} + d_{mx} f_{vx} + d_{my} f_{vy} + d_{mz} f_{vz})\,\mathbf{e}_{1234} \\ &+ (d_{vy} f_{pz} - d_{vz} f_{py} + d_{pz} f_{vy} - d_{py} f_{vz} + d_{mx} f_{pw} + d_{pw} f_{mx})\,\mathbf{e}_{4235} \\ &+ (d_{vz} f_{px} - d_{vx} f_{pz} + d_{px} f_{vz} - d_{pz} f_{vx} + d_{my} f_{pw} + d_{pw} f_{my})\,\mathbf{e}_{4315} \\ &+ (d_{vx} f_{py} - d_{vy} f_{px} + d_{py} f_{vx} - d_{px} f_{vy} + d_{mz} f_{pw} + d_{pw} f_{mz})\,\mathbf{e}_{4125} \\ &- (d_{mx} f_{px} + d_{my} f_{py} + d_{mz} f_{pz} + d_{px} f_{mx} + d_{py} f_{my} + d_{pz} f_{mz})\,\mathbf{e}_{3215}\end{aligned}$	

Table 4.15. These are the join operations between objects in CGA over 3D space.

zero. For two tangent spheres **s** and **t**, all we have to do is calculate $(\mathbf{s} \vee \mathbf{t}) \wedge \mathbf{e}_5$ to get the plane that's tangent to both spheres. (In practice, we don't actually have to multiply by $\mathbf{e}_5$ because we can just read the carrier plane's coordinates from the round part of the circle.)

The meet of flat objects works in the conformal algebra exactly as it did in the rigid algebra. The meet of two planes produces the line where they intersect, and the meet of a line and a plane produces the flat point where they intersect. Some of the most useful meet operations in the conformal algebra involve a flat geometry and a round geometry. In these cases, the result is a lower-dimensional round geometry that's real if an intersection actually occurs and imaginary otherwise. An example giving the details for the intersection of a line and a sphere is provided in Comparison Chart #4.

If both objects participating in a meet operation are trivectors, which represent circles and lines, then the result produced by their antiwedge product is a round point. In the case of two circles that don't share the same carrier plane, this round point is null if the circles touch at exactly one point, it's real if the containers of the circles intersect at more than one point, and it's imaginary otherwise. This property can be used to quickly determine whether two circles with arbitrary centers, attitudes, and radii are linked, and the details are laid out in Comparison Chart #5. In the case that one object is a circle and the other is a line not lying in the circle's carrier plane, the meet produces a round point that is null if the line hits the circle at exactly one point, real if the line passes through the middle of the circle, and imaginary if the line passes outside the circle. Finally, if the two objects are skew lines, then the meet produces the same crossing orientation as it did in the rigid algebra, except now it's a multiple of $\mathbf{e}_5$. This makes sense because both flat lines contain the point at infinity, so that's where they must intersect.

Math Library Notes

- The join and meet operations for objects in the conformal algebra are implemented by the `Wedge()` and `Antiwedge()` functions.
- Any of the join and meet operations can also be calculated by using the ^ symbol as an infix operator.

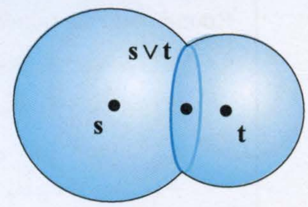
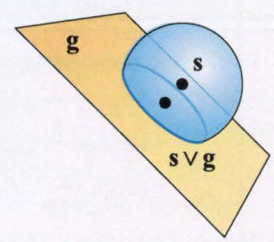
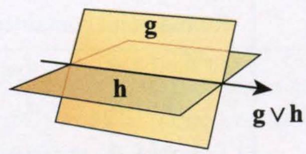
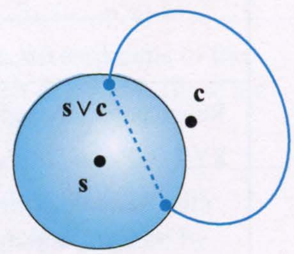
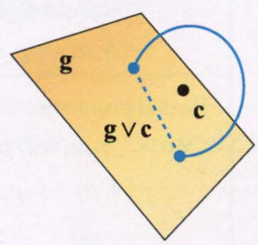
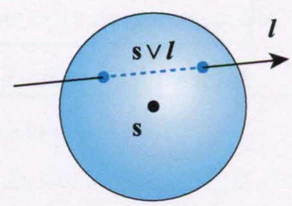
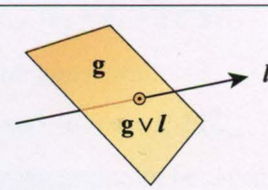
Meet Operation	Illustration
Circle where spheres **s** and **t** intersect. $\mathbf{s} \vee \mathbf{t} = (s_u t_x - s_x t_u)\mathbf{e}_{423} + (s_u t_y - s_y t_u)\mathbf{e}_{431}$ $+ (s_u t_z - s_z t_u)\mathbf{e}_{412} + (s_u t_w - s_w t_u)\mathbf{e}_{321}$ $+ (s_z t_y - s_y t_z)\mathbf{e}_{415} + (s_x t_z - s_z t_x)\mathbf{e}_{425} + (s_y t_x - s_x t_y)\mathbf{e}_{435}$ $+ (s_x t_w - s_w t_x)\mathbf{e}_{235} + (s_y t_w - s_w t_y)\mathbf{e}_{315} + (s_z t_w - s_w t_z)\mathbf{e}_{125}$	
Circle where sphere **s** and plane **g** intersect. $\mathbf{s} \vee \mathbf{g} = s_u g_x \mathbf{e}_{423} + s_u g_y \mathbf{e}_{431} + s_u g_z \mathbf{e}_{412} + s_u g_w \mathbf{e}_{321}$ $+ (s_z g_y - s_y g_z)\mathbf{e}_{415} + (s_x g_z - s_z g_x)\mathbf{e}_{425} + (s_y g_x - s_x g_y)\mathbf{e}_{435}$ $+ (s_x g_w - s_w g_x)\mathbf{e}_{235} + (s_y g_w - s_w g_y)\mathbf{e}_{315} + (s_z g_w - s_w g_z)\mathbf{e}_{125}$	
Line where planes **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\mathbf{e}_{415} + (g_x h_w - g_w h_x)\mathbf{e}_{235}$ $+ (g_x h_z - g_z h_x)\mathbf{e}_{425} + (g_y h_w - g_w h_y)\mathbf{e}_{315}$ $+ (g_y h_x - g_x h_y)\mathbf{e}_{435} + (g_z h_w - g_w h_z)\mathbf{e}_{125}$	
Dipole where sphere **s** and circle **c** intersect. $\mathbf{s} \vee \mathbf{c} = (s_y c_{gz} - s_z c_{gy} + s_u c_{vx})\mathbf{e}_{41} + (s_w c_{gx} - s_x c_{gw} + s_u c_{mx})\mathbf{e}_{23}$ $+ (s_z c_{gx} - s_x c_{gz} + s_u c_{vy})\mathbf{e}_{42} + (s_w c_{gy} - s_y c_{gw} + s_u c_{my})\mathbf{e}_{31}$ $+ (s_x c_{gy} - s_y c_{gx} + s_u c_{vz})\mathbf{e}_{43} + (s_w c_{gz} - s_z c_{gw} + s_u c_{mz})\mathbf{e}_{12}$ $+ (s_z c_{my} - s_y c_{mz} + s_w c_{vx})\mathbf{e}_{15} + (s_x c_{mz} - s_z c_{mx} + s_w c_{vy})\mathbf{e}_{25}$ $+ (s_y c_{mx} - s_x c_{my} + s_w c_{vz})\mathbf{e}_{35} - (s_x c_{vx} + s_y c_{vy} + s_z c_{vz})\mathbf{e}_{45}$	
Dipole where plane **g** and circle **c** intersect. $\mathbf{g} \vee \mathbf{c} = (g_y c_{gz} - g_z c_{gy})\mathbf{e}_{41} + (g_w c_{gx} - g_x c_{gw})\mathbf{e}_{23}$ $+ (g_z c_{gx} - g_x c_{gz})\mathbf{e}_{42} + (g_w c_{gy} - g_y c_{gw})\mathbf{e}_{31}$ $+ (g_x c_{gy} - g_y c_{gx})\mathbf{e}_{43} + (g_w c_{gz} - g_z c_{gw})\mathbf{e}_{12}$ $+ (g_z c_{my} - g_y c_{mz} + g_w c_{vx})\mathbf{e}_{15} + (g_x c_{mz} - g_z c_{mx} + g_w c_{vy})\mathbf{e}_{25}$ $+ (g_y c_{mx} - g_x c_{my} + g_w c_{vz})\mathbf{e}_{35} - (g_x c_{vx} + g_y c_{vy} + g_z c_{vz})\mathbf{e}_{45}$	
Dipole where sphere **s** and line ***l*** intersect. $\mathbf{s} \vee \boldsymbol{l} = s_u l_{vx}\mathbf{e}_{41} + s_u l_{vy}\mathbf{e}_{42} + s_u l_{vz}\mathbf{e}_{43}$ $+ s_u l_{mx}\mathbf{e}_{23} + s_u l_{my}\mathbf{e}_{31} + s_u l_{mz}\mathbf{e}_{12}$ $+ (s_z l_{my} - s_y l_{mz} + s_w l_{vx})\mathbf{e}_{15} + (s_x l_{mz} - s_z l_{mx} + s_w l_{vy})\mathbf{e}_{25}$ $+ (s_y l_{mx} - s_x l_{my} + s_w l_{vz})\mathbf{e}_{35} - (s_x l_{vx} + s_y l_{vy} + s_z l_{vz})\mathbf{e}_{45}$	
Flat point where plane **g** and line ***l*** intersect. $\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\mathbf{e}_{15} + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\mathbf{e}_{25}$ $+ (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\mathbf{e}_{35} - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\mathbf{e}_{45}$	

Table 4.16. These are the meet operations between objects in CGA over 3D space (part 1 of 2).

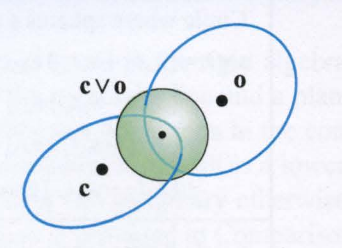
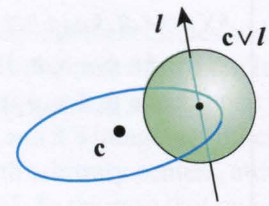
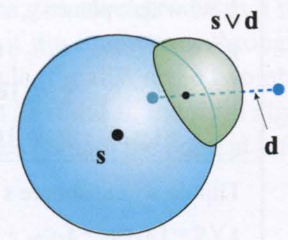
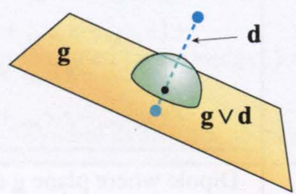
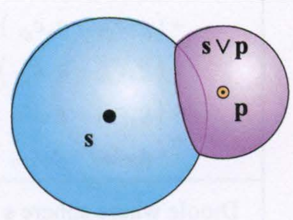
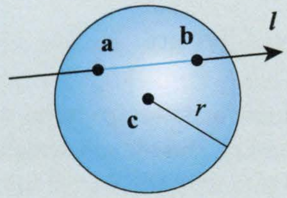
Meet Operation	**Illustration**
Round point contained by circles **c** and **o**. $\mathbf{c} \vee \mathbf{o} = (c_{gz}o_{my} - c_{gy}o_{mz} + c_{my}o_{gz} - c_{mz}o_{gy} + c_{vx}o_{gw} + g_{gw}o_{vx})\,\mathbf{e}_1$ $+ (c_{gx}o_{mz} - c_{gz}o_{mx} + c_{mz}o_{gx} - c_{mx}o_{gz} + c_{vy}o_{gw} + g_{gw}o_{vy})\,\mathbf{e}_2$ $+ (c_{gy}o_{mx} - c_{gx}o_{my} + c_{mx}o_{gy} - c_{my}o_{gx} + c_{vz}o_{gw} + g_{gw}o_{vz})\,\mathbf{e}_3$ $- (c_{gx}o_{vx} + c_{gy}o_{vy} + c_{gz}o_{vz} + c_{vx}o_{gx} + c_{vy}o_{gy} + c_{vz}o_{gz})\,\mathbf{e}_4$ $- (c_{mx}o_{vx} + c_{my}o_{vy} + c_{mz}o_{vz} + c_{vx}o_{mx} + c_{vy}o_{my} + c_{vz}o_{mz})\,\mathbf{e}_5$	
Round point centered on line $\boldsymbol{l}$ and contained by circle **c**. $\mathbf{c} \vee \boldsymbol{l} = (c_{gz}l_{my} - c_{gy}l_{mz} + c_{gw}l_{vx})\,\mathbf{e}_1 + (c_{gx}l_{mz} - c_{gz}l_{mx} + c_{gw}l_{vy})\,\mathbf{e}_2$ $+ (c_{gy}l_{mx} - c_{gx}l_{my} + c_{gw}l_{vz})\,\mathbf{e}_3 - (c_{gx}l_{vx} + c_{gy}l_{vy} + c_{gz}l_{vz})\,\mathbf{e}_4$ $- (c_{mx}l_{vx} + c_{my}l_{vy} + c_{mz}l_{vz} + c_{vx}l_{mx} + c_{vy}l_{my} + c_{vz}l_{mz})\,\mathbf{e}_5$	
Round point contained by sphere **s** and dipole **d**. $\mathbf{s} \vee \mathbf{d} = (s_y d_{mz} - s_z d_{my} - s_w d_{vx} + s_u d_{px})\,\mathbf{e}_1$ $+ (s_z d_{mx} - s_x d_{mz} - s_w d_{vy} + s_u d_{py})\,\mathbf{e}_2$ $+ (s_x d_{my} - s_y d_{mx} - s_w d_{vz} + s_u d_{pz})\,\mathbf{e}_3$ $+ (s_x d_{vx} + s_y d_{vy} + s_z d_{vz} + s_u d_{pw})\,\mathbf{e}_4$ $- (s_x d_{px} + s_y d_{py} + s_z d_{pz} + s_w d_{pw})\,\mathbf{e}_5$	
Round point centered in plane **g** and contained by dipole **d**. $\mathbf{g} \vee \mathbf{d} = (g_y d_{mz} - g_z d_{my} - g_w d_{vx})\,\mathbf{e}_1$ $+ (g_z d_{mx} - g_x d_{mz} - g_w d_{vy})\,\mathbf{e}_2$ $+ (g_x d_{my} - g_y d_{mx} - g_w d_{vz})\,\mathbf{e}_3$ $+ (g_x d_{vx} + g_y d_{vy} + d_z d_{vz})\,\mathbf{e}_4$ $- (g_x d_{px} + g_y d_{py} + g_z d_{pz} + g_w d_{pw})\,\mathbf{e}_5$	
Round point centered at flat point **p** and contained by sphere **s**. $\mathbf{s} \vee \mathbf{p} = s_u p_x \mathbf{e}_1 + s_u p_y \mathbf{e}_2 + s_u p_z \mathbf{e}_3 + s_u p_w \mathbf{e}_4$ $- (s_x p_x + s_y p_y + s_z p_z + s_w p_w)\,\mathbf{e}_5$	

Table 4.17. These are the meet operations between objects in CGA over 3D space (part 2 of 2).

Comparison Chart #4

Line-Sphere Intersection

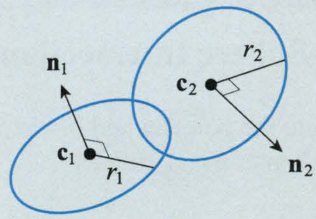


Calculate the points **a** and **b** where a line $\boldsymbol{l}$ intersects a sphere having center **c** and radius r.

Conventional Methods	Geometric Algebra
Let $\boldsymbol{l}(t) = \mathbf{p} + t\mathbf{v}$ be a parametric line containing the point **p** and running parallel to the direction vector **v**. Assume the direction is normalized so that $\lVert \mathbf{v} \rVert = 1$.	Let $\boldsymbol{l}$ be a flat line as defined in Equation (4.26) with direction $\boldsymbol{l}_{\mathbf{v}}$ and moment $\boldsymbol{l}_{\mathbf{m}}$. Assume the line is unitized so that $\lVert \boldsymbol{l}_{\mathbf{v}} \rVert = 1$. Assume the sphere defined by Equation (4.34) is unitized so that $s_u = -1$.
Translate the center of the sphere to the origin. Translate the line by subtracting **c** from **p**.	Translate the center of the sphere to the origin so it is given by $\mathbf{s} = -\mathbf{e}_{1234} + \frac{1}{2} r^2 \mathbf{e}_{3215}$. Translate the line by subtracting $\mathbf{c} \times \boldsymbol{l}_{\mathbf{v}}$ from its moment $\boldsymbol{l}_{\mathbf{m}}$.
The goal is to solve the equation $(\mathbf{p} + t\mathbf{v})^2 = r^2$ for values of t and plug them into the line $\mathbf{p} + t\mathbf{v}$ to obtain the intersection points **a** and **b**.	The goal is to calculate the endpoints of the dipole $\mathbf{d} = \mathbf{s} \vee \boldsymbol{l}$, the meet of **s** and $\boldsymbol{l}$. These are the intersection points **a** and **b**.
Expanding the quadratic equation, recognizing that $\mathbf{v}^2 = 1$, and collecting terms in powers of t, we have $$t^2 + 2(\mathbf{p} \cdot \mathbf{v})t + \mathbf{p}^2 - r^2 = 0.$$	Applying the formula in Table 4.16 with $s_x = s_y = s_z = 0$, the dipole **d** is given by $$\mathbf{d} = -l_{vx}\mathbf{e}_{41} - l_{vy}\mathbf{e}_{42} - l_{vz}\mathbf{e}_{43} - l_{mx}\mathbf{e}_{23} - l_{my}\mathbf{e}_{31} - l_{mz}\mathbf{e}_{12} + \tfrac{1}{2} r^2 (l_{vx}\mathbf{e}_{15} + l_{vy}\mathbf{e}_{25} + l_{vz}\mathbf{e}_{35}).$$
The discriminant δ of the polynomial is given by $$\delta = (\mathbf{p} \cdot \mathbf{v})^2 - \mathbf{p}^2 + r^2$$ If $\delta < 0$, then the line does not intersect the sphere.	Using $\boldsymbol{l}_{\mathbf{v}}^2 = 1$, the squared radius of the dipole is given by $$\lVert \mathbf{d} \rVert_{\odot}^2 = r^2 - \boldsymbol{l}_{\mathbf{m}}^2.$$ If $\lVert \mathbf{d} \rVert_{\odot}^2 < 0$, then the line does not intersect the sphere.
The parameter values where the line intersects the sphere are given by $t = -\mathbf{p} \cdot \mathbf{v} \pm \sqrt{\delta}$. Plugging these into the line $\boldsymbol{l}(t)$, the points **a** and **b** are obtained with $$\mathbf{a} = \mathbf{q} - \mathbf{v}\sqrt{\delta} \quad \text{and} \quad \mathbf{b} = \mathbf{q} + \mathbf{v}\sqrt{\delta},$$ where $\mathbf{q} = \mathbf{p} - (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + \mathbf{c}$ is the midpoint between them.	The center of the dipole is given by $$\operatorname{cen}(\mathbf{d}) = (l_{vy}l_{mz} - l_{vz}l_{my})\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\mathbf{e}_2 + (l_{vx}l_{my} - l_{vy}l_{mx})\mathbf{e}_3 + \mathbf{e}_4 + \tfrac{1}{2} r^2 \mathbf{e}_5.$$ The points **a** and **b** are obtained with $$\mathbf{a} = \mathbf{q} - \boldsymbol{l}_{\mathbf{v}} \lVert \mathbf{d} \rVert_{\odot} \quad \text{and} \quad \mathbf{b} = \mathbf{q} + \boldsymbol{l}_{\mathbf{v}} \lVert \mathbf{d} \rVert_{\odot},$$ where $\mathbf{q} = \operatorname{cen}(\mathbf{d}) + \mathbf{c}$ is the midpoint between them.

Comparison Chart #5

Linked Circles



Determine whether two circles with centers $\mathbf{c}_1$ and $\mathbf{c}_2$, radii r_1 and r_2, and plane normals $\mathbf{n}_1$ and $\mathbf{n}_2$ are linked.

Conventional Methods	Geometric Algebra
Let $d_1 = -\mathbf{n}_1 \cdot \mathbf{c}_1$ and $d_2 = -\mathbf{n}_2 \cdot \mathbf{c}_2$. The circles lie in the planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$.	Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be circles as defined in Equation (4.32).
Calculate a parametric line $\mathbf{p} + t\mathbf{v}$ where the two planes intersect using rows I and N in Table 1.1. The direction of the line is simply $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. If $\mathbf{v} = \mathbf{0}$, then the circles lie in parallel planes and cannot be linked. The point $\mathbf{p}$ is given by $$\mathbf{p} = \frac{d_1(\mathbf{v} \times \mathbf{n}_2) + d_2(\mathbf{n}_1 \times \mathbf{v})}{\mathbf{v}^2}.$$ The points where each circle intersects the plane of the other circle must lie on the line $\mathbf{p} + t\mathbf{v}$.	Calculate the round point $\mathbf{a} = \mathbf{c}_1 \vee \mathbf{c}_2$ where the circles meet using the formula in Table 4.17.
Solve for values of t such that $$(\mathbf{p} + t\mathbf{v} - \mathbf{c}_1)^2 = r_1^2.$$ These correspond to points on the line $\mathbf{p} + t\mathbf{v}$ that also lie on the circle centered at $\mathbf{c}_1$. Writing as a quadratic equation in t, we have $$\mathbf{v}^2 t^2 + 2(\mathbf{u} \cdot \mathbf{v})t + \mathbf{u}^2 - r_1^2 = 0,$$ where $\mathbf{u} = \mathbf{p} - \mathbf{c}_1$. The discriminant δ of the polynomial is $$\delta = (\mathbf{u} \cdot \mathbf{v})^2 - \mathbf{v}^2(\mathbf{u}^2 - r_1^2).$$ If $\delta < 0$, then neither circle intersects the plane of the other, so they are not linked. Otherwise, $$t_{1,2} = \frac{-(\mathbf{u} \cdot \mathbf{v}) \pm \sqrt{\delta}}{\mathbf{v}^2}.$$	The weighted squared radius of the round point is given by $$\lVert\mathbf{a}\rVert_{\odot}^2 = 2a_w a_u - a_x^2 - a_y^2 - a_z^2.$$ If $\lVert\mathbf{a}\rVert_{\odot}^2 > 0$, then the circles are linked.
Calculate the points $\mathbf{q}_1 = \mathbf{s} + t_1\mathbf{v}$ and $\mathbf{q}_2 = \mathbf{s} + t_2\mathbf{v}$, where $\mathbf{s} = \mathbf{p} - \mathbf{c}_2$. The circles are linked if $$(\mathbf{q}_1^2 - r_2^2)(\mathbf{q}_2^2 - r_2^2) < 0$$ because one point would be inside the circle centered at $\mathbf{c}_2$ and the other would be outside.	

4.8 Expansions

The final operation that we examine in the conformal algebra is the expansion. Just as it did in the rigid algebra, the expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ constructs an object that contains $\mathbf{a}$ and is orthogonal to $\mathbf{b}$. We could use either dual in the conformal algebra since they are just negatives of each other, but we choose the antidual in order to be consistent with the weight dual that was necessary for orthogonal projection in the rigid algebra. Tables 4.18, 4.19, and 4.20 illustrate the 18 different combinations in which one object can be expanded onto another object of higher grade in three dimensions. The tables also list the per-component calculations for each expansion.

Compared to the rigid algebra, the only difference for expansions in the conformal algebra is what it means to be orthogonal. The flat geometries that connected points to lines, points to planes, and lines to planes shown in Table 2.22 are still present in the conformal algebra when we expand a flat object onto another flat object, but the orthogonal objects created when we expand a round object onto another geometry are generalized to be spherical. The container of the expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ is always aligned to the containers of $\mathbf{a}$ and $\mathbf{b}$.

The practical application of the expansion operation is the ability to project a lower-dimensional object $\mathbf{a}$ onto a higher-dimensional object $\mathbf{b}$ by using the meet operation to intersect the expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ with $\mathbf{b}$. We calculate $\mathbf{b} \vee (\mathbf{a} \wedge \mathbf{b}^{\star})$ to perform such a projection in exactly the same manner used in the rigid algebra at the beginning of Section 2.13.6. In the conformal algebra, the projection of $\mathbf{a}$ onto $\mathbf{b}$ follows a generally round path that is a straight shot only when $\mathbf{a}$ is a flat object. As an example, consider the expansion of a circle $\mathbf{c}$ onto a plane $\mathbf{g}$, which is shown in the second row of Table 4.19. The expansion $\mathbf{c} \wedge \mathbf{g}^{\star}$ is the sphere that contains $\mathbf{c}$ and is orthogonal to $\mathbf{g}$. Containment and orthogonality both imply alignment, and which type depends on whether the objects are real of imaginary. In the case of a flat object, alignment to a round object requires that the flat object pass through the center of the round object. So the sphere in this case has a center in the plane $\mathbf{g}$ and contains the circle $\mathbf{c}$. When we intersect the sphere with the plane $\mathbf{g}$ by calculating $\mathbf{g} \vee (\mathbf{c} \wedge \mathbf{g}^{\star})$, the result is a circle that has been projected onto the plane. This kind of projection takes some getting used to, but it is a natural feature of the conformal algebra. The conventional meaning of orthogonal projection is impossible because there is no way to represent the noncircular ellipse that would generally result from the projection of a circle onto a plane.

We mentioned in Section 4.2.5 that the container of an object $\mathbf{u}$ is equivalent to the expansion of $\mathbf{u}$ onto its own carrier. Just as we could in the rigid algebra, we can expand an object onto another object that already contains it and still get a meaningful result. When we calculate $\mathbf{u} \wedge \operatorname{car}(\mathbf{u})^{\star}$, we always get a sphere, the container of $\mathbf{u}$, that has a center in the carrier of $\mathbf{u}$ and is simultaneously polar aligned to $\mathbf{u}$. If we were to project this container onto the carrier, then we just get $\mathbf{u}$ back, so we can write

$$\mathbf{u} = \operatorname{car}(\mathbf{u}) \vee \operatorname{con}(\mathbf{u}), \tag{4.63}$$

which can be interpreted as a sort of factorization of $\mathbf{u}$ into the antiwedge product of a flat object and a sphere.

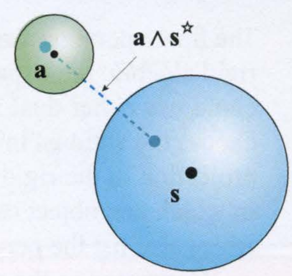
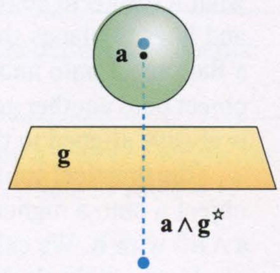
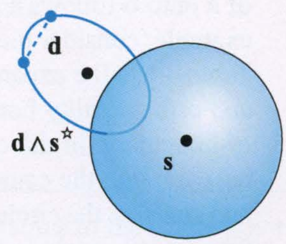
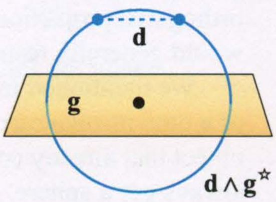
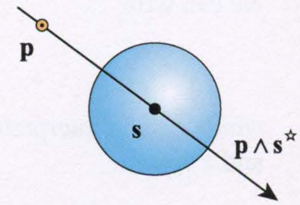
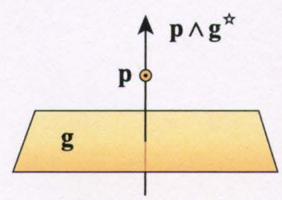
Expansion Operation	**Illustration**
Dipole containing round point **a** and orthogonal to sphere **s**. $$\begin{aligned} \mathbf{a}\wedge\mathbf{s}^{\star} = {} & (a_x s_u + a_w s_x)\,\mathbf{e}_{41} + (a_y s_z - a_z s_y)\,\mathbf{e}_{23} \\ & + (a_y s_u + a_w s_y)\,\mathbf{e}_{42} + (a_z s_x - a_x s_z)\,\mathbf{e}_{31} \\ & + (a_z s_u + a_w s_z)\,\mathbf{e}_{43} + (a_x s_y - a_y s_x)\,\mathbf{e}_{12} \\ & - (a_x s_w + a_u s_x)\,\mathbf{e}_{15} - (a_y s_w + a_u s_y)\,\mathbf{e}_{25} \\ & - (a_z s_w + a_u s_z)\,\mathbf{e}_{35} + (a_u s_u - a_w s_w)\,\mathbf{e}_{45} \end{aligned}$$	
Dipole containing round point **a** and orthogonal to plane **g**. $$\begin{aligned} \mathbf{a}\wedge\mathbf{g}^{\star} = {} & a_w g_x\,\mathbf{e}_{41} + (a_y g_z - a_z g_y)\,\mathbf{e}_{23} \\ & + a_w g_y\,\mathbf{e}_{42} + (a_z g_x - a_x g_z)\,\mathbf{e}_{31} \\ & + a_w g_z\,\mathbf{e}_{43} + (a_x g_y - a_y g_x)\,\mathbf{e}_{12} \\ & - (a_x g_w + a_u g_x)\,\mathbf{e}_{15} - (a_y g_w + a_u g_y)\,\mathbf{e}_{25} \\ & - (a_z g_w + a_u g_z)\,\mathbf{e}_{35} - a_w g_w\,\mathbf{e}_{45} \end{aligned}$$	
Circle containing dipole **d** and orthogonal to sphere **s**. $$\begin{aligned} \mathbf{d}\wedge\mathbf{s}^{\star} = {} & (d_{vy} s_z - d_{vz} s_y - d_{mx} s_u)\,\mathbf{e}_{423} + (d_{vz} s_x - d_{vx} s_z - d_{my} s_u)\,\mathbf{e}_{431} \\ & + (d_{vx} s_y - d_{vy} s_x - d_{mz} s_u)\,\mathbf{e}_{412} - (d_{mx} s_x + d_{my} s_y + d_{mz} s_z)\,\mathbf{e}_{321} \\ & - (d_{vx} s_w + d_{pw} s_x + d_{px} s_u)\,\mathbf{e}_{415} + (d_{pz} s_y - d_{py} s_z - d_{mx} s_w)\,\mathbf{e}_{235} \\ & - (d_{vy} s_w + d_{pw} s_y + d_{py} s_u)\,\mathbf{e}_{425} + (d_{px} s_z - d_{pz} s_x - d_{my} s_w)\,\mathbf{e}_{315} \\ & - (d_{vz} s_w + d_{pw} s_z + d_{pz} s_u)\,\mathbf{e}_{435} + (d_{py} s_x - d_{px} s_y - d_{mz} s_w)\,\mathbf{e}_{125} \end{aligned}$$	
Circle containing dipole **d** and orthogonal to plane **g**. $$\begin{aligned} \mathbf{d}\wedge\mathbf{g}^{\star} = {} & (d_{vy} g_z - d_{vz} g_y)\,\mathbf{e}_{423} + (d_{vz} g_x - d_{vx} g_z)\,\mathbf{e}_{431} \\ & + (d_{vx} g_y - d_{vy} g_x)\,\mathbf{e}_{412} - (d_{mx} g_x + d_{my} g_y + d_{mz} g_z)\,\mathbf{e}_{321} \\ & - (d_{vx} g_w + d_{pw} g_x)\,\mathbf{e}_{415} + (d_{pz} g_y - d_{py} g_z - d_{mx} g_w)\,\mathbf{e}_{235} \\ & - (d_{vy} g_w + d_{pw} g_y)\,\mathbf{e}_{425} + (d_{px} g_z - d_{pz} g_x - d_{my} g_w)\,\mathbf{e}_{315} \\ & - (d_{vz} g_w + d_{pw} g_z)\,\mathbf{e}_{435} + (d_{py} g_x - d_{px} g_y - d_{mz} g_w)\,\mathbf{e}_{125} \end{aligned}$$	
Line containing flat point **p** and orthogonal to sphere **s**. $$\begin{aligned} \mathbf{p}\wedge\mathbf{s}^{\star} = {} & -(p_w s_x + p_x s_u)\,\mathbf{e}_{415} + (p_z s_y - p_y s_z)\,\mathbf{e}_{235} \\ & - (p_w s_y + p_y s_u)\,\mathbf{e}_{425} + (p_x s_z - p_z s_x)\,\mathbf{e}_{315} \\ & - (p_w s_z + p_z s_u)\,\mathbf{e}_{435} + (p_y s_x - p_x s_y)\,\mathbf{e}_{125} \end{aligned}$$	
Line containing flat point **p** and orthogonal to plane **g**. $$\begin{aligned} \mathbf{p}\wedge\mathbf{g}^{\star} = {} & -p_w g_x\,\mathbf{e}_{415} + (p_z g_y - p_y g_z)\,\mathbf{e}_{235} \\ & - p_w g_y\,\mathbf{e}_{425} + (p_x g_z - p_z g_x)\,\mathbf{e}_{315} \\ & - p_w g_z\,\mathbf{e}_{435} + (p_y g_x - p_x g_y)\,\mathbf{e}_{125} \end{aligned}$$	

Table 4.18. These are the expansion operations in CGA over 3D space (part 1 of 3).

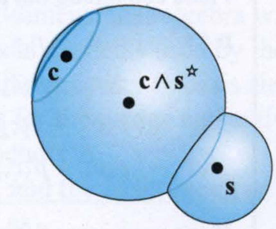
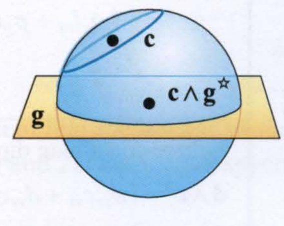
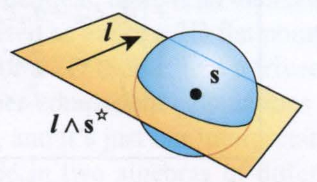
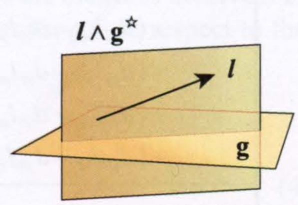
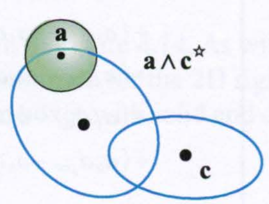
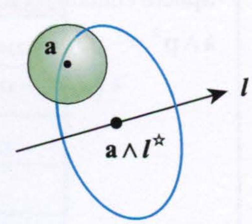
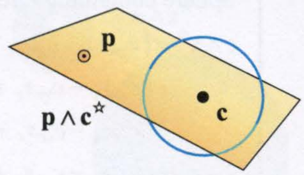
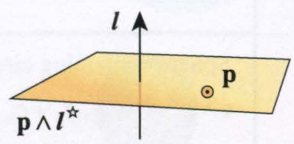
Expansion Operation	Illustration
Sphere containing circle **c** and orthogonal to sphere **s**. $\begin{aligned}\mathbf{c}\wedge\mathbf{s}^{\star} = {} & (c_{gw}s_u - c_{gx}s_x - c_{gy}s_y - c_{gz}s_z)\,\mathbf{e}_{1234} \\ & + (c_{vz}s_y - c_{vy}s_z + c_{mx}s_u - c_{gx}s_w)\,\mathbf{e}_{4235} \\ & + (c_{vx}s_z - c_{vz}s_x + c_{my}s_u - c_{gy}s_w)\,\mathbf{e}_{4315} \\ & + (c_{vy}s_x - c_{vx}s_y + c_{mz}s_u - c_{gz}s_w)\,\mathbf{e}_{4125} \\ & + (c_{mx}s_x + c_{my}s_y + c_{mz}s_z - c_{gw}s_w)\,\mathbf{e}_{3215}\end{aligned}$	
Sphere containing circle **c** and orthogonal to plane **g**. $\begin{aligned}\mathbf{c}\wedge\mathbf{g}^{\star} = {} & -(c_{gx}g_x + c_{gy}g_y + c_{gz}g_z)\,\mathbf{e}_{1234} \\ & + (c_{vz}g_y - c_{vy}g_z - c_{gx}g_w)\,\mathbf{e}_{4235} \\ & + (c_{vx}g_z - c_{vz}g_x - c_{gy}g_w)\,\mathbf{e}_{4315} \\ & + (c_{vy}g_x - c_{vx}g_y - c_{gz}g_w)\,\mathbf{e}_{4125} \\ & + (c_{mx}g_x + c_{my}g_y + c_{mz}g_z - c_{gw}g_w)\,\mathbf{e}_{3215}\end{aligned}$	
Plane containing line ***l*** and orthogonal to sphere **s**. $\begin{aligned}\boldsymbol{l}\wedge\mathbf{s}^{\star} = {} & (l_{vz}s_y - l_{vy}s_z + l_{mx}s_u)\,\mathbf{e}_{4235} + (l_{vx}s_z - l_{vz}s_x + l_{my}s_u)\,\mathbf{e}_{4315} \\ & + (l_{vy}s_x - l_{vx}s_y + l_{mz}s_u)\,\mathbf{e}_{4125} + (l_{mx}s_x + l_{my}s_y + l_{mz}s_z)\,\mathbf{e}_{3215}\end{aligned}$	
Plane containing line ***l*** and orthogonal to plane **g**. $\begin{aligned}\boldsymbol{l}\wedge\mathbf{g}^{\star} = {} & (l_{vz}g_y - l_{vy}g_z)\,\mathbf{e}_{4235} + (l_{vx}g_z - l_{vz}g_x)\,\mathbf{e}_{4315} \\ & + (l_{vy}g_x - l_{vx}g_y)\,\mathbf{e}_{4125} + (l_{mx}g_x + l_{my}g_y + l_{mz}g_z)\,\mathbf{e}_{3215}\end{aligned}$	
Circle containing round point **a** and orthogonal to circle **c**. $\begin{aligned}\mathbf{a}\wedge\mathbf{c}^{\star} = {} & (a_y c_{gz} - a_z c_{gy} - a_w c_{vx})\,\mathbf{e}_{423} + (a_z c_{gx} - a_x c_{gz} - a_w c_{vy})\,\mathbf{e}_{431} \\ & + (a_x c_{gy} - a_y c_{gx} - a_w c_{vz})\,\mathbf{e}_{412} + (a_x c_{vx} + a_y c_{vy} + a_z c_{vz})\,\mathbf{e}_{321} \\ & - (a_x c_{gw} + a_w c_{mx} + a_u c_{gx})\,\mathbf{e}_{415} + (a_z c_{my} - a_y c_{mz} - a_u c_{vx})\,\mathbf{e}_{235} \\ & - (a_y c_{gw} + a_w c_{my} + a_u c_{gy})\,\mathbf{e}_{425} + (a_x c_{mz} - a_z c_{mx} - a_u c_{vy})\,\mathbf{e}_{315} \\ & - (a_z c_{gw} + a_w c_{mz} + a_u c_{gz})\,\mathbf{e}_{435} + (a_y c_{mx} - a_x c_{my} - a_u c_{vz})\,\mathbf{e}_{125}\end{aligned}$	
Circle containing round point **a** and orthogonal to line ***l***. $\begin{aligned}\mathbf{a}\wedge\boldsymbol{l}^{\star} = {} & -a_w l_{vx}\,\mathbf{e}_{423} - a_w l_{vy}\,\mathbf{e}_{431} - a_w l_{vz}\,\mathbf{e}_{412} \\ & + (a_x l_{vx} + a_y l_{vy} + a_z l_{vz})\,\mathbf{e}_{321} \\ & - a_w l_{mx}\,\mathbf{e}_{415} + (a_z l_{my} - a_y l_{mz} - a_u l_{vx})\,\mathbf{e}_{235} \\ & - a_w l_{my}\,\mathbf{e}_{425} + (a_x l_{mz} - a_z l_{mx} - a_u l_{vy})\,\mathbf{e}_{315} \\ & - a_w l_{mz}\,\mathbf{e}_{435} + (a_y l_{mx} - a_x l_{my} - a_u l_{vz})\,\mathbf{e}_{125}\end{aligned}$	

Table 4.19. These are the expansion operations in CGA over 3D space (part 2 of 3).

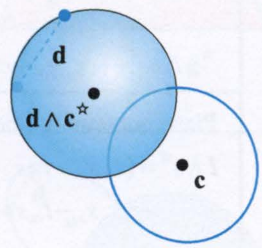
<table>
<tr><th>Expansion Operation</th><th>Illustration</th></tr>
<tr><td>
Plane containing flat point $\mathbf{p}$ and orthogonal to circle $\mathbf{c}$.

$$\begin{aligned}\mathbf{p}\wedge\mathbf{c}^{\star} = {} & (p_y c_{gz} - p_z c_{gy} - p_w c_{vx})\,\mathbf{e}_{4235} \\ & + (p_z c_{gx} - p_x c_{gz} - p_w c_{vy})\,\mathbf{e}_{4315} \\ & + (p_x c_{gy} - p_y c_{gx} - p_w c_{vz})\,\mathbf{e}_{4125} \\ & + (p_x c_{vx} + p_y c_{vy} + p_z c_{vz})\,\mathbf{e}_{3215}\end{aligned}$$
</td><td>


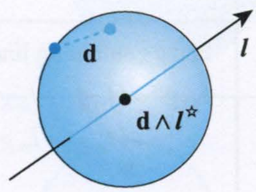
</td></tr>
<tr><td>
Plane containing flat point $\mathbf{p}$ and orthogonal to line $\boldsymbol{l}$.

$$\begin{aligned}\mathbf{p}\wedge\boldsymbol{l}^{\star} = {} & -p_w l_{vx}\,\mathbf{e}_{4235} - p_w l_{vy}\,\mathbf{e}_{4315} - p_w l_{vz}\,\mathbf{e}_{4125} \\ & + (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\,\mathbf{e}_{3215}\end{aligned}$$
</td><td>


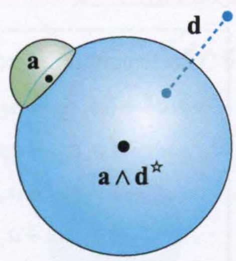
</td></tr>
<tr><td>
Sphere containing dipole $\mathbf{d}$ and orthogonal to circle $\mathbf{c}$.

$$\begin{aligned}\mathbf{d}\wedge\mathbf{c}^{\star} = {} & (d_{vx} c_{vx} + d_{vy} c_{vy} + d_{vz} c_{vz} + d_{mx} c_{gx} + d_{my} c_{gy} + d_{mz} c_{gz})\,\mathbf{e}_{1234} \\ & + (d_{vz} c_{my} - d_{vy} c_{mz} - d_{pw} c_{vx} + d_{py} c_{gz} - d_{pz} c_{gy} + d_{mx} c_{gw})\,\mathbf{e}_{4235} \\ & + (d_{vx} c_{mz} - d_{vz} c_{mx} - d_{pw} c_{vy} + d_{pz} c_{gx} - d_{px} c_{gz} + d_{my} c_{gw})\,\mathbf{e}_{4315} \\ & + (d_{vy} c_{mx} - d_{vx} c_{my} - d_{pw} c_{vz} + d_{px} c_{gy} - d_{py} c_{gx} + d_{mz} c_{gw})\,\mathbf{e}_{4125} \\ & + (d_{px} c_{vx} + d_{py} c_{vy} + d_{pz} c_{vz} + d_{mx} c_{mx} + d_{my} c_{my} + d_{mz} c_{mz})\,\mathbf{e}_{3215}\end{aligned}$$
</td><td>


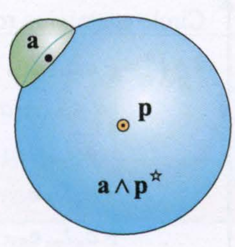
</td></tr>
<tr><td>
Sphere containing dipole $\mathbf{d}$ and orthogonal to line $\boldsymbol{l}$.

$$\begin{aligned}\mathbf{d}\wedge\boldsymbol{l}^{\star} = {} & (d_{vx} l_{vx} + d_{vy} l_{vy} + d_{vz} l_{vz})\,\mathbf{e}_{1234} \\ & + (d_{vz} l_{my} - d_{vy} l_{mz} - d_{pw} l_{vx})\,\mathbf{e}_{4235} \\ & + (d_{vx} l_{mz} - d_{vz} l_{mx} - d_{pw} l_{vy})\,\mathbf{e}_{4315} \\ & + (d_{vy} l_{mx} - d_{vx} l_{my} - d_{pw} l_{vz})\,\mathbf{e}_{4125} \\ & + (d_{px} l_{vx} + d_{py} l_{vy} + d_{pz} l_{vz} + d_{mx} l_{mx} + d_{my} l_{my} + d_{mz} l_{mz})\,\mathbf{e}_{3215}\end{aligned}$$
</td><td>


</td></tr>
<tr><td>
Sphere containing round point $\mathbf{a}$ and orthogonal to dipole $\mathbf{d}$.

$$\begin{aligned}\mathbf{a}\wedge\mathbf{d}^{\star} = {} & (a_x d_{vx} + a_y d_{vy} + a_z d_{vz} - a_w d_{pw})\,\mathbf{e}_{1234} \\ & + (a_z d_{my} - a_y d_{mz} + a_w d_{px} - a_u d_{vx})\,\mathbf{e}_{4235} \\ & + (a_x d_{mz} - a_z d_{mx} + a_w d_{py} - a_u d_{vy})\,\mathbf{e}_{4315} \\ & + (a_y d_{mx} - a_x d_{my} + a_w d_{pz} - a_u d_{vz})\,\mathbf{e}_{4125} \\ & + (a_u d_{pw} - a_x d_{px} - a_y d_{py} - a_z d_{pz})\,\mathbf{e}_{3215}\end{aligned}$$
</td><td>


</td></tr>
<tr><td>
Sphere containing round point $\mathbf{a}$ and centered at flat point $\mathbf{p}$.

$$\begin{aligned}\mathbf{a}\wedge\mathbf{p}^{\star} = {} & -a_w p_w\,\mathbf{e}_{1234} + a_w p_x\,\mathbf{e}_{4235} + a_w p_y\,\mathbf{e}_{4315} + a_w p_z\,\mathbf{e}_{4125} \\ & + (a_u p_w - a_x p_x - a_y p_y - a_z p_z)\,\mathbf{e}_{3215}\end{aligned}$$
</td><td>


</td></tr>
</table>

Table 4.20. These are the expansion operations in CGA over 3D space (part 3 of 3).

4.9 2D Round Geometry

The conformal geometric algebra over 2D Euclidean space is a four-dimensional algebra with the 16 basis elements listed in Table 4.21. These are exactly the same basis elements used by the rigid geometric algebra over 3D Euclidean space, but they are listed in a different order so basis elements without a factor of $\mathbf{e}_4$ come first for each grade. Multiplication tables for the wedge and antiwedge product showing the basis elements in this order are included in Appendix A.

In two dimensions, the basis vector $\mathbf{e}_3$ corresponds to the origin, and the basis vector $\mathbf{e}_4$ corresponds to the point at infinity. In terms of $\mathbf{e}_-$ and $\mathbf{e}_+$, they are defined as

$$\boxed{\begin{aligned}\mathbf{e}_3 &= \tfrac{1}{2}(\mathbf{e}_- - \mathbf{e}_+)\\ \mathbf{e}_4 &= \mathbf{e}_- + \mathbf{e}_+.\end{aligned}} \tag{4.64}$$

We have chosen to write the $\mathbf{e}_4$ factor on the left for each basis element that includes it so there is an exact match between the exterior algebras for 3D projective space and 2D conformal space. The volume element is $\mathbf{e}_{4321}$, which is equivalent to $\mathbf{e}_{1234}$.

Because the exterior algebras generated by the wedge and antiwedge products for the rigid algebra over 3D space and the conformal algebra over 2D space are identical, there is an interesting correspondence between the two structures. A vector can be interpreted as either a 3D flat point or a 2D round point, a bivector can be interpreted as either a 3D line or a 2D dipole, and a trivector can be interpreted as either a 3D plane or a 2D circle. The actual per-component calculations involved in performing join and meet operations are exactly the same, and it's just our interpretation of the results that is different. Geometric objects of the same grade in two algebras of different dimensionalities must have the same degrees of freedom, and that is discussed in Section 4.10.

The four-dimensional rigid and conformal algebras diverge where the metric is involved. In the conformal geometric algebra over 2D Euclidean space, the metric tensor $\mathfrak{g}$ with respect to the set of basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$ is the 4×4 matrix

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{4.65}$$

This metric tensor is extended to the full metric exomorphism $\mathbf{G}$ shown in Figure 4.14. As with the metric for 3D Euclidean space, $\mathbf{G}$ contains the metric and negative antimetric for the 2D rigid exterior algebra, and the pieces belonging to those metrics are enclosed in boxes with solid and dotted outlines.

Type	Grade	Basis Elements
Scalar	0	$\mathbf{1}$
Vectors	1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
Bivectors	2	$\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}$
Trivectors	3	$\mathbf{e}_{321}, \mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}$
Antiscalar	4	$\mathbb{1} = \mathbf{e}_{1234}$

Table 4.21. These are the 16 basis elements of the 4D conformal exterior algebra.

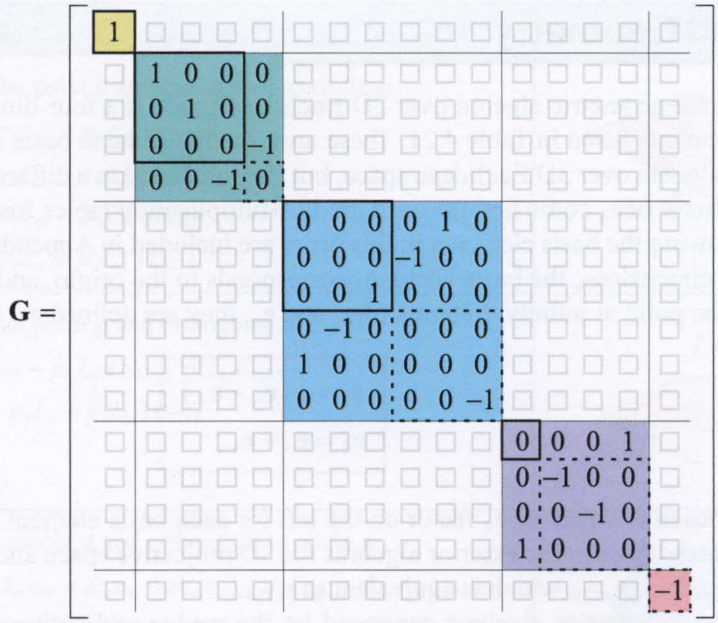


Figure 4.14. This is the metric exomorphism $\mathbf{G}$ for the 4D conformal exterior algebra, where rows and columns correspond to the basis elements in the order shown in Table 4.21. The metric antiexomorphism $\mathbb{G}$ is simply equal to $-\mathbf{G}$. The entries of the metric and the negative entries of the antimetric in the 3D rigid exterior algebra are enclosed in boxes with solid outlines and dotted outlines, respectively.

Objects representing two-dimensional flat geometries in the 3D rigid algebra are multiplied by $\mathbf{e}_4$ on the left to become flat geometries in the 4D conformal algebra. A flat point $\mathbf{p}$ is represented by the bivector

Flat point (2D)

$$\boxed{\mathbf{p} = p_x\mathbf{e}_{41} + p_y\mathbf{e}_{42} + p_z\mathbf{e}_{43},} \tag{4.66}$$

and a flat line $\mathbf{g}$ is represented by the trivector

Line (2D)

$$\boxed{\mathbf{g} = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}.} \tag{4.67}$$

Round objects in two-dimensional space have representations similar to round objects in three-dimensional space. The round part of an object is the collection of components that do not have a factor of $\mathbf{e}_4$, and the flat part is the collection of components that do have a factor of $\mathbf{e}_4$. There are three types of round objects in the 4D conformal algebra, and they are round points, dipoles, and circles. A round point $\mathbf{a}$ with center $\mathbf{p} = (p_x, p_y)$ and radius r has the form

Round point (2D)

$$\boxed{\mathbf{a} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + \mathbf{e}_3 + \frac{\mathbf{p}^2 + r^2}{2}\mathbf{e}_4.} \tag{4.68}$$

A round point $\mathbf{a}$ is expressed as a generic vector

$$\mathbf{a} = \underbrace{a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3}_{\text{Carrier Point}} + \underbrace{a_w\mathbf{e}_4}_{\substack{\text{Infinity} \\ (\text{when } a_x = a_y = a_z = 0)}} \tag{4.69}$$

with four components that are always labeled x, y, z, and w. The round part contains the flat carrier point coinciding with the center of the round point. If the round part is zero, leaving only the $\mathbf{e}_4$ component, then $\mathbf{a}$ is the point at infinity.

A dipole $\mathbf{d}$ with center $\mathbf{p} = (p_x, p_y)$, radius r, and normal vector $\mathbf{n} = (n_x, n_y)$ has the form

Dipole (2D)

$$\boxed{\begin{aligned}\mathbf{d} = n_x\mathbf{e}_{23} + n_y\mathbf{e}_{31} - (\mathbf{p}\cdot\mathbf{n})\,\mathbf{e}_{12} + \frac{\mathbf{p}^2 + r^2}{2}(n_y\mathbf{e}_{41} - n_x\mathbf{e}_{42})\\ -(p_x n_y - p_y n_x)(p_x\mathbf{e}_{41} + p_y\mathbf{e}_{42} + \mathbf{e}_{43}).\end{aligned}} \tag{4.70}$$

Note that the vector $\mathbf{n}$ does not have the same meaning for a dipole in two dimensions as it does for a dipole in three dimensions. In the same way that lines in two dimensions have a normal vector that is perpendicular to the line, so do dipoles, as shown in Figure 4.15. This is different from the direction $\mathbf{n}$ for a dipole in three dimensions, which points along the direction of the line carrying the dipole. A dipole is expressed as a generic bivector

$$\mathbf{d} = \underbrace{\overbrace{d_{gx}\mathbf{e}_{23} + d_{gy}\mathbf{e}_{31}}^{\text{Cocarrier Normal}} + d_{gz}\mathbf{e}_{12}}_{\text{Carrier Line}} + \underbrace{d_{px}\mathbf{e}_{41} + d_{py}\mathbf{e}_{42} + \overbrace{d_{pz}\mathbf{e}_{43}}^{\text{Cocarrier Position}}}_{\text{Flat Point (when } d_{gx} = d_{gy} = d_{gz} = 0)} \tag{4.71}$$

with components that are always labeled gx, gy, gz, px, py, and pz. The notation $\mathbf{d_g}$ means the vector $\mathbf{d_g} = (d_{gx}, d_{gy}, d_{gz})$, and the notation $\mathbf{d_p}$ means the vector $\mathbf{d_p} = (d_{px}, d_{py}, d_{pz})$. The dipole's carrier line occupies the round part of the bivector, and if the dipole has no round part, then it is a flat point.

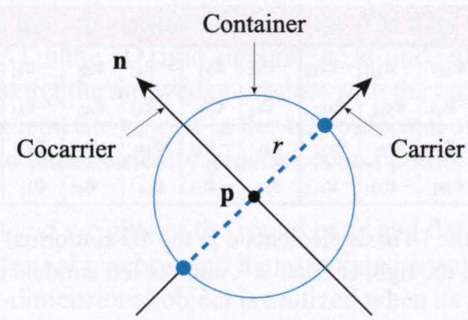


Figure 4.15. A dipole is shown here as the pair of blue points connected by a dashed line. The points of radius zero on its surface lie at a distance r from the center $\mathbf{p}$ in directions perpendicular to the normal vector $\mathbf{n}$.

A circle $\mathbf{c}$ with center $\mathbf{p} = (p_x, p_y)$ and radius r has the form

Circle (2D)

$$\boxed{\mathbf{c} = p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} - \mathbf{e}_{321} - \frac{\mathbf{p}^2 - r^2}{2}\mathbf{e}_{412}.} \tag{4.72}$$

As with spheres in three dimensions, the weight in the $\mathbf{e}_{321}$ component is intentionally negative so that the center can be directly read from the $\mathbf{e}_{423}$ and $\mathbf{e}_{431}$ components. A circle is expressed as a generic trivector

$$\mathbf{c} = \underbrace{c_w \mathbf{e}_{321}}_{\text{Carrier Plane}} + \underbrace{c_x \mathbf{e}_{423} + c_y \mathbf{e}_{431} + c_z \mathbf{e}_{412}}_{\substack{\text{Flat Line} \\ (\text{when } c_w = 0)}} \tag{4.73}$$

with four components labeled w, x, y, and z that are written with the round part first. If the round part is zero, then the circle is a line that includes the point at infinity.

Math Library Notes

- The `RoundPoint2D` class stores the four coordinates of a 4D vector representing a round point in two dimensions, and they are named `x`, `y`, `z`, and `w`.
- The `Dipole2D` class stores the six components of a 4D bivector representing a dipole in two dimensions. These components are divided into two parts named `g` and `p`, which are the carrier line as a `Line2D` and the flat point as a `FlatPoint2D`.
- The `Circle2D` class stores the four coordinates of a 4D trivector representing a circle in two dimensions, and they are named `w`, `x`, `y`, and `z`.
- The `FlatPoint2D` and `Line2D` classes are used for flat geometries in conformal geometric algebra as well as rigid geometric algebra.

Since the 4D conformal and rigid algebras are built on the same exterior algebra, the right and left complements are the same in both. However, duals are different because the metrics are different. The right and left duals and antiduals $\mathbf{u}^{\bigstar} = \overline{\mathbf{Gu}}$, $\mathbf{u}_{\bigstar} = \underline{\mathbf{Gu}}$, $\mathbf{u}^{\star} = \overline{\mathbb{G}\mathbf{u}}$, and $\mathbf{u}_{\star} = \underline{\mathbb{G}\mathbf{u}}$ are listed in Table 4.22 for all 16 basis elements in the algebra. As always in conformal geometric algebras, the antidual is the negative dual because $\mathbb{G} = -\mathbf{G}$. The duals of the five types of geometric object in the 4D conformal algebra are listed in Table 4.23.

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{321}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbb{1}$
$\mathbf{u}^{\bigstar}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$\mathbf{e}_1$	$\mathbf{e}_2$	$-\mathbf{e}_4$	$-\mathbf{1}$
$\mathbf{u}_{\bigstar}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$\mathbf{e}_3$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_4$	$-\mathbf{1}$
$\mathbf{u}^{\star}$	$-\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$\mathbf{e}_3$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_4$	$\mathbf{1}$
$\mathbf{u}_{\star}$	$-\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_3$	$\mathbf{e}_1$	$\mathbf{e}_2$	$-\mathbf{e}_4$	$\mathbf{1}$

Table 4.22. For each of the 16 basis elements $\mathbf{u}$ in the 4D conformal exterior algebra, this table lists the right dual $\mathbf{u}^{\bigstar}$, the left dual $\mathbf{u}_{\bigstar}$, the right antidual $\mathbf{u}^{\star}$, and the left antidual $\mathbf{u}_{\star}$.

Type	Dual
Flat point $\mathbf{p}$	$\mathbf{p}^{\bigstar} = p_z \mathbf{e}_{12} - p_y \mathbf{e}_{41} + p_x \mathbf{e}_{42}$
Line $\mathbf{g}$	$\mathbf{g}^{\bigstar} = g_x \mathbf{e}_1 + g_y \mathbf{e}_2 - g_z \mathbf{e}_4$
Round point $\mathbf{a}$	$\mathbf{a}^{\bigstar} = -a_z \mathbf{e}_{321} + a_x \mathbf{e}_{423} + a_y \mathbf{e}_{431} - a_w \mathbf{e}_{412}$
Dipole $\mathbf{d}$	$\mathbf{d}^{\bigstar} = d_{gy} \mathbf{e}_{23} - d_{gx} \mathbf{e}_{31} + d_{pz} \mathbf{e}_{12} - d_{py} \mathbf{e}_{41} + d_{px} \mathbf{e}_{42} - d_{gz} \mathbf{e}_{43}$
Circle $\mathbf{c}$	$\mathbf{c}^{\bigstar} = c_x \mathbf{e}_1 + c_y \mathbf{e}_2 - c_w \mathbf{e}_3 - c_z \mathbf{e}_4$

Table 4.23. These are the duals of the geometric objects arising in the conformal geometric algebra over 2D space. The antidual is always the negation of the dual in this algebra.

The carrier of a round object $\mathbf{u}$ is extracted by multiplying it by $\mathbf{e}_4$ with the wedge product. We define the carrier in two dimensions as

Carrier (2D)

$$\boxed{\operatorname{car}(\mathbf{u}) = \mathbf{e}_4 \wedge \mathbf{u},} \tag{4.74}$$

where $\mathbf{e}_4$ multiplies $\mathbf{u}$ on the left because $\mathbf{e}_4$ comes first in subscript ordering for all the basis elements. Multiplying on the right would not change the underlying geometry, and we only multiply on the left to avoid extraneous sign flips. As in three-dimensions, the cocarrier is the carrier of the antidual, and we define it as

Cocarrier (2D)

$$\boxed{\operatorname{ccr}(\mathbf{u}) = \mathbf{e}_4 \wedge \mathbf{u}^{\star}.} \tag{4.75}$$

The carriers and cocarriers for round objects in two dimensions are listed in Table 4.24. They are nontrivial only for a dipole, and that case is illustrated in Figure 4.15.

Type	Carrier	Cocarrier
Round point $\mathbf{a}$	$\operatorname{car}(\mathbf{a}) = a_x \mathbf{e}_{41} + a_y \mathbf{e}_{42} + a_z \mathbf{e}_{43}$	$\operatorname{ccr}(\mathbf{a}) = a_z \mathbb{1}$
Dipole $\mathbf{d}$	$\operatorname{car}(\mathbf{d}) = d_{gx} \mathbf{e}_{423} + d_{gy} \mathbf{e}_{431} + d_{gz} \mathbf{e}_{412}$	$\operatorname{ccr}(\mathbf{d}) = -d_{gy} \mathbf{e}_{423} + d_{gx} \mathbf{e}_{431} - d_{pz} \mathbf{e}_{412}$
Circle $\mathbf{c}$	$\operatorname{car}(\mathbf{c}) = c_w \mathbb{1}$	$\operatorname{ccr}(\mathbf{c}) = -c_x \mathbf{e}_{41} - c_y \mathbf{e}_{42} + c_w \mathbf{e}_{43}$

Table 4.24. These are the carriers and cocarriers of the round geometric objects arising in the conformal geometric algebra over 2D space, as defined by Equations (4.74) and (4.75).

Centers, containers, and partners have the same definitions in two dimensions as those given by Equations (4.39), (4.40), and (4.41) in three dimensions. They are listed for the two-dimensional round objects in Table 4.25. As in the 3D rigid algebra, the attitude of an object $\mathbf{u}$ in the 4D conformal algebra is extracted by taking the antiwedge product with the complement of the origin $\mathbf{e}_3$. The attitudes of the five types of geometric objects in the 4D conformal algebra are listed in Table 4.26. These attitudes can be used to parametrically generate round points contained by an object as described in Section 4.6.

Table 4.27 lists the bulks and weights of the round parts and flat parts of the five types of geometric objects in the 4D conformal algebra, and the associated norms are listed in Table 4.28. Just as in three dimensions, a two-dimensional object is unitized when its round weight has a magnitude of one. The round bulk norm corresponds to the weighted perpendicular distance between the origin and the object's carrier, and the flat weight norm corresponds to the weighted distance between the carrier's support and the object's center. The center norm of a round object $\mathbf{u}$ can be defined as either the combination of these distances as given by Equation (4.43) or by the dot product of $\mathbf{u}$ and its conformal conjugate as given by Equation (4.51). The radius norm continues to be defined in two dimensions with the antidot product as shown in Equation (4.45). The center norms and radius norms for round objects in the 4D conformal algebra are listed in Table 4.29.

There are only three ways to join two objects in two dimensions, and they are illustrated in Table 4.30. These are the 2D analogs of the first three 3D join operations listed in Table 4.15. Even though the exterior algebras for the 4D rigid algebra and 4D conformal algebra are equivalent, there is one more join operation listed here compared to those listed in Table 2.7 because we have made a distinction between a general dipole and a flat point, which is a specific kind of dipole with one end at infinity.

There are six nontrivial meet operations, and they are shown in Tables 4.31 and 4.32. These operations are analogs of 3D meet operations shown in Tables 4.16 and 4.17, but for cases where one or both objects are of dimensionality one greater. For instance, the meet of two circles in 2D is analogous to the meet of two spheres in 3D. The details of this particular case are discussed further in Comparison Chart #6. Finally, there are eight distinct expansion operations in two dimensions, and they are listed in Tables 4.33 and 4.34. As in three dimensions, a 2D object **a** can be projected onto another 2D object **b** of higher grade by calculating the expansion $\mathbf{a} \wedge \mathbf{b}^{\star}$ and then taking the meet of the result with **b**.

Type	Center
Round point **a**	$\operatorname{cen}(\mathbf{a}) = a_x a_z \mathbf{e}_1 + a_y a_z \mathbf{e}_2 + a_z^2 \mathbf{e}_3 + a_z a_w \mathbf{e}_4$
Dipole **d**	$\operatorname{cen}(\mathbf{d}) = -(d_{gy} d_{pz} + d_{gx} d_{gz})\mathbf{e}_1 + (d_{gx} d_{pz} - d_{gy} d_{gz})\mathbf{e}_2$ $+ (d_{gx}^2 + d_{gy}^2)\mathbf{e}_3 + (d_{pz}^2 - d_{gx} d_{py} + d_{gy} d_{px})\mathbf{e}_4$
Circle **c**	$\operatorname{cen}(\mathbf{c}) = -c_x c_w \mathbf{e}_1 - c_y c_w \mathbf{e}_2 + c_w^2 \mathbf{e}_3 + (c_x^2 + c_y^2 - c_z c_w)\mathbf{e}_4$

Type	Container
Round point **a**	$\operatorname{con}(\mathbf{a}) = -a_z^2 \mathbf{e}_{321} + a_x a_z \mathbf{e}_{423} + a_y a_z \mathbf{e}_{431} + (a_z a_w - a_x^2 - a_y^2)\mathbf{e}_{412}$
Dipole **d**	$\operatorname{con}(\mathbf{d}) = -(d_{gx}^2 + d_{gy}^2)\mathbf{e}_{321} - (d_{gx} d_{gz} + d_{gy} d_{pz})\mathbf{e}_{423}$ $+ (d_{gx} d_{pz} - d_{gy} d_{gz})\mathbf{e}_{431} + (d_{gy} d_{px} - d_{gx} d_{py} - d_{gz}^2)\mathbf{e}_{412}$
Circle **c**	$\operatorname{con}(\mathbf{c}) = -c_w^2 \mathbf{e}_{321} - c_x c_w \mathbf{e}_{423} - c_y c_w \mathbf{e}_{431} - c_z c_w \mathbf{e}_{412}$

Type	Partner
Round point **a**	$\operatorname{par}(\mathbf{a}) = a_x a_z^2 \mathbf{e}_1 + a_y a_z^2 \mathbf{e}_2 + a_z^3 \mathbf{e}_3 + (a_x^2 + a_y^2 - a_z a_w) a_z \mathbf{e}_4$
Dipole **d**	$\operatorname{par}(\mathbf{d}) = (d_{gx}^2 + d_{gy}^2)(d_{gx} \mathbf{e}_{23} + d_{gy} \mathbf{e}_{31} + d_{gz} \mathbf{e}_{12} + d_{pz} \mathbf{e}_{43})$ $+ (d_{gz}^2 - d_{pz}^2 + d_{gx} d_{py} - d_{gy} d_{px})(d_{gy} \mathbf{e}_{41} - d_{gx} \mathbf{e}_{42}) - d_{gz} d_{pz}(d_{gx} \mathbf{e}_{41} + d_{gy} \mathbf{e}_{42})$
Circle **c**	$\operatorname{par}(\mathbf{c}) = c_w^3 \mathbf{e}_{321} + c_x c_w^2 \mathbf{e}_{423} + c_y c_w^2 \mathbf{e}_{431} + (c_x^2 + c_y^2 - c_z c_w) c_w \mathbf{e}_{412}$

Table 4.25. These are the centers, containers, and partners of the round geometric objects arising in the conformal geometric algebra over 2D space.

Type	Attitude
Flat point **p**	$\operatorname{att}(\mathbf{p}) = -p_z \mathbf{e}_4$
Line **g**	$\operatorname{att}(\mathbf{g}) = -g_y \mathbf{e}_{41} + g_x \mathbf{e}_{42}$
Round point **a**	$\operatorname{att}(\mathbf{a}) = a_z \mathbf{1}$
Dipole **d**	$\operatorname{att}(\mathbf{d}) = d_{gy} \mathbf{e}_1 - d_{gx} \mathbf{e}_2 - d_{pz} \mathbf{e}_4$
Circle **c**	$\operatorname{att}(\mathbf{c}) = -c_w \mathbf{e}_{12} - c_y \mathbf{e}_{41} + c_x \mathbf{e}_{42}$

Table 4.26. These are the attitudes of the geometric objects arising in the conformal geometric algebra over 2D space, as defined by Equation (2.178).

Type	Round bulk and weight	Flat bulk and weight
Flat point $\mathbf{p}$	$\mathbf{p}_\bullet = 0$ $\mathbf{p}_\circ = 0$	$\mathbf{p}_\blacksquare = p_x\mathbf{e}_{41} + p_y\mathbf{e}_{42}$ $\mathbf{p}_\square = p_z\mathbf{e}_{43}$
Line $\mathbf{g}$	$\mathbf{g}_\bullet = 0$ $\mathbf{g}_\circ = 0$	$\mathbf{g}_\blacksquare = g_z\mathbf{e}_{412}$ $\mathbf{g}_\square = g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431}$
Round point $\mathbf{a}$	$\mathbf{a}_\bullet = a_x\mathbf{e}_1 + a_y\mathbf{e}_2$ $\mathbf{a}_\circ = a_z\mathbf{e}_3$	$\mathbf{a}_\blacksquare = a_w\mathbf{e}_4$ $\mathbf{a}_\square = 0$
Dipole $\mathbf{d}$	$\mathbf{d}_\bullet = d_{gz}\mathbf{e}_{12}$ $\mathbf{d}_\circ = d_{gx}\mathbf{e}_{23} + d_{gy}\mathbf{e}_{31}$	$\mathbf{d}_\blacksquare = d_{px}\mathbf{e}_{41} + d_{py}\mathbf{e}_{42}$ $\mathbf{d}_\square = d_{pz}\mathbf{e}_{43}$
Circle $\mathbf{c}$	$\mathbf{c}_\bullet = 0$ $\mathbf{c}_\circ = c_w\mathbf{e}_{321}$	$\mathbf{c}_\blacksquare = c_z\mathbf{e}_{412}$ $\mathbf{c}_\square = c_x\mathbf{e}_{423} + c_y\mathbf{e}_{431}$

Table 4.27. For each type of geometric object $\mathbf{u}$ in the conformal geometric algebra over 2D space, this table lists the round bulk $\mathbf{u}_\bullet$, the round weight $\mathbf{u}_\circ$, the flat bulk $\mathbf{u}_\blacksquare$, and the flat weight $\mathbf{u}_\square$.

Type	Round bulk norm	Round weight norm	Flat bulk norm	Flat weight norm
Flat point $\mathbf{p}$	$\lVert\mathbf{p}\rVert_\bullet = 0$	$\lVert\mathbf{p}\rVert_\circ = 0$	$\lVert\mathbf{p}\rVert_\blacksquare = \sqrt{p_x^2 + p_y^2}$	$\lVert\mathbf{p}\rVert_\square = \lvert p_z\rvert$
Line $\mathbf{g}$	$\lVert\mathbf{g}\rVert_\bullet = 0$	$\lVert\mathbf{g}\rVert_\circ = 0$	$\lVert\mathbf{g}\rVert_\blacksquare = \lvert g_z\rvert$	$\lVert\mathbf{g}\rVert_\square = \sqrt{g_x^2 + g_y^2}$
Round point $\mathbf{a}$	$\lVert\mathbf{a}\rVert_\bullet = \sqrt{a_x^2 + a_y^2}$	$\lVert\mathbf{a}\rVert_\circ = \lvert a_z\rvert$	$\lVert\mathbf{a}\rVert_\blacksquare = \lvert a_w\rvert$	$\lVert\mathbf{a}\rVert_\square = 0$
Dipole $\mathbf{d}$	$\lVert\mathbf{d}\rVert_\bullet = \lvert d_{gz}\rvert$	$\lVert\mathbf{d}\rVert_\circ = \sqrt{d_{gx}^2 + d_{gy}^2}$	$\lVert\mathbf{d}\rVert_\blacksquare = \sqrt{d_{px}^2 + d_{py}^2}$	$\lVert\mathbf{d}\rVert_\square = \lvert d_{pz}\rvert$
Circle $\mathbf{c}$	$\lVert\mathbf{c}\rVert_\bullet = 0$	$\lVert\mathbf{c}\rVert_\circ = \lvert c_w\rvert$	$\lVert\mathbf{c}\rVert_\blacksquare = \lvert c_z\rvert$	$\lVert\mathbf{c}\rVert_\square = \sqrt{c_x^2 + c_y^2}$

Table 4.28. These are the round bulk norms, round weight norms, flat bulk norms, and flat weight norms of geometric objects in the conformal geometric algebra over 2D space.

Type	Center Norm	Radius Norm
Round point $\mathbf{a}$	$\lVert\mathbf{a}\rVert_\odot = \sqrt{a_x^2 + a_y^2}$	$\lVert\mathbf{a}\rVert_\circledcirc = \sqrt{2a_z a_w - a_x^2 - a_y^2}$
Dipole $\mathbf{d}$	$\lVert\mathbf{d}\rVert_\odot = \sqrt{d_{gz}^2 + d_{pz}^2}$	$\lVert\mathbf{d}\rVert_\circledcirc = \sqrt{d_{pz}^2 - d_{gz}^2 - 2(d_{gx}d_{py} - d_{gy}d_{px})}$
Circle $\mathbf{c}$	$\lVert\mathbf{c}\rVert_\odot = \sqrt{c_x^2 + c_y^2}$	$\lVert\mathbf{c}\rVert_\circledcirc = \sqrt{c_x^2 + c_y^2 - 2c_z c_w}$

Table 4.29. This table lists the center norm $\lVert\mathbf{u}\rVert_\odot$ and radius norm $\lVert\mathbf{u}\rVert_\circledcirc$ for the round objects in the conformal geometric algebra over 2D space.

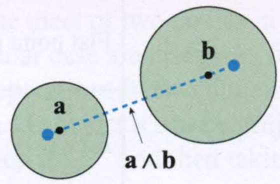
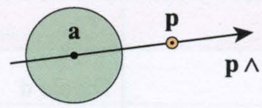
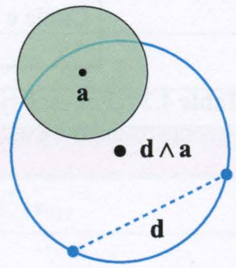
Join Operation	Illustration
Dipole containing round points **a** and **b**. $\mathbf{a} \wedge \mathbf{b} = (a_y b_z - a_z b_y)\,\mathbf{e}_{23} + (a_z b_x - a_x b_z)\,\mathbf{e}_{31} + (a_x b_y - a_y b_x)\,\mathbf{e}_{12} + (a_w b_x - a_x b_w)\,\mathbf{e}_{41} + (a_w b_y - a_y b_w)\,\mathbf{e}_{42} + (a_w b_z - a_z b_w)\,\mathbf{e}_{43}$	
Line containing flat point **p** and round point **a**. $\mathbf{p} \wedge \mathbf{a} = (p_y a_z - p_z a_y)\,\mathbf{e}_{423} + (p_z a_x - p_x a_z)\,\mathbf{e}_{431} + (p_x a_y - p_y a_x)\,\mathbf{e}_{412}$	
Circle containing dipole **d** and round point **a**. $\mathbf{d} \wedge \mathbf{a} = -(d_{gx} a_x + d_{gy} a_y + d_{gz} a_z)\,\mathbf{e}_{321} + (d_{py} a_z - d_{pz} a_y + d_{gx} a_w)\,\mathbf{e}_{423} + (d_{pz} a_x - d_{px} a_z + d_{gy} a_w)\,\mathbf{e}_{431} + (d_{px} a_y - d_{py} a_x + d_{gz} a_w)\,\mathbf{e}_{412}$	

Table 4.30. These are the join operations in CGA over 2D space.

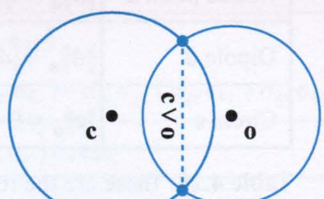
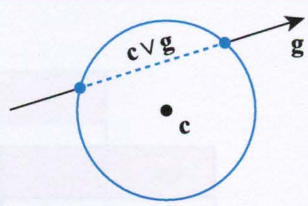
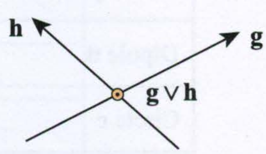
Meet Operation	Illustration
Dipole where circles **c** and **o** intersect. $\mathbf{c} \vee \mathbf{o} = (c_x o_w - c_w o_x)\,\mathbf{e}_{23} + (c_y o_w - c_w o_y)\,\mathbf{e}_{31} + (c_z o_w - c_w o_z)\,\mathbf{e}_{12} + (c_z o_y - c_y o_z)\,\mathbf{e}_{41} + (c_x o_z - c_z o_x)\,\mathbf{e}_{42} + (c_y o_x - c_x o_y)\,\mathbf{e}_{43}$	
Dipole where circle **c** and line **g** intersect. $\mathbf{c} \vee \mathbf{g} = -c_w g_x \mathbf{e}_{23} - c_w g_y \mathbf{e}_{31} - c_w g_z \mathbf{e}_{12} + (c_z g_y - c_y g_z)\,\mathbf{e}_{41} + (c_x g_z - c_z g_x)\,\mathbf{e}_{42} + (c_y g_x - c_x g_y)\,\mathbf{e}_{43}$	
Flat point where lines **g** and **h** intersect. $\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43}$	

Table 4.31. These are the meet operations in CGA over 2D space (part 1 of 2).

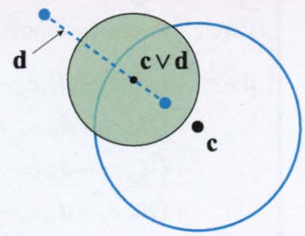
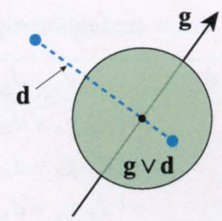
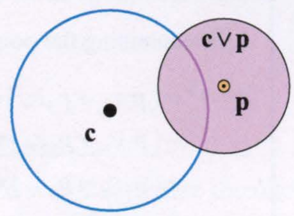
Meet Operation	Illustration
Round point contained by circle $\mathbf{c}$ and dipole $\mathbf{d}$. $\begin{aligned}\mathbf{c} \vee \mathbf{d} = &(c_z d_{gy} - c_y d_{gz} + c_w d_{px})\,\mathbf{e}_1 \\ &+ (c_x d_{gz} - c_z d_{gx} + c_w d_{py})\,\mathbf{e}_2 \\ &+ (c_y d_{gx} - c_x d_{gy} + c_w d_{pz})\,\mathbf{e}_3 \\ &- (c_x d_{px} + c_y d_{py} + c_z d_{pz})\,\mathbf{e}_4\end{aligned}$	
Round point contained by line $\mathbf{g}$ and dipole $\mathbf{d}$. $\begin{aligned}\mathbf{g} \vee \mathbf{d} = &(g_z d_{gy} - g_y d_{gz})\,\mathbf{e}_1 \\ &+ (g_x d_{gz} - g_z d_{gx})\,\mathbf{e}_2 \\ &+ (g_y d_{gx} - g_x d_{gy})\,\mathbf{e}_3 \\ &- (g_x d_{px} + g_y d_{py} + g_z d_{pz})\,\mathbf{e}_4\end{aligned}$	
Round point centered at flat point $\mathbf{p}$ and contained by circle $\mathbf{c}$. $\mathbf{c} \vee \mathbf{p} = c_w p_x \mathbf{e}_1 + c_w p_y \mathbf{e}_2 + c_w p_z \mathbf{e}_3 - (c_x p_x + c_y p_y + c_z p_z)\,\mathbf{e}_4$	

Table 4.32. These are the meet operations in CGA over 2D space (part 2 of 2).

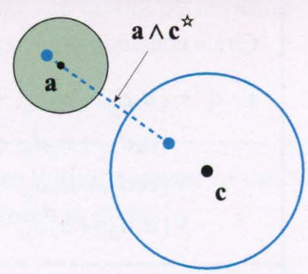
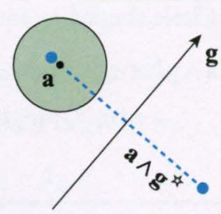
Expansion Operation	Illustration
Dipole containing round point $\mathbf{a}$ and orthogonal to circle $\mathbf{c}$. $\begin{aligned}\mathbf{a} \wedge \mathbf{c}^{\star} = &(a_z c_y + a_y c_w)\,\mathbf{e}_{23} - (a_z c_x + a_x c_w)\,\mathbf{e}_{31} + (a_y c_x - a_x c_y)\,\mathbf{e}_{12} \\ &- (a_x c_z + a_w c_x)\,\mathbf{e}_{41} - (a_y c_z + a_w c_y)\,\mathbf{e}_{42} + (a_w c_w - a_z c_z)\,\mathbf{e}_{43}\end{aligned}$	
Dipole containing round point $\mathbf{a}$ and orthogonal to line $\mathbf{g}$. $\begin{aligned}\mathbf{a} \wedge \mathbf{g}^{\star} = &a_z g_y \mathbf{e}_{23} - a_z g_x \mathbf{e}_{31} + (a_y g_x - a_x g_y)\,\mathbf{e}_{12} \\ &- (a_x g_z + a_w g_x)\,\mathbf{e}_{41} - (a_y g_z + a_w g_y)\,\mathbf{e}_{42} - a_z g_z \mathbf{e}_{43}\end{aligned}$	

Table 4.33. These are the expansion operations in CGA over 2D space (part 1 of 2).

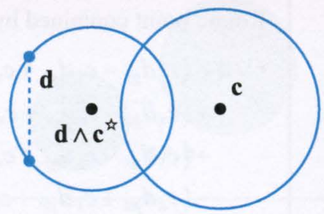
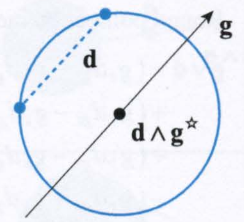
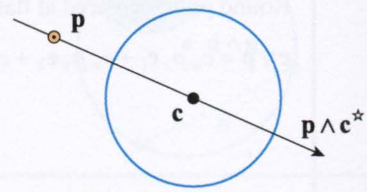
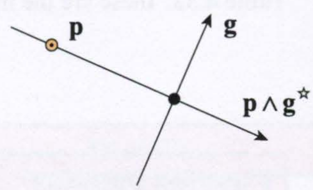
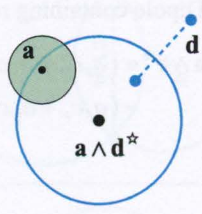
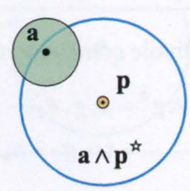
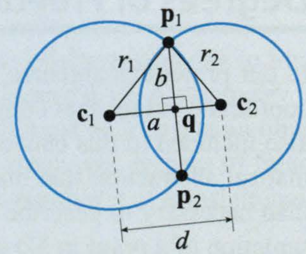
Expansion Operation	Illustration
Circle containing dipole **d** and orthogonal to circle **c**. $\mathbf{d} \wedge \mathbf{c}^{\star} = (d_{gx}c_x + d_{gy}c_y - d_{gz}c_w)\,\mathbf{e}_{321}$ $+ (d_{gx}c_z + d_{pz}c_y + d_{py}c_w)\,\mathbf{e}_{423}$ $+ (d_{gy}c_z - d_{pz}c_x - d_{px}c_w)\,\mathbf{e}_{431}$ $+ (d_{py}c_x - d_{px}c_y + d_{gz}c_z)\,\mathbf{e}_{412}$	
Circle containing dipole **d** and orthogonal to line **g**. $\mathbf{d} \wedge \mathbf{g}^{\star} = (d_{gx}g_x + d_{gy}g_y)\,\mathbf{e}_{321}$ $+ (d_{gx}g_z + d_{pz}g_y)\,\mathbf{e}_{423}$ $+ (d_{gy}g_z - d_{pz}g_x)\,\mathbf{e}_{431}$ $+ (d_{py}g_x - d_{px}g_y + d_{gz}g_z)\,\mathbf{e}_{412}$	
Line containing flat point **p** and orthogonal to circle **c**. $\mathbf{p} \wedge \mathbf{c}^{\star} = (p_z c_y + p_y c_w)\,\mathbf{e}_{423}$ $- (p_z c_x + p_x c_w)\,\mathbf{e}_{431}$ $+ (p_y c_x - p_x c_y)\,\mathbf{e}_{412}$	
Line containing flat point **p** and orthogonal to line **g**. $\mathbf{p} \wedge \mathbf{g}^{\star} = p_z g_y \mathbf{e}_{423} - p_z g_x \mathbf{e}_{431} + (p_y g_x - p_x g_y)\,\mathbf{e}_{412}$	
Circle containing round point **a** and orthogonal to dipole **d**. $\mathbf{a} \wedge \mathbf{d}^{\star} = (a_x d_{gy} - a_y d_{gx} + a_z d_{pz})\,\mathbf{e}_{321}$ $- (a_z d_{px} + a_y d_{gz} + a_w d_{gy})\,\mathbf{e}_{423}$ $+ (a_x d_{gz} + a_w d_{gx} - a_z d_{py})\,\mathbf{e}_{431}$ $+ (a_x d_{px} + a_y d_{py} - a_w d_{pz})\,\mathbf{e}_{412}$	
Circle containing round point **a** and centered at flat point **p**. $\mathbf{a} \wedge \mathbf{p}^{\star} = a_z p_z \mathbf{e}_{321} - a_z p_x \mathbf{e}_{423} - a_z p_y \mathbf{e}_{431}$ $+ (a_x p_x + a_y p_y - a_w p_z)\,\mathbf{e}_{412}$	

Table 4.34. These are the expansion operations in CGA over 2D space (part 2 of 2).

Comparison Chart #6

Intersection of 2D Circles

Determine whether a circle with center $\mathbf{c}_1$ and radius r_1 and another circle with center $\mathbf{c}_2$ and radius r_2 intersect. If so, calculate the points of intersection $\mathbf{p}_1$ and $\mathbf{p}_2$.



Conventional Methods	Geometric Algebra
Let $\mathbf{v} = \mathbf{c}_2 - \mathbf{c}_1$ be the difference between the two centers, and let $d = \lVert\mathbf{v}\rVert$. Let $\mathbf{q}$ be the point where the line connecting the centers $\mathbf{c}_1$ and $\mathbf{c}_2$ intersects the line connecting the points $\mathbf{p}_1$ and $\mathbf{p}_2$. The point $\mathbf{q}$ is halfway between the points $\mathbf{p}_1$ and $\mathbf{p}_2$. Let a be the distance between $\mathbf{q}$ and $\mathbf{c}_1$, and let b be the distance between $\mathbf{q}$ and $\mathbf{p}_1$.	Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be circles as defined by Equation (4.72).
We have the Pythagorean relationships $a^2 + b^2 = r_1^2 \text{ and } (d-a)^2 + b^2 = r_2^2.$ Eliminating b^2 lets us write a as $a = \dfrac{d^2 + r_1^2 - r_2^2}{2d}.$ Plugging this into $a^2 + b^2 = r_1^2$ gives us $b^2 = r_1^2 - \dfrac{\left(d^2 + r_1^2 - r_2^2\right)^2}{4d^2}.$ If $b^2 < 0$, then the circles do not intersect.	Calculate the dipole $\mathbf{d} = \mathbf{c}_1 \vee \mathbf{c}_2$ where the circles meet using the formula in Table 4.31. The squared radius of the dipole is given by $\dfrac{\lVert\mathbf{d}\rVert_\circledcirc^2}{\lVert\mathbf{d}\rVert_\bigcirc^2} = \dfrac{d_{pz}^2 - d_{gz}^2 - 2\left(d_{gx}d_{py} - d_{gy}d_{px}\right)}{d_{gx}^2 + d_{gy}^2}.$ If $\lVert\mathbf{d}\rVert_\circledcirc^2 < 0$, then the circles do not intersect. Otherwise, we unitize the dipole by dividing its components by $\lVert\mathbf{d}\rVert_\bigcirc$.
The point $\mathbf{q}$ is given by $\mathbf{q} = \mathbf{c}_1 + (a/d)\,\mathbf{v}.$ The points $\mathbf{p}_1$ and $\mathbf{p}_2$ are then given by $\mathbf{p}_{1,2} = \mathbf{q} \pm (b/d)\,\mathbf{u},$ where $\mathbf{u} = (-v_y, v_x)$.	The point $\mathbf{q}$ coincides with the center of the unitized dipole $\mathbf{d}$, which is given by $\operatorname{cen}(\mathbf{d}) = -\left(d_{gx}d_{gz} + d_{gy}d_{pz}\right)\mathbf{e}_1 + \left(d_{gx}d_{pz} - d_{gy}d_{gz}\right)\mathbf{e}_2 + \mathbf{e}_3 + \left(d_{pz}^2 - d_{gx}d_{py} + d_{gy}d_{px}\right)\mathbf{e}_4.$ The attitude of the dipole $\mathbf{d}$ is given by $\operatorname{att}(\mathbf{d}) = d_{gy}\mathbf{e}_1 - d_{gx}\mathbf{e}_2 - d_{pz}\mathbf{e}_4.$ The points $\mathbf{p}_1$ and $\mathbf{p}_2$ are then given by $\mathbf{p}_{1,2} = \operatorname{cen}(\mathbf{d}) \pm \dfrac{\lVert\mathbf{d}\rVert_\circledcirc}{\lVert\mathbf{d}\rVert_\bigcirc}\operatorname{att}(\mathbf{d}),$ where we only calculate the x and y coordinates.

4.10 Degrees of Freedom

In our projective algebras, every distinct instance of a geometric object has a specific number of coordinate values that completely encode its position and attitude. For round objects, the radius is also included in this encoding. The projective representation of each type of geometry is a mathematical abstraction that makes computation convenient, but it actually contains more information than necessary to describe any geometry with a finite position. The simplest example is the representation of a point in 3D space by a homogeneous vector with four components. Even though we are using four numbers to represent a point, there is a redundancy that leads to infinitely many 4D vectors collapsing to the same location in space when we divide by the weight. There are ultimately only three numerical values necessary to identify any specific point in 3D space, and this minimum number is called the point's *degrees of freedom*. The number of values necessary to identify lines, planes, and round objects is also less than the number of coordinates in their representations, and not just by a difference of one. The term "degrees of freedom", often abbreviated DOF, generally refers to the minimum number of independent numerical values required to fully describe the state of a system. In the case of a specific type of geometric object, degrees of freedom means the minimum number of values necessary to distinguish among all instances of that type.

We can determine a general formula for the degrees of freedom possessed by a k-dimensional flat geometry existing inside an n-dimensional ambient space by considering its position and attitude separately. First, we identify the support of the geometry, which is the point contained in the geometry that is closest to the origin. There are no restrictions on the location of the support, so it has n degrees of translational freedom because it has n coordinates. If the geometry is itself a point, then we're already done because there is no attitude to consider. Otherwise, we imagine that the geometry is attached to the support and restricted to be perpendicular to the direction $\mathbf{v}$ connecting the support to the origin. This establishes part of the attitude, but the geometry is still free to rotate within the $(n-1)$-dimensional subspace perpendicular to $\mathbf{v}$. We can figure out how many degrees of freedom are contained in this rotation by considering how a k-dimensional subspace can be embedded inside an m-dimensional ambient space. We can choose k basis vectors to span the subspace, and each one has m components determining its direction in the ambient space. This establishes an absolute maximum of km degrees of freedom. But there are infinitely many ways to choose a basis for the same subspace, so we need to remove all linear transformations of the subspace onto itself to account for the redundancy. For a basis aligned to the subspace, these transformations correspond to all invertible $k \times k$ matrices, so k^2 degrees should be subtracted for a reduced count of $k(m-k)$ degrees of freedom for the rotation. After adding this to the n degrees of translational freedom and substituting $m = n-1$, we have $n + k(n-k-1)$ total degrees of freedom. This can be simplified slightly to

Degrees of freedom

$$\boxed{\mathrm{DOF}(n,k) = (n-k)(k+1),} \tag{4.76}$$

where the notation $\mathrm{DOF}(n,k)$ means the degrees of freedom that a k-dimensional flat geometry has inside an n-dimensional space.

A k-dimensional round object has a greater number of degrees of freedom than a k-dimensional flat object, and these degrees of freedom come from three sources. First, the object's carrier is a $(k+1)$-dimensional flat geometry that is free to have any position and attitude in space, so its degrees of freedom are given by Equation (4.76), but with k increased by one. Second, the center of the round object can exist anywhere on the carrier, and this adds another $k+1$ degrees of freedom. Finally, one more degree of freedom must be added to account for an arbitrary radius. The total number of degrees of freedom for a k-dimensional round object in an n-dimensional space is thus given by $(n-k-1)(k+2)+k+2$. This number is also be equal to $\mathrm{DOF}(n+1,k+1)$, which reflects a correspondence between the representations of k-dimensional flat objects in n dimensions and $(k-1)$-dimensional round objects in $n-1$ dimensions.

The degrees of freedom possessed by all flat objects in spaces of dimension one to four and all round objects in spaces of dimension zero to three are shown in Figure 4.16. A symmetry is clearly visible in the diagram because $\mathrm{DOF}(n,k) = \mathrm{DOF}(n, n-k-1)$. It must be true that an object occupying k dimensions inside an n-dimensional space has the same rotational degrees of freedom as a object occupying all except k dimensions in addition to the same number n of translational degrees of freedom.

Geometric reasoning allowed us to derive Equation (4.76) and figure out how many degrees of freedom each type of object must have. We can arrive at the same results through a purely algebraic route by considering the components of each type's projective representation. A k-dimensional flat object existing inside an n-dimensional Euclidean space is represented in a rigid geometric algebra by a homogeneous $(k+1)$-vector in an $(n+1)$-dimensional projective space. The number of coordinates in such a representation is thus $\binom{n+1}{k+1}$, and this places an upper limit on the possible degrees of freedom. One degree is always removed to account for homogeneity because any nonzero scaling of the algebraic representation has no effect on the actual geometry. For vectors and antivectors,

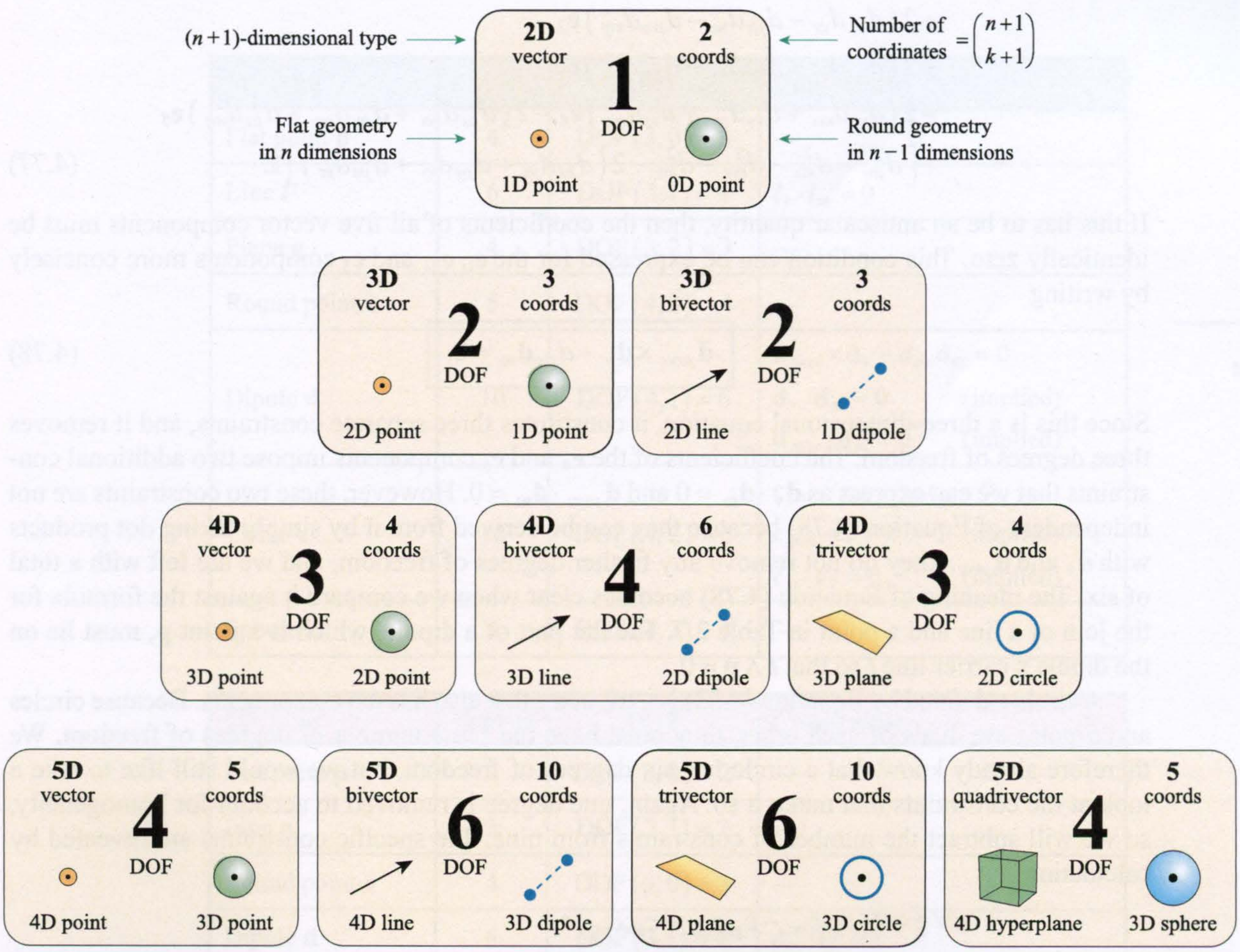


Figure 4.16. The four rows in this diagram show the degrees of freedom possessed by flat geometric objects in spaces of dimension $n=1$ to $n=4$. In each row, the dimensionality of the geometry begins at $k=0$ in the leftmost cell and increases one at a time going to the right. The same four rows also contain the degrees of freedom possessed by round geometric objects in spaces of one dimension lower. The dimensionality of each round object is one less than the flat object sharing the same cell.

this reduces the degrees of freedom to the dimension n, which we know must be the lower limit for a point. For any other type of object that has a grade in between that of vectors and antivectors, there are internal constraints imposed on the object's components by Equation (3.68) or (3.73). Each constraint takes away a degree of freedom because it means that the values of the components cannot be chosen arbitrarily and must be dependent on each other in some way. For a line $\boldsymbol{l}$ in 3D space, we have already seen in Equation (2.37) that the constraint $\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$ must always be true, and this reduces the degrees of freedom by one in addition to the one removed for homogeneity. The six components of a 3D line therefore have only four degrees of freedom, and this is shown in the third row of Figure 4.16. The same constraint applies to a dipole in 2D Euclidean space because it is a six-component bivector in the conformal algebra.

In the conformal algebra over a 3D Euclidean space, dipoles and circles both have representations that are not vectors or antivectors. A dipole $\mathbf{d}$ defined by Equation (4.31) is a bivector with ten components and thus a maximum of ten degrees of freedom. Removing one degree for homogeneity brings us down to nine degrees. The specific constraints on the components can be determined by calculating $\mathbf{d} \vee \underset{\sim}{\mathbf{d}}$ and requiring that it be equal to the antiscalar given by $\mathbf{d} \circ \mathbf{d}$. The geometric antiproduct gives us

$$\begin{aligned}
\mathbf{d} \vee \underset{\sim}{\mathbf{d}} = {} & 2\left(d_{py}d_{vz} - d_{pz}d_{vy} - d_{pw}d_{mx}\right)\mathbf{e}_1 \\
& + 2\left(d_{pz}d_{vx} - d_{px}d_{vz} - d_{pw}d_{my}\right)\mathbf{e}_2 \\
& + 2\left(d_{px}d_{vy} - d_{py}d_{vx} - d_{pw}d_{mz}\right)\mathbf{e}_3 \\
& - 2\left(d_{vx}d_{mx} + d_{vy}d_{my} + d_{vz}d_{mz}\right)\mathbf{e}_4 - 2\left(d_{px}d_{mx} + d_{py}d_{my} + d_{pz}d_{mz}\right)\mathbf{e}_5 \\
& + \left[d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - 2\left(d_{px}d_{vx} + d_{py}d_{vy} + d_{pz}d_{vz}\right)\right]\mathbb{1}.
\end{aligned} \tag{4.77}$$

If this has to be an antiscalar quantity, then the coefficients of all five vector components must be identically zero. This condition can be expressed for the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ components more concisely by writing

Dipole constraint

$$\boxed{\mathbf{d}_{pxyz} \times \mathbf{d}_{\mathbf{v}} - d_{pw}\mathbf{d}_{\mathbf{m}} = 0.} \tag{4.78}$$

Since this is a three-dimensional equation, it constitutes three separate constraints, and it removes three degrees of freedom. The coefficients of the $\mathbf{e}_4$ and $\mathbf{e}_5$ components impose two additional constraints that we can express as $\mathbf{d}_{\mathbf{v}} \cdot \mathbf{d}_{\mathbf{m}} = 0$ and $\mathbf{d}_{pxyz} \cdot \mathbf{d}_{\mathbf{m}} = 0$. However, these two constraints are not independent of Equation (4.78) because they can be derived from it by simply taking dot products with $\mathbf{d}_{\mathbf{v}}$ and $\mathbf{d}_{pxyz}$. They do not remove any further degrees of freedom, and we are left with a total of six. The meaning of Equation (4.78) becomes clear when we compare it against the formula for the join of a line and a point in Table 2.7. The flat part of a dipole, which is a point $\mathbf{p}$, must lie on the dipole's carrier line $\boldsymbol{l}$ so that $\boldsymbol{l} \wedge \mathbf{p} = 0$.

A circle $\mathbf{c}$ defined by Equation (4.33) is a trivector that also has ten components. Because circles and dipoles are duals of each other, they must have the same number of degrees of freedom. We therefore already know that a circle has six degrees of freedom, but we would still like to take a look at the constraints that make it so. Again, one degree is removed to account for homogeneity, so we will subtract the number of constraints from nine. The specific constraints are revealed by calculating

$$\begin{aligned}
\mathbf{c} \vee \underset{\sim}{\mathbf{c}} = {} & 2\left(c_{gy}c_{mz} - c_{gz}c_{my} - c_{gw}c_{vx}\right)\mathbf{e}_1 \\
& + 2\left(c_{gz}c_{mx} - c_{gx}c_{mz} - c_{gw}c_{vy}\right)\mathbf{e}_2 \\
& + 2\left(c_{gx}c_{my} - c_{gy}c_{mx} - c_{gw}c_{vz}\right)\mathbf{e}_3 \\
& + 2\left(c_{gx}c_{vx} + c_{gy}c_{vy} + c_{gz}c_{vz}\right)\mathbf{e}_4 + 2\left(c_{mx}c_{vx} + c_{my}c_{vy} + c_{mz}c_{vz}\right)\mathbf{e}_5 \\
& + \left[c_{vx}^2 + c_{vy}^2 + c_{vz}^2 - c_{gw}^2 + 2\left(c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz}\right)\right]\mathbb{1}.
\end{aligned} \tag{4.79}$$

As before, all five vector coefficients must be identically zero, and they correspond to three independent constraints altogether. From the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ components, we have

Circle constraint

$$\boxed{\mathbf{c}_{gxyz} \times \mathbf{c}_{\mathbf{m}} - c_{gw}\mathbf{c}_{\mathbf{v}} = 0.} \tag{4.80}$$

This equation removes three degrees of freedom, reducing the total to six. The coefficients of the $\mathbf{e}_4$ and $\mathbf{e}_5$ components impose the additional constraints $\mathbf{c}_{gxyz} \cdot \mathbf{c}_{\mathbf{v}} = 0$ and $\mathbf{c}_{\mathbf{m}} \cdot \mathbf{c}_{\mathbf{v}} = 0$, but these can be derived from Equation (4.80) by taking dot products with $\mathbf{c}_{gxyz}$ and $\mathbf{c}_{\mathbf{m}}$. As with dipoles, Equation (4.80) has a meaning when we compare it to a formula in Table 2.7, this time the meet of a plane and a line. The flat part of a circle, which is a line $\boldsymbol{l}$, must be contained in the circle's carrier plane $\mathbf{g}$ so that $\mathbf{g} \vee \boldsymbol{l} = 0$.

The fourth row of Figure 4.16 includes the six degrees of freedom shared by dipoles and circles in 3D space. Table 4.35 lists all of the algebraic constraints that we have derived for flat and round objects in 3D space, including the additional dependent constraints that are implied by Equations (4.78) and (4.80). The table also highlights, for both 3D space and 2D space, the differences between the number of coordinates in the representation of each geometry type and the degrees of freedom due to homogeneity and any internal constraints on the coordinate values.

3D Type	Coords	DOF	Constraints	
Flat point $\mathbf{p}$	4	$\operatorname{DOF}(3,0) = 3$	—	
Line $\boldsymbol{l}$	6	$\operatorname{DOF}(3,1) = 4$	$\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$	
Plane $\mathbf{g}$	4	$\operatorname{DOF}(3,2) = 3$	—	
Round point $\mathbf{a}$	5	$\operatorname{DOF}(4,0) = 4$	—	
Dipole $\mathbf{d}$	10	$\operatorname{DOF}(4,1) = 6$	$\mathbf{d}_{pxyz} \times \mathbf{d}_{\mathbf{v}} - d_{pw}\mathbf{d}_{\mathbf{m}} = 0$	
			$\mathbf{d}_{\mathbf{v}} \cdot \mathbf{d}_{\mathbf{m}} = 0$	(implied)
			$\mathbf{d}_{pxyz} \cdot \mathbf{d}_{\mathbf{m}} = 0$	(implied)
Circle $\mathbf{c}$	10	$\operatorname{DOF}(4,2) = 6$	$\mathbf{c}_{gxyz} \times \mathbf{c}_{\mathbf{m}} - c_{gw}\mathbf{c}_{\mathbf{v}} = 0$	
			$\mathbf{c}_{gxyz} \cdot \mathbf{c}_{\mathbf{v}} = 0$	(implied)
			$\mathbf{c}_{\mathbf{m}} \cdot \mathbf{c}_{\mathbf{v}} = 0$	(implied)
Sphere $\mathbf{s}$	5	$\operatorname{DOF}(4,3) = 4$	—	

2D Type	Coords	DOF	Constraints
Flat point $\mathbf{p}$	3	$\operatorname{DOF}(2,0) = 2$	—
Line $\mathbf{g}$	3	$\operatorname{DOF}(2,1) = 2$	—
Round point $\mathbf{a}$	4	$\operatorname{DOF}(3,0) = 3$	—
Dipole $\mathbf{d}$	6	$\operatorname{DOF}(3,1) = 4$	$\mathbf{d}_{\mathbf{g}} \cdot \mathbf{d}_{\mathbf{p}} = 0$
Circle $\mathbf{c}$	4	$\operatorname{DOF}(3,2) = 3$	—

Table 4.35. These tables summarize the number of homogeneous coordinates, the degrees of freedom (DOF), and the algebraic constraints for the types of flat objects and round objects appearing in the rigid algebra and conformal algebra over 3D (blue) and 2D (red) Euclidean spaces.

Chapter 5

Conformal Transformations

We end the book with a short look at the kinds of transformations that can be performed in the conformal geometric algebra. Chapter 3 provided a comprehensive examination of the rigid transformations that arise in projective geometric algebra, which include all Euclidean isometries that preserve distances and angles. Importantly, we saw that proper and improper isometries performed with sandwich products required a bit more computation than the equivalent matrix formulations. The conformal geometric algebra introduced in Chapter 4 gets its name because the transformations that can be performed within it make up a much larger set of transformations that preserve angles but not necessarily distances. It is beyond the scope of this chapter to cover the wide variety of conformal transformations in its entirety, but we do highlight some of the most interesting characteristics. The sandwich products that implement conformal transformations begin crossing into the realm of the computationally absurd, reaching a point where it would be a challenge to fit some per-component calculations onto a single page. We will provide a small taste of the complexity for relatively simple transformations like translation and dilation to demonstrate that it would be much more practical to implement them with conventional methods. As such, the material presented in this chapter should be considered primarily an exhibition of mathematical beauty.

5.1 Generalized Operators

In the rigid geometric algebra, a rigid transformation is performed on an object $\mathbf{u}$ by calculating the sandwich product $\mathbf{X} ⟇ \mathbf{u} ⟇ \underset{\sim}{\mathbf{X}}$ for some operator $\mathbf{X}$ that could be a motor or flector. Complement isometries are performed by using the geometric product instead of the geometric antiproduct. Conformal transformations are performed in the same manner except now, $\mathbf{u}$ can be a flat object or round object, and in the case of a round object, $\mathbf{u}$ can be real, imaginary, or null. The operator $\mathbf{X}$ is much more complicated because it can have up to 16 components, twice as many as the eight components that a general operator has in the rigid algebra.

Calculating the geometric product between elements using the basis vectors $\mathbf{e}_4$ and $\mathbf{e}_5$ can be tricky due to the nondiagonal metric tensor. In order to calculate a geometric product in the conformal algebra by hand using the method described in Section 3.1, it's necessary to first use Equation (4.15) to convert all factors of $\mathbf{e}_4$ and $\mathbf{e}_5$ to the basis that includes the vectors $\mathbf{e}_-$ and $\mathbf{e}_+$. Then the multiplication can be carried out with a diagonal metric tensor, and the result can be converted back to the $\mathbf{e}_4$-$\mathbf{e}_5$ basis with Equation (4.10). The complete 1024-entry multiplication tables for the geometric product and antiproduct with respect to the $\mathbf{e}_4$-$\mathbf{e}_5$ basis are each displayed across two pages in Appendix A.

The reverses of all 32 basis elements in the 5D conformal algebra are listed in Table 5.1. The reverse and antireverse operations are always the same in this case because the exponent in Equation (3.54) is always even. We still write the tilde below the operator in the sandwich antiproduct

$\mathbf{u}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\tilde{\mathbf{u}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{45}$

$\mathbf{u}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
$\tilde{\mathbf{u}}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{415}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{435}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$

Table 5.1. For each of the 32 basis elements $\mathbf{u}$ in the 5D conformal exterior algebra, these are the reverses $\tilde{\mathbf{u}}$. (Reverses and antireverses are equivalent in five dimensions.)

$\mathbf{X} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{X}}$ to indicate antireverse so we're consistent with algebras of other dimensionalities. Keep in mind that reverse and antireverse are not equivalent operations in the 4D conformal algebra discussed below in Section 5.4.

5.1.1 Rigid Transformations

All of the rigid transformations developed in Chapter 3 continue to function in the same way in the conformal algebra. The only difference is that all rigid operators are multiplied by $\mathbf{e}_5$ just like the flat geometric objects were in Equations (4.25), (4.26), and (4.27). Thus, a general rigid motor $\mathbf{Q}$ in the conformal algebra over three-dimensional space has the form

Rigid motor

$$\boxed{\mathbf{Q} = Q_{vx}\,\mathbf{e}_{415} + Q_{vy}\,\mathbf{e}_{425} + Q_{vz}\,\mathbf{e}_{435} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{235} + Q_{my}\,\mathbf{e}_{315} + Q_{mz}\,\mathbf{e}_{125} + Q_{mw}\,\mathbf{e}_5,} \tag{5.1}$$

and general rigid flector $\mathbf{F}$ has the form

Rigid flector

$$\boxed{\mathbf{F} = F_{px}\,\mathbf{e}_{15} + F_{py}\,\mathbf{e}_{25} + F_{pz}\,\mathbf{e}_{35} + F_{pw}\,\mathbf{e}_{45} + F_{gx}\,\mathbf{e}_{4235} + F_{gy}\,\mathbf{e}_{4315} + F_{gz}\,\mathbf{e}_{4125} + F_{gw}\,\mathbf{e}_{3215}.} \tag{5.2}$$

The components of $\mathbf{Q}$ and $\mathbf{F}$ have the same meanings here as they did previously in Equations (3.94) and (3.159). The sandwich antiproducts $\mathbf{Q} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{Q}}$ and $-\mathbf{F} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{F}}$ can perform Euclidean isometries on all objects in the conformal algebra, so $\mathbf{u}$ can be a round point, dipole, circle, or sphere. Flat points, lines, and planes are now special cases in which a dipole, circle, or sphere has an infinite radius.

The homogeneous magnitude $\delta\mathbf{1} + \phi\mathbb{1}$ that represents the pitch of a screw motion in the rigid algebra where $\mathbb{1} = \mathbf{e}_{1234}$ is also multiplied by $\mathbf{e}_5$ to bring it into the conformal algebra. The exponential form of a rigid motor $\mathbf{Q}$ is given by

$$\mathbf{Q} = \exp_{\vee}\left[(\delta\mathbf{e}_5 + \phi\mathbb{1}) \vee \boldsymbol{l}\right], \tag{5.3}$$

where the antiscalar unit $\mathbb{1}$ is now $\mathbf{e}_{12345}$. The quantity $s\mathbf{e}_5 + t\mathbb{1}$ behaves as a dual number under the geometric antiproduct. When we remove the factors of $\mathbf{e}_5$, the quantity $s\mathbf{1} + t\mathbf{e}_{1234}$ behaves as a dual number under the geometric product.

The amount of computation necessary to transform a flat object with a rigid motor $\mathbf{Q}$ or rigid flector $\mathbf{F}$ does not change in the conformal algebra. For example, the formulas listed in Table 3.7 for translating flat points, lines, and planes are still perfectly valid. However, it does require a greater amount of computation to transform a round point, and much more for dipoles and circles. The reason that transforming a round point is more expensive than transforming a flat point is that its position and radius are mixed together in the $\mathbf{e}_5$ component. When the point is moved rigidly, this component needs to be recomputed in such a way that the radius does not change. This extra computation is unavoidable even if the radius is zero because the $\mathbf{e}_5$ component of a round point depends on its position.

The translation operator $\mathbf{T}$ in the conformal algebra has the form

Translation operator

$$\boxed{\mathbf{T} = \tau_x \mathbf{e}_{235} + \tau_y \mathbf{e}_{315} + \tau_z \mathbf{e}_{125} + \mathbb{1},} \tag{5.4}$$

which again is just the translation operator from the rigid algebra multiplied by $\mathbf{e}_5$. When we apply this operator to a vector representing a round point $\mathbf{a}$ with the sandwich antiproduct $\mathbf{T} \vee \mathbf{a} \vee \underset{\sim}{\mathbf{T}}$, the effect is that $\mathbf{a}$ is multiplied by the 5×5 matrix

$$\begin{bmatrix} 1 & 0 & 0 & t_x & 0 \\ 0 & 1 & 0 & t_y & 0 \\ 0 & 0 & 1 & t_z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & \frac{1}{2}t^2 & 1 \end{bmatrix}, \tag{5.5}$$

where $\mathbf{t} = 2\boldsymbol{\tau}$ is the displacement vector. The upper-left 4×4 portion of this matrix is precisely the same matrix that translates four-dimensional homogeneous points. The zeros in the rightmost column of the first four rows mean that the x, y, z, and w components of the translated point do not depend on the u component. The bottom row of this matrix is responsible for calculating the new value of the u component, and it depends on all five components of the point.

The per-component formulas that arise when the translation operator $\mathbf{T}$ is applied to all seven types of geometries in three dimensions are listed in Table 5.2. The formulas for flat points, lines, and planes are identical to the versions listed in Table 3.7 for the rigid algebra. The formula for a round point implements the matrix given by Equation (5.5), and the formula for a sphere is very similar due to the fact that round points and spheres are duals. The formulas for dipoles and circles are much more involved. Notice that the carriers, whose components are always listed first, are translated with exactly the same formulas as the corresponding flat objects. That is, the carrier line of a dipole is translated just like a flat line, and the carrier plane of a circle is translated just like a flat plane. The large amount of extra work necessary to calculate the round components (those with a factor of $\mathbf{e}_5$) make the implementation of operators in the conformal algebra rather impractical. For example, it would be easier to store the seven floating-point values defining the center, radius, and normal of a circle, transform those with conventional methods as necessary, and only construct a ten-component geometric algebra representation with Equation (4.32) when it's time to do calculations in the exterior algebra using the wedge and antiwedge products. The computation necessary for something as simple as a translation operator is already excessive, and it gets much worse for more complicated conformal motions.

5.1.2 Sphere Inversion

In the rigid algebra, all Euclidean isometries are built up from one or more reflections across planes. If the planes are configured in a special way, such as two planes being parallel to each other, then the result is a specific type of rigid transformation, a translation in that case. In conformal geometric algebra, planes are generalized to spheres, and all conformal transformations are built up from one or more *inversions* through spheres. If the spheres are configured in a special way, such as two spheres as being concentric, then the result is a specific type of conformal transformation. In this case, the concentric spheres generate a dilation, which is described in detail later in Section 5.2.

A sphere inversion is not the same kind of inversion as the improper isometry discussed in Section 3.5.4 that reflects through a single point. A sphere inversion is a reciprocation of space that exchanges everything inside a sphere with everything outside. The surface of the sphere doesn't

Type	Translation Formula
Flat point $\mathbf{p}$	$\mathbf{T} \vee \mathbf{p} \vee \underset{\sim}{\mathbf{T}} = (p_x + 2\tau_x p_w)\,\mathbf{e}_{15} + (p_y + 2\tau_y p_w)\,\mathbf{e}_{25} + (p_z + 2\tau_z p_w)\,\mathbf{e}_{35} + p_w\,\mathbf{e}_{45}$
Line $\boldsymbol{l}$	$\mathbf{T} \vee \boldsymbol{l} \vee \underset{\sim}{\mathbf{T}} = l_{vx}\,\mathbf{e}_{415} + l_{vy}\,\mathbf{e}_{425} + l_{vz}\,\mathbf{e}_{435} + [l_{mx} + 2(\tau_y l_{vz} - \tau_z l_{vy})]\,\mathbf{e}_{235}$ $+ [l_{my} + 2(\tau_z l_{vx} - \tau_x l_{vz})]\,\mathbf{e}_{315} + [l_{mz} + 2(\tau_x l_{vy} - \tau_y l_{vx})]\,\mathbf{e}_{125}$
Plane $\mathbf{g}$	$\mathbf{T} \vee \mathbf{g} \vee \underset{\sim}{\mathbf{T}} = g_x\,\mathbf{e}_{4235} + g_y\,\mathbf{e}_{4315} + g_z\,\mathbf{e}_{4125} + (g_w - 2\boldsymbol{\tau} \cdot \mathbf{g}_{xyz})\,\mathbf{e}_{3215}$
Round point $\mathbf{a}$	$\mathbf{T} \vee \mathbf{a} \vee \underset{\sim}{\mathbf{T}} = (a_x + 2\tau_x a_w)\,\mathbf{e}_1 + (a_y + 2\tau_y a_w)\,\mathbf{e}_2 + (a_z + 2\tau_z a_w)\,\mathbf{e}_3 + a_w\,\mathbf{e}_4$ $+ (a_u + 2\boldsymbol{\tau} \cdot \mathbf{a}_{xyz} + 2\boldsymbol{\tau}^2 a_w)\,\mathbf{e}_5$
Dipole $\mathbf{d}$	$\mathbf{T} \vee \mathbf{d} \vee \underset{\sim}{\mathbf{T}} = d_{vx}\,\mathbf{e}_{41} + d_{vy}\,\mathbf{e}_{42} + d_{vz}\,\mathbf{e}_{43} + [d_{mx} + 2(\tau_y d_{vz} - \tau_z d_{vy})]\,\mathbf{e}_{23}$ $+ [d_{my} + 2(\tau_z d_{vx} - \tau_x d_{vz})]\,\mathbf{e}_{31} + [d_{mz} + 2(\tau_x d_{vy} - \tau_y d_{vx})]\,\mathbf{e}_{12}$ $+ [d_{px} + 2(\tau_y d_{mz} - \tau_z d_{my} + \tau_x d_{pw} + 2\tau_x \boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}} - \boldsymbol{\tau}^2 d_{vx})]\,\mathbf{e}_{15}$ $+ [d_{py} + 2(\tau_z d_{mx} - \tau_x d_{mz} + \tau_y d_{pw} + 2\tau_y \boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}} - \boldsymbol{\tau}^2 d_{vy})]\,\mathbf{e}_{25}$ $+ [d_{pz} + 2(\tau_x d_{my} - \tau_y d_{mx} + \tau_z d_{pw} + 2\tau_z \boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}} - \boldsymbol{\tau}^2 d_{vz})]\,\mathbf{e}_{35}$ $+ (d_{pw} + 2\boldsymbol{\tau} \cdot \mathbf{d}_{\mathbf{v}})\,\mathbf{e}_{45}$
Circle $\mathbf{c}$	$\mathbf{T} \vee \mathbf{c} \vee \underset{\sim}{\mathbf{T}} = c_{gx}\,\mathbf{e}_{423} + c_{gy}\,\mathbf{e}_{431} + c_{gz}\,\mathbf{e}_{412}$ $+ (c_{gw} - 2\boldsymbol{\tau} \cdot \mathbf{c}_{gxyz})\,\mathbf{e}_{321} + [c_{vx} + 2(\tau_y c_{gz} - \tau_z c_{gy})]\,\mathbf{e}_{415}$ $+ [c_{vy} + 2(\tau_z c_{gx} - \tau_x c_{gz})]\,\mathbf{e}_{425} + [c_{vz} + 2(\tau_x c_{gy} - \tau_y c_{gx})]\,\mathbf{e}_{435}$ $+ [c_{mx} + 2(\tau_y c_{vz} - \tau_z c_{vy} - \tau_x c_{gw} + 2\tau_x \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gx})]\,\mathbf{e}_{235}$ $+ [c_{my} + 2(\tau_z c_{vx} - \tau_x c_{vz} - \tau_y c_{gw} + 2\tau_y \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gy})]\,\mathbf{e}_{315}$ $+ [c_{mz} + 2(\tau_x c_{vy} - \tau_y c_{vx} - \tau_z c_{gw} + 2\tau_z \boldsymbol{\tau} \cdot \mathbf{c}_{gxyz} - \boldsymbol{\tau}^2 c_{gz})]\,\mathbf{e}_{125}$
Sphere $\mathbf{s}$	$\mathbf{T} \vee \mathbf{s} \vee \underset{\sim}{\mathbf{T}} = s_u\,\mathbf{e}_{1234} + (s_x - 2\tau_x s_u)\,\mathbf{e}_{4235} + (s_y - 2\tau_y s_u)\,\mathbf{e}_{4315} + (s_z - 2\tau_z s_u)\,\mathbf{e}_{4125}$ $+ (s_w - 2\boldsymbol{\tau} \cdot \mathbf{s}_{xyz} + 2\boldsymbol{\tau}^2 s_u)\,\mathbf{e}_{3215}$

Table 5.2. The multivector $\mathbf{T} = \tau_x\mathbf{e}_{235} + \tau_y\mathbf{e}_{315} + \tau_z\mathbf{e}_{125} + \mathbb{1}$ acts as a translation operator for flat and round objects under the geometric antiproduct in three dimensions. These formulas translate by the displacement vector $2\boldsymbol{\tau}$. The operator $\mathbf{T}$ is always unitized, and the geometries being translated can have any weight.

move just like points on a plane don't move when space is reflected across it. An inversion can be performed with a sphere having any center and radius, even a radius that is zero or imaginary. As long as the radius isn't zero, any inversion is an involution that returns everything to its starting location when performed twice. Because the entire infinite expanse of space outside the sphere has to end up squished inside the finite volume of the sphere after an inversion, distances are clearly not preserved.

A sphere $\mathbf{s}$ is a flector in the conformal algebra, and an object $\mathbf{u}$ is inverted across a sphere by calculating $-\mathbf{s} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{s}}$. (The antireverse operation has no effect on antivectors, so $\mathbf{s} = \underset{\sim}{\mathbf{s}}$, and we write the antireverse here just to be consistent with other operators.) To get an idea of what an inversion looks like, suppose that a round point $\mathbf{a}$ lies at a distance x from the center of a sphere of radius r, as shown in Figure 5.1(a). When the point is inverted across the sphere, it is moved along the line connecting it to the center of the sphere to a distance r^2/x from the center. The product of the old distance from center and new distance from center is equal to the squared radius of the sphere. In this way, inversion in a sphere performs a sort of geometric reciprocal. When the sphere is real, the new position of the point is on the same side of the sphere's center as the old position. When the sphere is imaginary, the new position is on the opposite side at a *negative* distance so the product with the original positive distance is the negative value given by r^2.

Several examples of sphere inversion are shown in Figure 5.1. In each part, the two objects are images of each other under an inversion across the blue sphere. As illustrated by the circles in part (b), the surfaces of round objects are inverted pointwise, and this has the effect of changing the radius. Part (c) demonstrates that flat objects do not remain flat unless they contain the center of the sphere. All flat objects contain the point at infinity $\mathbf{e}_5$, and that point must end up at a distance of zero from the center of the sphere after the inversion. As a result, the inversion of any flat object is an object that contains the center of the sphere. In the opposite sense, the inversion of any object that contains the center of the sphere must contain the point at infinity and must therefore be a flat object. The circle and line shown in part (c) are sphere inversions of each other. Another example of this relationship is shown in part (d), and this time, the line intersects the sphere. Points on the surface of the sphere do not move under an inversion, so the circle corresponding to the line's inversion must contain the center of the sphere and the two points where the line pierces its surface. The final example in part (e) demonstrates a similar but less intuitive relationship between flat points and dipoles. A flat point is really a dipole with one end at some position $\mathbf{p}$ and the other end at infinity. When a flat point is inverted across the sphere, the endpoint at infinity must be moved to the center of the sphere, and the endpoint at $\mathbf{p}$ is inverted just like any other point. The resulting dipole is the lower-dimensional analog of the circle shown in part (c).

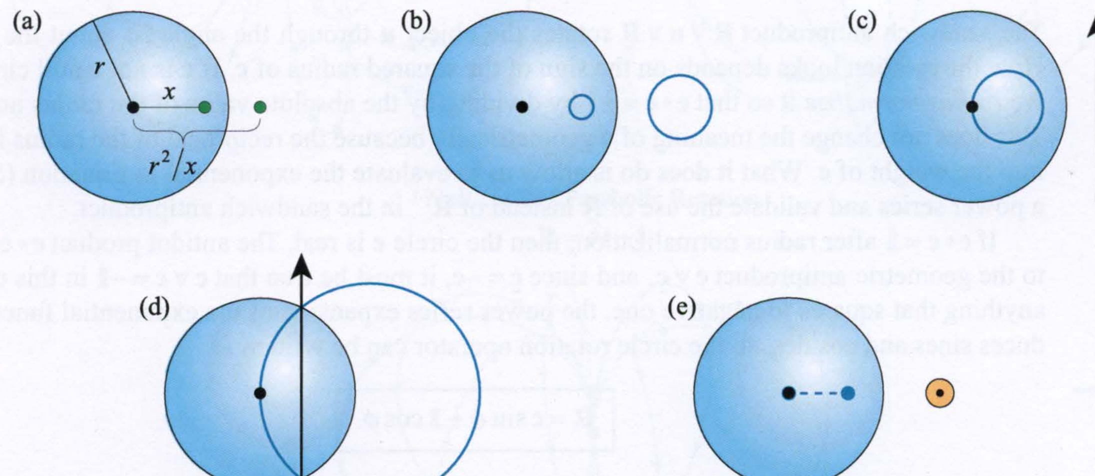


Figure 5.1. These are several examples of sphere inversion. The two objects shown in each case are inversions of each other across the blue sphere. (a) A point at a distance x from the center of a sphere of radius r is moved to the distance r^2/x from the center after inversion. (b) Surfaces of round objects are inverted pointwise. (c) Since flat objects like the line shown contain the point at infinity $\mathbf{e}_5$, their inversions must contain the center of the sphere. (d) An object intersects the sphere at the same points as its inversion. (e) A flat point is a dipole with one end at infinity, so its inversion must be a dipole with one end at the center of the sphere.

Sphere inversion is a linear transformation in the 5D projective space, which means it can be implemented by a 5×5 matrix. The matrix corresponding to the inversion across a sphere of radius r centered at the point (m_x, m_y, m_z) is

$$\begin{bmatrix} r^2 - 2m_x^2 & -2m_x m_y & -2m_x m_z & (\mathbf{m}^2 - r^2)\, m_x & 2m_x \\ -2m_x m_y & r^2 - 2m_y^2 & -2m_y m_z & (\mathbf{m}^2 - r^2)\, m_y & 2m_y \\ -2m_x m_z & -2m_y m_z & r^2 - 2m_z^2 & (\mathbf{m}^2 - r^2)\, m_z & 2m_z \\ -2m_x & -2m_y & -2m_z & \mathbf{m}^2 & 2 \\ -(\mathbf{m}^2 - r^2)\, m_x & -(\mathbf{m}^2 - r^2)\, m_y & -(\mathbf{m}^2 - r^2)\, m_z & \frac{1}{2}(\mathbf{m}^2 - r^2)^2 & \mathbf{m}^2 \end{bmatrix}. \tag{5.6}$$

This matrix transforms a 5D vector representing a round point. Though not obvious by looking at the entries of this matrix, inversion across a sphere of radius zero, a null sphere, moves all points to the center of the sphere but with some weighting. The matrix simplifies considerably when the sphere is centered at the origin, and it becomes clear that the origin $\mathbf{e}_4$ and the point at infinity $\mathbf{e}_5$ are exchanged in that case.

5.1.3 Circle Rotation

When two plane reflections are composed in the rigid algebra, the result is a rotation about the line where the two planes intersect. This is generalized in the conformal algebra by composing two sphere inversions to produce a rotation about the circle where the two spheres intersect. The axis of rotation is now generalized to a circle, and depending on the spatial relationship of the two spheres multiplied together, that circle can be real, imaginary, or null. The conformal motions associated with these three cases are shown in Figure 5.2.

A circle rotation operator $\mathbf{R}$ can be expressed as the exponential

$$\mathbf{R} = \exp_{\vee}(\phi \mathbf{c}). \tag{5.7}$$

The sandwich antiproduct $\mathbf{R} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{R}}$ rotates the object $\mathbf{u}$ through the angle 2ϕ about the circle $\mathbf{c}$. How the rotation looks depends on the sign of the squared radius of $\mathbf{c}$. If $\mathbf{c}$ is not a null circle, then we *radius normalize* it so that $\mathbf{c} \circ \mathbf{c} = \pm\mathbb{1}$ by dividing by the absolute value of the radius norm $\|\mathbf{c}\|_{\circ}$. This does not change the meaning of $\mathbf{c}$ geometrically because the reciprocal of the radius is moved into the weight of $\mathbf{c}$. What it does do is allow us to evaluate the exponential in Equation (5.7) with a power series and validate the use of $\underset{\sim}{\mathbf{R}}$ instead of $\mathbf{R}^{-1}$ in the sandwich antiproduct.

If $\mathbf{c} \circ \mathbf{c} = \mathbb{1}$ after radius normalization, then the circle $\mathbf{c}$ is real. The antidot product $\mathbf{c} \circ \mathbf{c}$ is equal to the geometric antiproduct $\mathbf{c} \vee \underset{\sim}{\mathbf{c}}$, and since $\underset{\sim}{\mathbf{c}} = -\mathbf{c}$, it must be true that $\mathbf{c} \vee \mathbf{c} = -\mathbb{1}$ in this case. For anything that squares to negative one, the power series expansion of the exponential function produces sines and cosines, so the circle rotation operator can be written as

Elliptic rotation operator

$$\boxed{\mathbf{R} = \mathbf{c}\sin\phi + \mathbb{1}\cos\phi.} \tag{5.8}$$

Compared to the rotation operator shown in Equation (3.86), the only difference is that the line $\boldsymbol{l}$ has been generalized to a circle $\mathbf{c}$. This operation is called an *elliptic rotation*, and it is shown on the left in Figure 5.2. An elliptic rotation is the motion that we would intuitively expect if we were asked to generalize a rotation about a line to a rotation about a circle. Throughout a full revolution as ϕ ranges from zero to π radians, all of the space outside the circle flows through the interior of the circle at some point in time. In particular, the point at infinity arrives at the center of the circle halfway through a revolution when $\phi = \pi/2$. The point beginning at the center reaches infinity at the same time, so they exchange places.

Now suppose that $\mathbf{c} \circ \mathbf{c} = -\mathbb{1}$ because the circle $\mathbf{c}$ is imaginary. In this case, $\mathbf{c} \vee \mathbf{c} = \mathbb{1}$, and the power series expansion of the exponential function produces hyperbolic sines and cosines. This time, the circle rotation operator can be written as

Hyperbolic rotation operator

$$\boxed{\mathbf{R} = \mathbf{c}\sinh\phi + \mathbb{1}\cosh\phi.} \tag{5.9}$$

This operation is called a *hyperbolic rotation*, and it is shown on the right in Figure 5.2. Whereas an elliptic rotation fixes the real circle $\mathbf{c}$, a hyperbolic rotation fixes the real dipole $\mathbf{c}^\star$, the dual of the imaginary circle $\mathbf{c}$. As shown in the figure, the ends of the dipole serve as the source and sink for the motion that a hyperbolic rotation generates such that points in space move away from the source and toward the sink. This motion is not cyclic as in the elliptic rotation, and points just move closer to the sink at a slowing rate as the angle ϕ increases indefinitely.

The final case is that $\mathbf{c} \circ \mathbf{c} = 0$, which corresponds to a null circle. Since $\mathbf{c} \vee \mathbf{c} = 0$ as well, the power series expansion of the exponential has only two terms, and the circle rotation operator now becomes

Parabolic rotation operator

$$\boxed{\mathbf{R} = \phi\mathbf{c} + \mathbb{1}.} \tag{5.10}$$

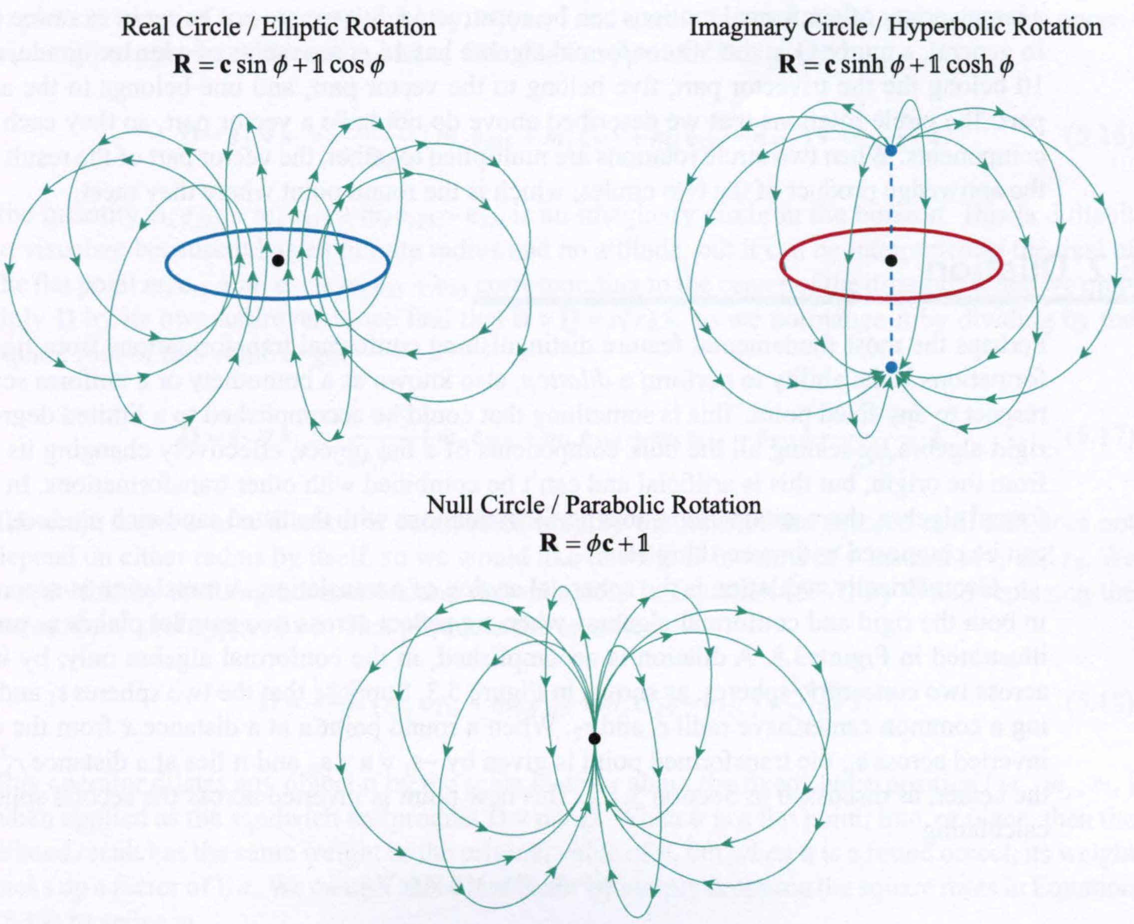


Figure 5.2. (Top left) The motion generated by a real circle $\mathbf{c}$ is an elliptic rotation in which points revolve about a fixed circular axis. (Top right) The motion generated by an imaginary circle $\mathbf{c}$ is a hyperbolic rotation in which points flow between the fixed ends of the dipole $\mathbf{c}^\star$. (Bottom) The motion generated by a null circle $\mathbf{c}$ is a parabolic rotation in which points follow noncyclic loops from one side of the fixed center to the other.

This operation is called a *parabolic rotation*, and it is shown at the bottom of Figure 5.2. The only fixed point is the center of the circle, and other points in space flow along loops from one side to the other. As in hyperbolic rotations, this motion is not cyclic, and points move toward the center at a slowing rate as the angle ϕ increases indefinitely. A parabolic rotation represents the boundary between elliptic rotations and hyperbolic rotations. As the absolute size of the circle shrinks in the real and imaginary cases because the squared radius approaches zero, the shapes of the elliptic and hyperbolic rotations both approach the shape of the parabolic rotation.

For all three circle rotations described above, the inverse of the operator $\mathbf{R}$ is simply $\underset{\sim}{\mathbf{R}}$. In the elliptic case, $\mathbf{c} \vee \underset{\sim}{\mathbf{c}} = \mathbb{1}$, and we have

$$\mathbf{R} \vee \underset{\sim}{\mathbf{R}} = (\mathbf{c} \vee \underset{\sim}{\mathbf{c}}) \sin^2 \phi + \mathbf{c} \sin \phi \cos \phi + \underset{\sim}{\mathbf{c}} \sin \phi \cos \phi + \mathbb{1} \cos^2 \phi = \mathbb{1}, \tag{5.11}$$

where the fact that $\mathbf{c} = -\underset{\sim}{\mathbf{c}}$ causes the terms containing $\sin \phi \cos \phi$ to cancel out. In the hyperbolic case, $\mathbf{c} \vee \underset{\sim}{\mathbf{c}} = -\mathbb{1}$, and we have

$$\mathbf{R} \vee \underset{\sim}{\mathbf{R}} = (\mathbf{c} \vee \underset{\sim}{\mathbf{c}}) \sinh^2 \phi + \mathbf{c} \sinh \phi \cosh \phi + \underset{\sim}{\mathbf{c}} \sinh \phi \cosh \phi + \mathbb{1} \cosh^2 \phi = \mathbb{1}. \tag{5.12}$$

And in the parabolic case, $\mathbf{c} \vee \underset{\sim}{\mathbf{c}} = 0$, and we have

$$\mathbf{R} \vee \underset{\sim}{\mathbf{R}} = \mathbf{c} \vee \underset{\sim}{\mathbf{c}} + \phi \mathbf{c} + \phi \underset{\sim}{\mathbf{c}} + \mathbb{1} = \mathbb{1}. \tag{5.13}$$

By composing arbitrary numbers of circle rotations of various kinds and rigid screw motions, a large variety of conformal motions can be constructed, but we are not going to examine them all. In general, a motor $\mathbf{Q}$ in the 5D conformal algebra has 16 components of even antigrade, of which 10 belong the the trivector part, five belong to the vector part, and one belongs to the antiscalar part. The circle rotations that we described above do not have a vector part, so they each have 11 components. When two circle rotations are multiplied together, the vector part of the result contains the antiwedge product of the two circles, which is the round point where they meet.

5.2 Dilation

Perhaps the most fundamental feature distinguishing conformal transformations from rigid transformations is the ability to perform a *dilation*, also known as a homothety or a uniform scale, with respect to any fixed point. This is something that could be accomplished to a limited degree in the rigid algebra by scaling all the bulk components of a flat object, effectively changing its distance from the origin, but this is artificial and can't be combined with other transformations. In the conformal algebra, there are operators that perform dilations with the usual sandwich product, and they can be composed with everything else.

Geometrically, a dilation is the spherical analog of a translation. A translation is accomplished in both the rigid and conformal algebras when we reflect across two parallel planes as previously illustrated in Figure 3.8. A dilation is accomplished, in the conformal algebra only, by inverting across two concentric spheres, as shown in Figure 5.3. Suppose that the two spheres $\mathbf{s}_1$ and $\mathbf{s}_2$ sharing a common center have radii r_1 and r_2. When a round point $\mathbf{a}$ at a distance x from the center is inverted across $\mathbf{s}_1$, the transformed point is given by $-\mathbf{s}_1 \vee \mathbf{a} \vee \underset{\sim}{\mathbf{s}}_1$, and it lies at a distance r_1^2 / x from the center, as discussed in Section 5.1.2. This new point is inverted across the second sphere $\mathbf{s}_2$ by calculating

$$\mathbf{a}' = \mathbf{s}_2 \vee \mathbf{s}_1 \vee \mathbf{a} \vee \underset{\sim}{\mathbf{s}}_1 \vee \underset{\sim}{\mathbf{s}}_2, \tag{5.14}$$

and the final result $\mathbf{a}'$ must lie at a distance from the center given by

$$\frac{r_2^2}{r_1^2 / x} = \frac{r_2^2}{r_1^2} x. \tag{5.15}$$

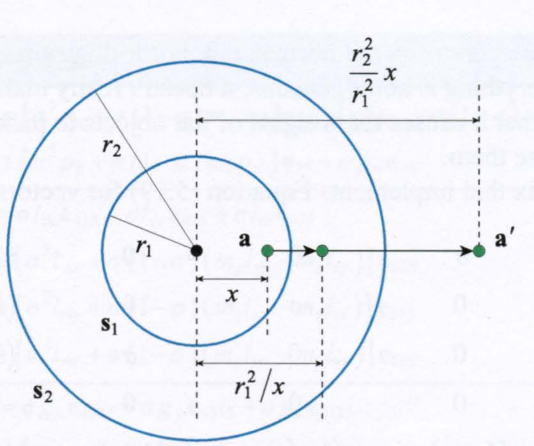


Figure 5.3. When a round point $\mathbf{a}$ is inverted across two concentric spheres $\mathbf{s}_1$ and $\mathbf{s}_2$ having radii r_1 and r_2, it is dilated by a factor of r_2^2/r_1^2 with respect to the center of the spheres to become the point $\mathbf{a}'$.

The total effect is that the distance between the original point $\mathbf{a}$ and the center of the spheres has been scaled by the ratio r_2^2/r_1^2 of the spheres' squared radii.

The geometric antiproduct of two concentric spheres yields a dilation operator $\mathbf{D}$ that we can express in terms of a center position (m_x, m_y, m_z) and a scale factor σ. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be two concentric unitized spheres with radii r_1 and r_2. The dilation operator $\mathbf{D}$ is given by

$$\mathbf{D} = \mathbf{s}_2 \mathbin{\vee\!\!\!\!\!\;-} \mathbf{s}_1 = \frac{r_1^2 - r_2^2}{2}(m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}) + \frac{r_1^2 + r_2^2}{2}\mathbb{1}. \tag{5.16}$$

The quantity $m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}$ is an imaginary circle in the horizon. This is difficult to visualize because it has an infinite radius and no attitude, but it can be interpreted as the dual of the flat point $m_x \mathbf{e}_{15} + m_y \mathbf{e}_{25} + m_z \mathbf{e}_{35} + \mathbf{e}_{45}$ corresponding to the center of the dilation. When we multiply $\mathbf{D}$ by its own antireverse, we find that $\mathbf{D} \mathbin{\vee\!\!\!\!\!\;-} \underset{\sim}{\mathbf{D}} = r_1^2 r_2^2 \mathbb{1}$, so we normalize it by dividing by the square root of this value to get

$$\mathbf{D} = \mathbf{s}_2 \mathbin{\vee\!\!\!\!\!\;-} \mathbf{s}_1 = \frac{r_1^2 - r_2^2}{2r_1 r_2}(m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}) + \frac{r_1^2 + r_2^2}{2r_1 r_2}\mathbb{1}. \tag{5.17}$$

The scale factor σ of a dilation is determined only by the ratio of the squared radii and does not depend on either radius by itself, so we would like to write $\mathbf{D}$ in terms of σ instead of r_1 and r_2. We can do this by dividing numerators and denominators in Equation (5.17) by r_1^2 and replacing the ratios r_2^2/r_1^2 that appear by σ. That produces the operator

$$\mathbf{D} = \frac{1-\sigma}{2\sqrt{\sigma}}(m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}) + \frac{1+\sigma}{2\sqrt{\sigma}}\mathbb{1}. \tag{5.18}$$

This operator dilates any object $\mathbf{u}$ by the scale factor σ about the fixed center position (m_x, m_y, m_z) when applied as the sandwich antiproduct $\mathbf{D} \mathbin{\vee\!\!\!\!\!\;-} \mathbf{u} \mathbin{\vee\!\!\!\!\!\;-} \underset{\sim}{\mathbf{D}}$. When $\mathbf{u}$ is a flat point, line, or plane, then the dilated result has the same weight as the original value of $\mathbf{u}$, but when $\mathbf{u}$ is a round object, its weight picks up a factor of $1/\sigma$. We can get rid of this factor by simply dropping the square roots in Equation (5.18) to arrive at

Dilation operator

$$\boxed{\mathbf{D} = \frac{1-\sigma}{2}(m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}) + \frac{1+\sigma}{2}\mathbb{1}.} \tag{5.19}$$

This version of the operator is not normalized, but it dilates round objects without changing their weights. Since everything is homogeneous, it doesn't really matter that $\mathbf{D} \vee \underset{\sim}{\mathbf{D}} = \sigma \mathbb{1}$ in this case. The only downside is that it causes the weights of flat objects to pick up a factor of σ now, so it may be necessary to unitize them.

The 5×5 matrix that implements Equation (5.19) for vectors representing round points is

$$\begin{bmatrix} \sigma & 0 & 0 & (1-\sigma)\,m_x & 0 \\ 0 & \sigma & 0 & (1-\sigma)\,m_y & 0 \\ 0 & 0 & \sigma & (1-\sigma)\,m_z & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sigma(1-\sigma)\,m_x & \sigma(1-\sigma)\,m_y & \sigma(1-\sigma)\,m_z & \tfrac{1}{2}(1-\sigma)^2\mathbf{m}^2 & \sigma^2 \end{bmatrix}. \tag{5.20}$$

This matrix has a form similar to the translation matrix given by Equation (5.5) in that the fourth row is a constant $[0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]$ and the rightmost column has a nonzero entry only in the last row. The upper-left 4×4 portion correctly dilates a 4D homogeneous point by a scale factor σ about the center $\mathbf{m} = (m_x, m_y, m_z)$. The fifth row recalculates the radius of a round point, and it leaves a radius of zero unchanged.

The dilation operator is a motor in the conformal algebra, and it can be expressed as an exponential of the circle $\mathbf{c} = m_x \mathbf{e}_{235} + m_y \mathbf{e}_{315} + m_z \mathbf{e}_{125} - \mathbf{e}_{321}$ that defines its center. As mentioned earlier, this circle is imaginary, and it always squares as $\mathbf{c} \vee \mathbf{c} = \mathbb{1}$. That being the case, the hyperbolic rotation formula in Equation (5.9) applies, and we can write

$$\mathbf{D} = \exp_{\vee}(\delta \mathbf{c}) = \mathbf{c} \sinh \delta + \mathbb{1} \cosh \delta. \tag{5.21}$$

We just need to figure out how the angle δ relates to the scale factor σ. When we expand the hyperbolic sine and cosine in terms of exponentials, we can express $\mathbf{D}$ as

$$\mathbf{D} = \frac{e^{\delta} - e^{-\delta}}{2}\mathbf{c} + \frac{e^{\delta} + e^{-\delta}}{2}\mathbb{1}. \tag{5.22}$$

By multiplying numerators and denominators by $e^{-\delta}$, this can be rewritten as

$$\mathbf{D} = \frac{1 - e^{-2\delta}}{2e^{-\delta}}\mathbf{c} + \frac{1 + e^{-2\delta}}{2e^{-\delta}}\mathbb{1}. \tag{5.23}$$

We have a match with Equation (5.18) when we set $e^{-2\delta} = \sigma$, and that means

$$\delta = -\frac{1}{2}\ln \sigma. \tag{5.24}$$

Table 5.3 lists the per-component formulas that arise when the version of the dilation operator $\mathbf{D}$ given by Equation (5.19) is applied to all seven types of geometries in three dimensions. The formulas for flat points, lines, and planes each contain a superfluous homogeneous factor of σ that would typically be dropped in a direct implementation. The formula for a round point is equivalent to multiplication by the 5×5 matrix in Equation (5.20). As with the translation operator shown earlier in Table 5.2, the formulas for dilating dipoles and circles are rather ridiculous considering the simple effect they have. This makes dilation with a sandwich product impractical by itself since it would be much easier to just scale an object's center and radius. Geometric algebra representations could then be rebuilt from definitions whenever they are going to be used with operations of the exterior algebra.

Type	Dilation Formula
Flat point **p**	$\begin{aligned}\mathbf{D} ⟇ \mathbf{p} ⟇ \underset{\sim}{\mathbf{D}} &= [\sigma^2 p_x + \sigma(1-\sigma) m_x p_w]\,\mathbf{e}_{15} + [\sigma^2 p_y + \sigma(1-\sigma) m_y p_w]\,\mathbf{e}_{25} \\ &+ [\sigma^2 p_z + \sigma(1-\sigma) m_z p_w]\,\mathbf{e}_{35} + \sigma p_w \mathbf{e}_{45}\end{aligned}$
Line $\boldsymbol{l}$	$\begin{aligned}\mathbf{D} ⟇ \boldsymbol{l} ⟇ \underset{\sim}{\mathbf{D}} &= \sigma l_{vx}\mathbf{e}_{415} + \sigma l_{vy}\mathbf{e}_{425} + \sigma l_{vz}\mathbf{e}_{435} \\ &+ [\sigma^2 l_{mx} + \sigma(1-\sigma)(m_y l_{vz} - m_z l_{vy})]\,\mathbf{e}_{235} \\ &+ [\sigma^2 l_{my} + \sigma(1-\sigma)(m_z l_{vx} - m_x l_{vz})]\,\mathbf{e}_{315} \\ &+ [\sigma^2 l_{mz} + \sigma(1-\sigma)(m_x l_{vy} - m_y l_{vx})]\,\mathbf{e}_{125}\end{aligned}$
Plane **g**	$\begin{aligned}\mathbf{D} ⟇ \mathbf{g} ⟇ \underset{\sim}{\mathbf{D}} &= \sigma g_x \mathbf{e}_{4235} + \sigma g_y \mathbf{e}_{4315} + \sigma g_z \mathbf{e}_{4125} \\ &+ [\sigma^2 g_w - \sigma(1-\sigma)\,\mathbf{m}\cdot\mathbf{g}_{xyz}]\,\mathbf{e}_{3215}\end{aligned}$
Round point **a**	$\begin{aligned}\mathbf{D} ⟇ \mathbf{a} ⟇ \underset{\sim}{\mathbf{D}} &= (\sigma a_x + (1-\sigma) m_x a_w)\,\mathbf{e}_1 + (\sigma a_y + (1-\sigma) m_y a_w)\,\mathbf{e}_2 \\ &+ (\sigma a_z + (1-\sigma) m_z a_w)\,\mathbf{e}_3 + a_w \mathbf{e}_4 \\ &+ [\sigma^2 a_u + \sigma(1-\sigma)\,\mathbf{m}\cdot\mathbf{a}_{xyz} + \tfrac{1}{2}(1-\sigma)^2\,\mathbf{m}^2 a_w]\,\mathbf{e}_5\end{aligned}$
Dipole **d**	$\begin{aligned}\mathbf{D} ⟇ \mathbf{d} ⟇ \underset{\sim}{\mathbf{D}} &= d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + [\sigma d_{mx} + (1-\sigma)(m_y d_{vz} - m_z d_{vy})]\,\mathbf{e}_{23} \\ &+ [\sigma d_{my} + (1-\sigma)(m_z d_{vx} - m_x d_{vz})]\,\mathbf{e}_{31} + [\sigma d_{mz} + (1-\sigma)(m_x d_{vy} - m_y d_{vx})]\,\mathbf{e}_{12} \\ &+ [\sigma^2 d_{px} + \sigma(1-\sigma)(m_y d_{mz} - m_z d_{my} + m_x d_{pw}) + \tfrac{1}{2}(1-\sigma)^2(2m_x\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}} - \mathbf{m}^2 d_{vx})]\,\mathbf{e}_{15} \\ &+ [\sigma^2 d_{py} + \sigma(1-\sigma)(m_z d_{mx} - m_x d_{mz} + m_y d_{pw}) + \tfrac{1}{2}(1-\sigma)^2(2m_y\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}} - \mathbf{m}^2 d_{vy})]\,\mathbf{e}_{25} \\ &+ [\sigma^2 d_{py} + \sigma(1-\sigma)(m_x d_{my} - m_y d_{mx} + m_z d_{pw}) + \tfrac{1}{2}(1-\sigma)^2(2m_z\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}} - \mathbf{m}^2 d_{vz})]\,\mathbf{e}_{35} \\ &+ [\sigma d_{pw} + (1-\sigma)\,\mathbf{m}\cdot\mathbf{d}_{\mathbf{v}}]\,\mathbf{e}_{45}\end{aligned}$
Circle **c**	$\begin{aligned}\mathbf{D} ⟇ \mathbf{c} ⟇ \underset{\sim}{\mathbf{D}} &= c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412} \\ &+ [\sigma c_{gw} - (1-\sigma)\,\mathbf{m}\cdot\mathbf{c}_{gxyz}]\,\mathbf{e}_{321} + [\sigma c_{vx} + (1-\sigma)(m_y c_{gz} - m_z c_{gy})]\,\mathbf{e}_{415} \\ &+ [\sigma c_{vy} + (1-\sigma)(m_z c_{gx} - m_x c_{gz})]\,\mathbf{e}_{425} + [\sigma c_{vz} + (1-\sigma)(m_x c_{gy} - m_y c_{gx})]\,\mathbf{e}_{435} \\ &+ [\sigma^2 c_{mx} + \sigma(1-\sigma)(m_y c_{vz} - m_z c_{vy} - m_x c_{gw}) + \tfrac{1}{2}(1-\sigma)^2(2m_x\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gx})]\,\mathbf{e}_{235} \\ &+ [\sigma^2 c_{my} + \sigma(1-\sigma)(m_z c_{vx} - m_x c_{vz} - m_y c_{gw}) + \tfrac{1}{2}(1-\sigma)^2(2m_y\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gy})]\,\mathbf{e}_{315} \\ &+ [\sigma^2 c_{mz} + \sigma(1-\sigma)(m_x c_{vy} - m_y c_{vx} - m_z c_{gw}) + \tfrac{1}{2}(1-\sigma)^2(2m_z\mathbf{m}\cdot\mathbf{c}_{gxyz} - \mathbf{m}^2 c_{gz})]\,\mathbf{e}_{125}\end{aligned}$
Sphere **s**	$\begin{aligned}\mathbf{D} ⟇ \mathbf{s} ⟇ \underset{\sim}{\mathbf{D}} &= s_u \mathbf{e}_{1234} + (\sigma s_x - (1-\sigma) m_x s_u)\,\mathbf{e}_{4235} \\ &+ (\sigma s_y - (1-\sigma) m_y s_u)\,\mathbf{e}_{4315} + (\sigma s_z - (1-\sigma) m_z s_u)\,\mathbf{e}_{4125} \\ &+ [\sigma^2 s_w - \sigma(1-\sigma)\,\mathbf{m}\cdot\mathbf{s}_{xyz} + \tfrac{1}{2}(1-\sigma)^2\,\mathbf{m}^2 s_u]\,\mathbf{e}_{3215}\end{aligned}$

Table 5.3. The multivector $\mathbf{D} = \frac{1-\sigma}{2}(m_x\mathbf{e}_{235} + m_y\mathbf{e}_{315} + m_z\mathbf{e}_{125} - \mathbf{e}_{321}) + \frac{1+\sigma}{2}\mathbb{1}$ acts as a dilation operator for flat and round objects under the geometric antiproduct in three dimensions. These formulas dilate by the scale factor σ about the fixed center $\mathbf{m} = (m_x, m_y, m_z)$. The operator $\mathbf{D}$ is weighted so that the weights of round objects are preserved.

5.3 Duals and Complements

Because the metric is not degenerate in the conformal algebra, transformations that are performed with the geometric antiproduct can also be performed with the geometric product, but with a dual operator. Suppose that an operator $\mathbf{X}$ performs a specific transformation with a sandwich of geometric antiproducts such that $\mathbf{u}' = \mathbf{X} \vee \mathbf{u} \vee \underset{\sim}{\mathbf{X}}$. Then the dual of $\mathbf{X}$ performs the exact same transformation with the sandwich of geometric products

$$\mathbf{u}' = \mathbf{X}^\star \wedge \mathbf{u} \wedge \widetilde{\mathbf{X}^\star}. \tag{5.25}$$

This means that every operator essentially has two different representations, $\mathbf{X}$ and $\mathbf{X}^\star$. For example, the translation operator $\mathbf{T}$ given by Equation (5.4) has the dual form

$$\mathbf{T}^\star = \tau_x \mathbf{e}_{15} + \tau_y \mathbf{e}_{25} + \tau_z \mathbf{e}_{35} - \mathbf{1}. \tag{5.26}$$

The operator $\mathbf{T}^\star$ works with the geometric product such that the sandwich $\mathbf{T}^\star \wedge \mathbf{u} \wedge \widetilde{\mathbf{T}^\star}$ translates the object $\mathbf{u}$ by the displacement vector $2\boldsymbol{\tau}$. (Notice that $\mathbf{T}^\star$ contains a subtraction whereas $\mathbf{T}$ does not). Even though this is a rigid transformation, there is no corresponding dual operator in the rigid algebra that also performs a translation. As established in Chapter 3, operations that perform Euclidean isometries must use the geometric antiproduct. For the sake of consistency, we prefer to use the antiproduct in the conformal algebra as well, but it is not a requirement.

As another example, the dilation operator $\mathbf{D}$ given by Equation (5.19) has the dual form

$$\mathbf{D} = \frac{1-\sigma}{2}\left(m_x \mathbf{e}_{15} + m_y \mathbf{e}_{25} + m_z \mathbf{e}_{35} + \mathbf{e}_{45}\right) - \frac{1+\sigma}{2}\mathbf{1}. \tag{5.27}$$

The sandwich product $\mathbf{D} \wedge \mathbf{u} \wedge \widetilde{\mathbf{D}^\star}$ dilates the object $\mathbf{u}$ by the scale factor σ about the fixed center $\mathbf{m} = (m_x, m_y, m_z)$. The interpretation of the fixed geometry in this form of the operator is much more intuitive because it can easily be read off as the flat point $m_x \mathbf{e}_{15} + m_y \mathbf{e}_{25} + m_z \mathbf{e}_{35} + \mathbf{e}_{45}$. This highlights a special feature of the conformal algebra that makes it possible to always construct an operator that contains the invariant of a transformation as a real geometry or null geometry. We can do that because the dual of any imaginary object is real, so any operator based on an imaginary object $\mathbf{x}$ and applied with one product can be dualized into an equivalent operator based on the dual real object $\mathbf{x}^\star$ and applied with the complementary product.

It's always the real geometry, either $\mathbf{x}$ or $\mathbf{x}^\star$, that is invariant regardless of which product is used to perform an operation. This is exemplified by the hyperbolic rotation shown in Figure 5.2. Because the circle $\mathbf{c}$ is imaginary in that case, it is not the invariant geometry. Instead, its dual $\mathbf{c}^\star$ is the invariant geometry because it is the real dipole dual to the imaginary circle. We can write the operator $\mathbf{R}$ in terms of the real invariant as

$$\mathbf{R}^\star = \mathbf{c}^\star \sinh\phi - \mathbf{1}\cosh\phi, \tag{5.28}$$

which is the dual of Equation (5.9), and then transform objects with the sandwich $\mathbf{R}^\star \wedge \mathbf{u} \wedge \widetilde{\mathbf{R}^\star}$.

In Section 3.9.2, it was demonstrated that for any operator $\mathbf{X}$ that performs a Euclidean isometry with the geometric antiproduct, the complement of $\mathbf{X}$ performs a different kind of transformation with the geometric product, a complement isometry. This led to the two complementary sets of transformation groups shown in Figure 3.16. The geometric product could only perform transformations that fixed the origin, and the geometric antiproduct could only perform transformations that fixed the horizon. Any transformation that fixed both the origin and the horizon could be performed by both products. These transformation groups also exist in the conformal algebra, and they are shown in the purple boxes in Figure 5.4.

As in the rigid algebra, taking the complement of an operator in the conformal algebra has the same effect as taking the inverse transpose of the equivalent matrix formulation, but this is now extended to the larger 5×5 matrices such as those shown in Equations (5.5) and (5.20). This has the effect of turning an operator that fixes the origin into one that fixes the horizon, and vice versa. The dilation operator $\mathbf{D}$ in the conformal algebra fixes the horizon, and its complement operator $\overline{\mathbf{D}}$ fixes the origin. By adjoining the group $\mathrm{D}(n)$ of dilations about the origin to the groups from the rigid algebra, we double the number of closed subgroups in the conformal algebra that fix either the origin or horizon. The new subgroups that combine a dilation with a Euclidean isometry or complement isometry are shown in the green boxes in Figure 5.4.

Since every operator in the conformal algebra has two forms, complement operators can also be performed using either the geometric product or antiproduct. In the rigid algebra, a complement operator $\overline{\mathbf{X}}$ was necessarily performed with the geometric product, but in the conformal algebra, we can take its dual and perform the same operation with the geometric antiproduct. The dual of the complement is given by

$$\overline{\mathbf{X}}^{\star} = \overline{\mathbf{G}\overline{\mathbf{X}}} = \overline{\overline{\mathbb{G}\mathbf{X}}} = -\mathbf{G}\mathbf{X}, \tag{5.29}$$

where $\mathbf{G}$ is the metric exomorphism, and we have used the facts that $\mathbb{G} = -\mathbf{G}$ and the double complement is the identity in five dimensions. Sign changes don't matter for operators, so we would just drop any negation that arises, including the one in Equation (5.29). What this means is that for an operator $\mathbf{X}$ performed with one of the geometric products, the operator $\mathbf{G}\mathbf{X}$ performs the complement operation with the same product.

The form of the 5×5 matrix corresponding to the type of transformation performed by the members of each subgroup are also shown in Figure 5.4. Every matrix on the left side of the figure has zeros in its fourth column everywhere except the fourth row, and this means that the origin $\mathbf{e}_4$ is mapped to itself. Every matrix on the right side of the figure has zeros in its fifth column everywhere except the fifth row, and this means that the point at infinity $\mathbf{e}_5$ is mapped to itself. In the center of the figure, both $\mathbf{e}_4$ and $\mathbf{e}_5$ are fixed.

In each of the complement groups shown on the left side of Figure 5.4, all entries in the rightmost column of the matrix representations are potentially nonzero, and this means that the position of a round point after the transformation is applied depends on the point's radius. Null points having radius zero do not follow the same paths shown for flat points undergoing complement isometries in Figures 3.14 and 3.15. To recover the same complement isometries in the conformal algebra, a round point must have a zero u component so that the upper-left 4×4 portion of the transformation matrix is effectively applied to it. Such a point is an imaginary round point with a squared radius equal to its negative squared distance from the origin.

In general, operations in the conformal algebra fix neither the origin nor the horizon. Both are moved by a sphere inversion unless the surface of the sphere happens to contain the origin. All of the circle rotations also move both the origin and horizon except in special cases when the origin is part of the invariant geometry. Arbitrary combinations of these operations generate the much larger conformal group represented by the blue box at the top of Figure 5.4.

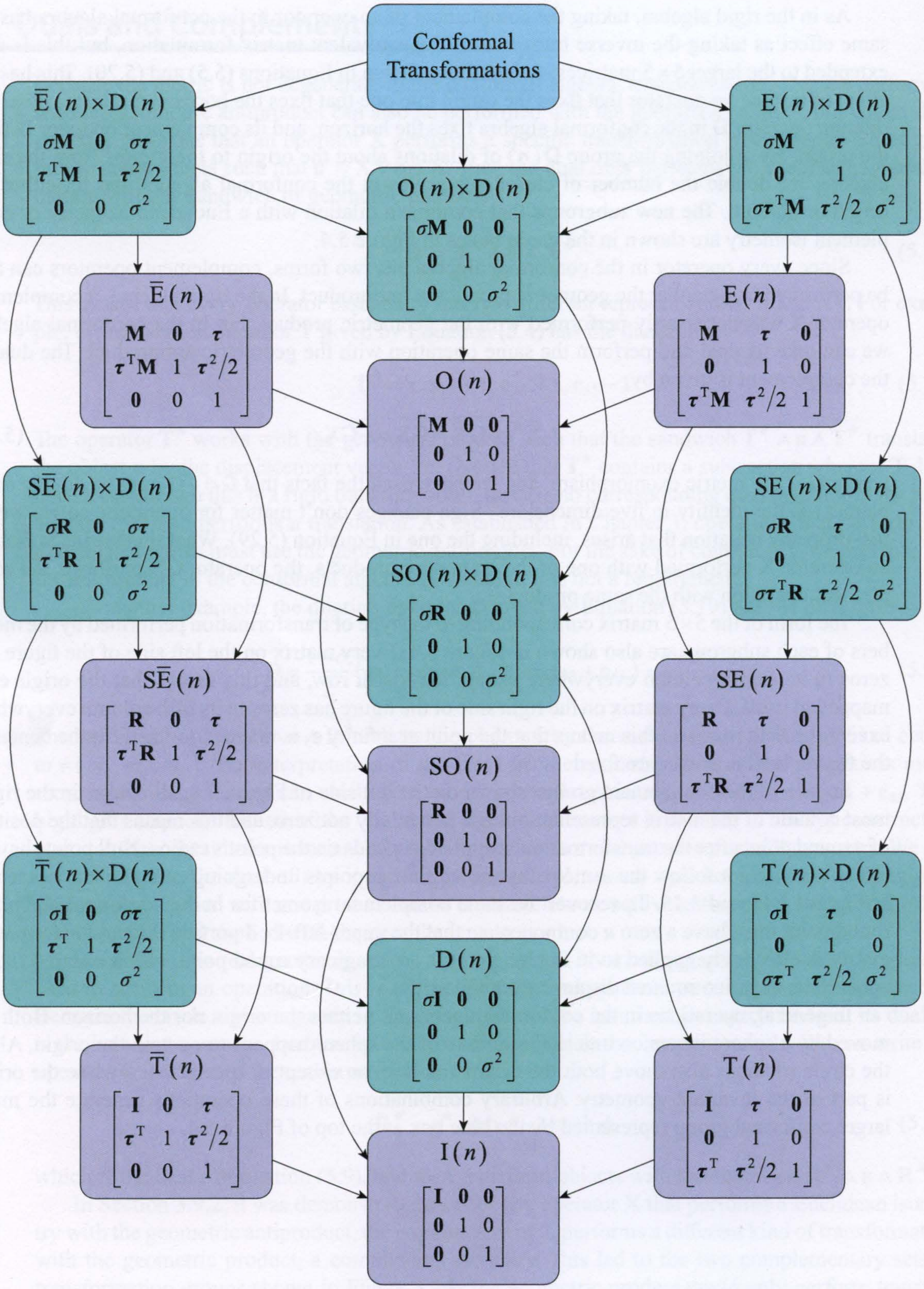


Figure 5.4. These are particular transformation subgroups in conformal geometric algebra. The matrix $\mathbf{I}$ is the $n\times n$ identity, the matrix $\mathbf{R}$ is a rotation, and the matrix $\mathbf{M}$ is merely orthogonal. The vector $\boldsymbol{\tau}$ is a translation, and the scalar σ is a positive scale value. Groups on the left fix the origin, groups on the right fix the point at infinity, and groups in the center fix both. Groups with a purple background are equivalent to those with the same name shown in Figure 3.16.

5.4 2D Conformal Transformations

There are many conformal motions possible that we have not discussed. Some of the more general motions can be understood well in two dimensions, so we take a look at a specific kind of motor in the 4D conformal geometric algebra introduced in Section 4.9. The full multiplication tables for the geometric product and antiproduct in this algebra are included in Appendix A, and the reverses are given in Table 3.5 since they are the same in the 4D rigid geometric algebra.

The conformal motor $\mathbf{Q}$ that we are interested in has the exponential form

$$\mathbf{Q} = \exp_{\vee}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \mathbf{d}\right], \tag{5.30}$$

where $\mathbf{d}$ is a 2D dipole, and the values of δ and ϕ are scalar values. This expression is algebraically identical to that given in Section 3.6.2. The dual number $\delta\mathbf{1} + \phi\mathbb{1}$ is the same, and the 2D dipole $\mathbf{d}$ has the same six-component bivector representation as the 3D line $\boldsymbol{l}$ in the rigid algebra. The 4D conformal algebra over two-dimensional Euclidean space and the 4D rigid algebra over three-dimensional Euclidean space have the same exterior structures and differ only in their metric tensors. The degenerate metric in the rigid algebra was responsible for some simplifications when the exponential form of $\mathbf{Q}$ was expanded, but the conformal metric will cause the expansion of Equation (5.30) to be more complicated.

We assume that $\mathbf{d}$ is a real dipole that has been radius normalized so that $\mathbf{d} \vee \mathbf{d} = -\mathbb{1}$. It will make things easier if we split the exponential into a product of exponentials and write

$$\mathbf{Q} = \exp_{\vee}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \mathbf{d}\right] = \exp_{\vee}\left(-\delta\mathbf{d}^{\star}\right) \vee \exp_{\vee}(\phi\mathbf{d}). \tag{5.31}$$

This relies on a couple of facts, the first of which is that $\mathbf{d}$ and $\mathbf{1}$ commute under the antiproduct. Also, since $\underset{\sim}{\mathbf{d}} = -\mathbf{d}$, the product $\mathbf{1} \vee \mathbf{d}$ is then equal to $-\mathbf{d}^{\star}$ according to Equation (3.63), so

$$\exp_{\vee}\left[(\delta\mathbf{1} + \phi\mathbb{1}) \vee \mathbf{d}\right] = \exp_{\vee}\left(-\delta\mathbf{d}^{\star} + \phi\mathbf{d}\right). \tag{5.32}$$

Now, using Equation (3.63) again, it's easy to show that $\mathbf{d} \vee \mathbf{d}^{\star} = \mathbf{d}^{\star} \vee \mathbf{d} = \mathbf{1}$, so $\mathbf{d}$ and $\mathbf{d}^{\star}$ commute, and we are allowed to convert an exponential of the sum $-\delta\mathbf{d}^{\star} + \phi\mathbf{d}$ into a product of exponentials. The exponential $\exp_{\vee}(\phi\mathbf{d})$ on the right side of Equation (5.31) expands into sines and cosines because $\mathbf{d}$ is a real dipole. The other exponential $\exp_{\vee}(-\delta\mathbf{d}^{\star})$ expands into hyperbolic sines and cosines because $\mathbf{d}^{\star}$ is an imaginary dipole, and therefore $\mathbf{d}^{\star} \vee \mathbf{d}^{\star} = \mathbb{1}$. We can now write $\mathbf{Q}$ as

$$\mathbf{Q} = \left(-\mathbf{d}^{\star}\sinh\delta + \mathbb{1}\cosh\delta\right) \vee \left(\mathbf{d}\sin\phi + \mathbb{1}\cos\phi\right), \tag{5.33}$$

and this finally multiplies out to

$$\boxed{\mathbf{Q} = \mathbf{d}\cosh\delta\sin\phi - \mathbf{d}^{\star}\sinh\delta\cos\phi - \mathbf{1}\sinh\delta\sin\phi + \mathbb{1}\cosh\delta\cos\phi.} \tag{5.34}$$

The screw motion represented by Equation (3.109) is given by the same equation after we make the substitutions $\mathbf{d} = \boldsymbol{l}$, $\cosh\delta = 1$, and $\sinh\delta = \delta$.

In the conformal motion that the motor $\mathbf{Q}$ in Equation (5.34) represents, the dipole $\mathbf{d}$ determines the fixed points in the 2D plane, and the homogeneous magnitude $\delta\mathbf{1} + \phi\mathbb{1}$ controls the shape of the motion. The quotient δ/ϕ is called the *loxodromic parameter*, and it corresponds to a continuous transition between elliptic and hyperbolic motions. The motions produced by several different parameter values are shown in Figure 5.5. When $\phi = 0$, the motion is entirely hyperbolic, as shown in the upper-left part of the figure. When $\delta = 0$, the motion is entirely elliptic, as shown in the upper-right part of the figure. Otherwise, the motor $\mathbf{Q}$ produces a blend of hyperbolic and elliptic motions called a loxodromic transformation. Three intermediate steps are shown at the bottom of the figure.

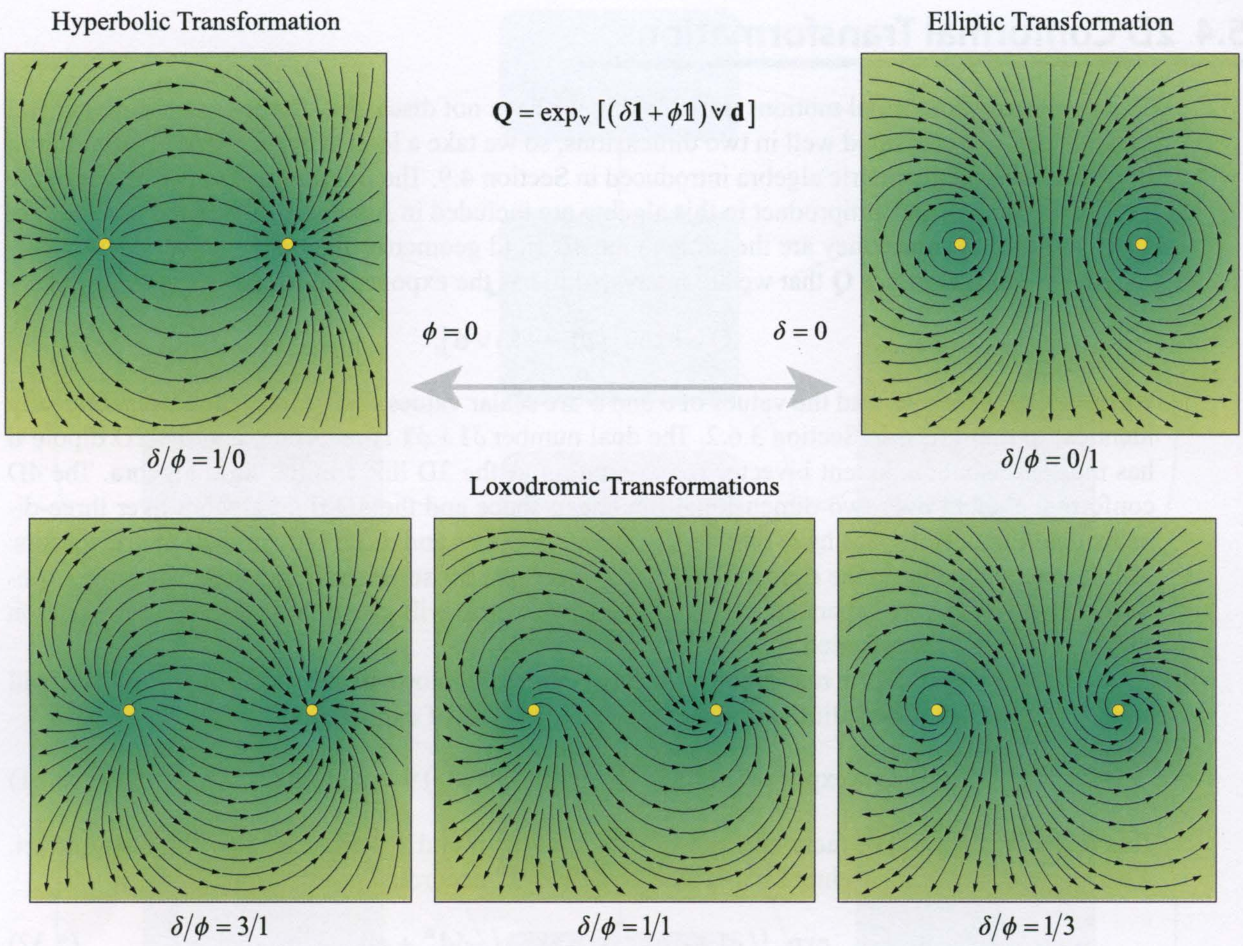


Figure 5.5. These are the mixtures of hyperbolic and elliptic conformal transformations produced by the motor $\mathbf{Q}$ from Equation (5.34). The yellow dots are the fixed endpoints of dipole $\mathbf{d}$. When both the angles δ and ϕ are nonzero, the result is a loxodromic transformation.

When the dipole $\mathbf{d}$ used to construct the motor $\mathbf{Q}$ in Equation (5.30) is a flat point $\mathbf{p}$, the only fixed point at a finite position is $\mathbf{p}$ itself. The point at infinity $\mathbf{e}_4$ is also fixed because that's where the other end of the dipole is. In this case, an algebraically equivalent but conceptually different set of transformations are produced, and examples for the same loxodromic parameters are shown in Figure 5.6. This demonstrates that dilations and rotations are on opposite ends of the same spectrum and that it's possible to continuously transform one of those types of motion into the other. As shown in the upper-left part of the figure, the motion is a dilation when $\phi = 0$. We have already seen in Section 5.2 that dilations are hyperbolic in nature. As shown in the upper-right part of the figure, the motion is a rotation when $\delta = 0$. Otherwise, the motor $\mathbf{Q}$ produces a swirling motion that blends a dilation and rotation together as shown in three examples at the bottom of the figure.

There are two special cases corresponding to dipoles of radius zero. These result in parabolic motions when exponentiated to produce the motor

$$\mathbf{Q} = \exp_{\vee}(\phi\mathbf{d}) = \phi\mathbf{d} + \mathbb{1}. \tag{5.35}$$

The transformation performed by $\mathbf{Q}$ in this case is the parabolic motion shown on the left side of Figure 5.7. When $\mathbf{d}$ is a flat point $\mathbf{p} = p_x\mathbf{e}_{41} + p_y\mathbf{e}_{42}$ in the horizon, the transformation performed by $\mathbf{Q}$ is a translation by the displacement vector $(2p_y, -2p_x)$.

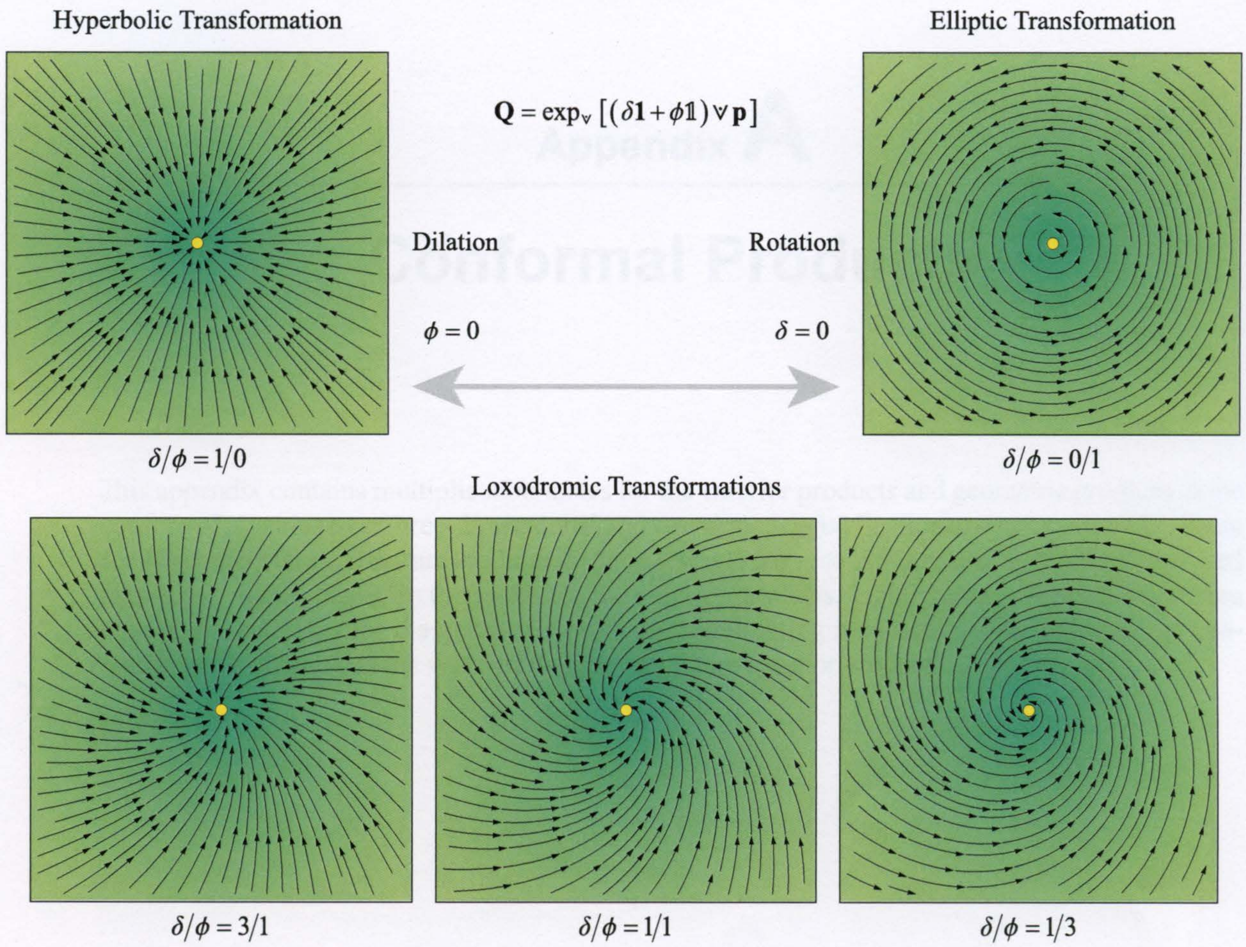


Figure 5.6. These are the mixtures of dilations and rotations produced by the motor $\mathbf{Q}$ from Equation (5.34) when the dipole is a flat point $\mathbf{p}$ shown as a yellow dot. When both the angles δ and ϕ are nonzero, the result is a loxodromic transformation.

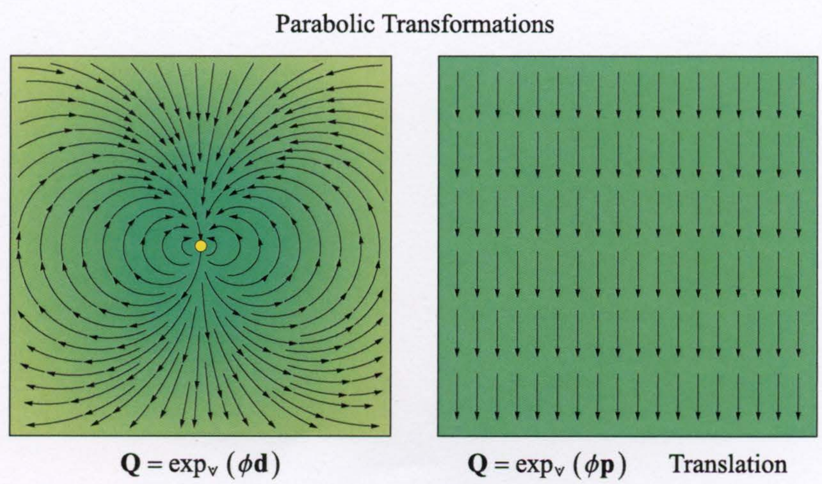


Figure 5.7. These are parabolic conformal transformations in two dimensions, corresponding to dipoles of radius zero. When the dipole is a flat point $\mathbf{p}$ in the horizon, the transformation is a translation.

Appendix A

Conformal Products

This appendix contains multiplication tables for the exterior products and geometric products in the conformal algebras over three-dimensional and two-dimensional Euclidean spaces. All of the tables for 3D conformal algebras are too large to fit on a single page, so they are split in half and displayed across two facing pages. In the tables for the exterior products, highlighting is applied to nonzero entries. In the tables for the geometric products, highlighting indicates that the value of the geometric product includes the value of the corresponding exterior product.

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

3D Conformal Exterior Algebra

$\mathbf{a} \backslash \mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{5}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{5}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	0	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{15}$	0	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	0	0	$\mathbf{e}_{125}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{415}$
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{25}$	$\mathbf{e}_{412}$	0	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{321}$	0	$-\mathbf{e}_{125}$	0	$\mathbf{e}_{235}$	$-\mathbf{e}_{425}$
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	0	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{315}$	$-\mathbf{e}_{235}$	0	$-\mathbf{e}_{435}$
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	$\mathbf{e}_{45}$	0	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	0
$\mathbf{e}_{5}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{45}$	0	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	0	0	0	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	$\mathbf{e}_{415}$	0	0	0	$-\mathbf{e}_{1234}$	0	0	0	$\mathbf{e}_{4125}$	$-\mathbf{e}_{4315}$	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{423}$	0	$\mathbf{e}_{425}$	0	0	0	0	$-\mathbf{e}_{1234}$	0	$-\mathbf{e}_{4125}$	0	$\mathbf{e}_{4235}$	0
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	0	0	$\mathbf{e}_{435}$	0	0	0	0	0	$-\mathbf{e}_{1234}$	$\mathbf{e}_{4315}$	$-\mathbf{e}_{4235}$	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	0	0	$\mathbf{e}_{423}$	$\mathbf{e}_{235}$	$-\mathbf{e}_{1234}$	0	0	0	0	0	$-\mathbf{e}_{3215}$	0	0	$\mathbf{e}_{4235}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{431}$	$\mathbf{e}_{315}$	0	$-\mathbf{e}_{1234}$	0	0	0	0	0	$-\mathbf{e}_{3215}$	0	$\mathbf{e}_{4315}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{125}$	0	0	$-\mathbf{e}_{1234}$	0	0	0	0	0	$-\mathbf{e}_{3215}$	$\mathbf{e}_{4125}$
$\mathbf{e}_{15}$	$\mathbf{e}_{15}$	0	$-\mathbf{e}_{125}$	$\mathbf{e}_{315}$	$\mathbf{e}_{415}$	0	0	$-\mathbf{e}_{4125}$	$\mathbf{e}_{4315}$	$-\mathbf{e}_{3215}$	0	0	0	0	0	0
$\mathbf{e}_{25}$	$\mathbf{e}_{25}$	$\mathbf{e}_{125}$	0	$-\mathbf{e}_{235}$	$\mathbf{e}_{425}$	0	$\mathbf{e}_{4125}$	0	$-\mathbf{e}_{4235}$	0	$-\mathbf{e}_{3215}$	0	0	0	0	0
$\mathbf{e}_{35}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{235}$	0	$\mathbf{e}_{435}$	0	$-\mathbf{e}_{4315}$	$\mathbf{e}_{4235}$	0	0	0	$-\mathbf{e}_{3215}$	0	0	0	0
$\mathbf{e}_{45}$	$\mathbf{e}_{45}$	$-\mathbf{e}_{415}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{435}$	0	0	0	0	0	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	0	0	0	0
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{1234}$	0	0	0	$\mathbf{e}_{4235}$	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	0	$-\mathbf{e}_{1234}$	0	0	$\mathbf{e}_{4315}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	0	0	$-\mathbf{e}_{1234}$	0	$\mathbf{e}_{4125}$	0	0	0	0	0	0	0	0	$-\mathbb{1}$	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	0	0	0	$-\mathbf{e}_{1234}$	$\mathbf{e}_{3215}$	0	0	0	0	0	0	0	0	0	$-\mathbb{1}$
$\mathbf{e}_{415}$	$\mathbf{e}_{415}$	0	$-\mathbf{e}_{4125}$	$\mathbf{e}_{4315}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{425}$	$\mathbf{e}_{425}$	$\mathbf{e}_{4125}$	0	$-\mathbf{e}_{4235}$	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{435}$	$\mathbf{e}_{435}$	$-\mathbf{e}_{4315}$	$\mathbf{e}_{4235}$	0	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0
$\mathbf{e}_{235}$	$\mathbf{e}_{235}$	$\mathbf{e}_{3215}$	0	0	$-\mathbf{e}_{4235}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{315}$	$\mathbf{e}_{315}$	0	$\mathbf{e}_{3215}$	0	$-\mathbf{e}_{4315}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{125}$	$\mathbf{e}_{125}$	0	0	$\mathbf{e}_{3215}$	$-\mathbf{e}_{4125}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{1234}$	$\mathbf{e}_{1234}$	0	0	0	0	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{4235}$	$\mathbf{e}_{4235}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{4315}$	$\mathbf{e}_{4315}$	0	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{4125}$	$\mathbf{e}_{4125}$	0	0	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{3215}$	$\mathbf{e}_{3215}$	0	0	0	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

3D Conformal Exterior Algebra

a \ b	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
1	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
$\mathbf{e}_1$	$\mathbf{e}_{1234}$	0	0	0	0	$-\mathbf{e}_{4125}$	$\mathbf{e}_{4315}$	$-\mathbf{e}_{3215}$	0	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_2$	0	$\mathbf{e}_{1234}$	0	0	$\mathbf{e}_{4125}$	0	$-\mathbf{e}_{4235}$	0	$-\mathbf{e}_{3215}$	0	0	0	$\mathbb{1}$	0	0	0
$\mathbf{e}_3$	0	0	$\mathbf{e}_{1234}$	0	$-\mathbf{e}_{4315}$	$\mathbf{e}_{4235}$	0	0	0	$-\mathbf{e}_{3215}$	0	0	0	$\mathbb{1}$	0	0
$\mathbf{e}_4$	0	0	0	$\mathbf{e}_{1234}$	0	0	0	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	0	0	0	0	$\mathbb{1}$	0
$\mathbf{e}_5$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{4315}$	$-\mathbf{e}_{4125}$	$-\mathbf{e}_{3215}$	0	0	0	0	0	0	$\mathbb{1}$	0	0	0	0	0
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0
$\mathbf{e}_{23}$	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	0	0	0	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{15}$	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{25}$	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{35}$	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{45}$	0	0	0	$-\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{431}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{412}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{321}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{415}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{425}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{435}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{235}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{315}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{125}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{1234}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{4235}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{4315}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{4125}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{3215}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbb{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

3D Conformal Exterior Algebra

$\mathbf{a}$ \ $\mathbf{b}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\mathbf{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{23}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{31}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{12}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{25}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{35}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{45}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{423}$	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0
$\mathbf{e}_{431}$	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0
$\mathbf{e}_{412}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0
$\mathbf{e}_{321}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
$\mathbf{e}_{415}$	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0
$\mathbf{e}_{425}$	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0
$\mathbf{e}_{435}$	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0
$\mathbf{e}_{235}$	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
$\mathbf{e}_{315}$	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0
$\mathbf{e}_{125}$	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0
$\mathbf{e}_{1234}$	0	0	0	0	0	1	0	0	0	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$
$\mathbf{e}_{4235}$	0	1	0	0	0	0	$\mathbf{e}_4$	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_5$	0	0	0
$\mathbf{e}_{4315}$	0	0	1	0	0	0	0	$\mathbf{e}_4$	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$-\mathbf{e}_5$	0	0
$\mathbf{e}_{4125}$	0	0	0	1	0	0	0	0	$\mathbf{e}_4$	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	0	$-\mathbf{e}_5$	0
$\mathbf{e}_{3215}$	0	0	0	0	1	0	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	0	0	0	0	0	0	$-\mathbf{e}_5$
$\mathbb{1}$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

3D Conformal Exterior Algebra

a \ b	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**
$\mathbf{e}_1$	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	$\mathbf{e}_1$
$\mathbf{e}_2$	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	$\mathbf{e}_2$
$\mathbf{e}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	$\mathbf{e}_3$
$\mathbf{e}_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	$\mathbf{e}_4$
$\mathbf{e}_5$	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	$\mathbf{e}_5$
$\mathbf{e}_{41}$	0	0	0	0	0	0	0	**−1**	0	0	0	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	0	0	0	0	0	**−1**	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	0	0	0	0	0	0	**−1**	0	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	0	0	0	0	**−1**	0	0	0	0	0	0	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	0	0	0	0	0	**−1**	0	0	0	0	0	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	0	0	0	0	0	0	**−1**	0	0	0	0	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	$\mathbf{e}_{12}$
$\mathbf{e}_{15}$	**−1**	0	0	0	0	0	0	0	0	0	$-\mathbf{e}_1$	$\mathbf{e}_5$	0	0	0	$\mathbf{e}_{15}$
$\mathbf{e}_{25}$	0	**−1**	0	0	0	0	0	0	0	0	$-\mathbf{e}_2$	0	$\mathbf{e}_5$	0	0	$\mathbf{e}_{25}$
$\mathbf{e}_{35}$	0	0	**−1**	0	0	0	0	0	0	0	$-\mathbf{e}_3$	0	0	$\mathbf{e}_5$	0	$\mathbf{e}_{35}$
$\mathbf{e}_{45}$	0	0	0	**−1**	0	0	0	0	0	0	$-\mathbf{e}_4$	0	0	0	$\mathbf{e}_5$	$\mathbf{e}_{45}$
$\mathbf{e}_{423}$	0	0	0	0	$-\mathbf{e}_4$	0	0	0	$-\mathbf{e}_3$	$\mathbf{e}_2$	0	0	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	0	0	0	$-\mathbf{e}_4$	0	$\mathbf{e}_3$	0	$-\mathbf{e}_1$	0	$\mathbf{e}_{43}$	0	$-\mathbf{e}_{41}$	$\mathbf{e}_{31}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	0	0	0	$-\mathbf{e}_4$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{12}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	0	0	0	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	0	0	0	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	0	$\mathbf{e}_{321}$
$\mathbf{e}_{415}$	$-\mathbf{e}_4$	0	0	$\mathbf{e}_1$	0	0	0	$-\mathbf{e}_5$	0	0	$\mathbf{e}_{41}$	$-\mathbf{e}_{45}$	0	0	$\mathbf{e}_{15}$	$\mathbf{e}_{415}$
$\mathbf{e}_{425}$	0	$-\mathbf{e}_4$	0	$\mathbf{e}_2$	0	0	0	0	$-\mathbf{e}_5$	0	$\mathbf{e}_{42}$	0	$-\mathbf{e}_{45}$	0	$\mathbf{e}_{25}$	$\mathbf{e}_{425}$
$\mathbf{e}_{435}$	0	0	$-\mathbf{e}_4$	$\mathbf{e}_3$	0	0	0	0	0	$-\mathbf{e}_5$	$\mathbf{e}_{43}$	0	0	$-\mathbf{e}_{45}$	$\mathbf{e}_{35}$	$\mathbf{e}_{435}$
$\mathbf{e}_{235}$	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	0	$-\mathbf{e}_5$	0	0	0	0	0	$\mathbf{e}_{23}$	0	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	0	$\mathbf{e}_{235}$
$\mathbf{e}_{315}$	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	0	0	$-\mathbf{e}_5$	0	0	0	0	$\mathbf{e}_{31}$	$-\mathbf{e}_{35}$	0	$\mathbf{e}_{15}$	0	$\mathbf{e}_{315}$
$\mathbf{e}_{125}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	0	0	0	$-\mathbf{e}_5$	0	0	0	$\mathbf{e}_{12}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	0	0	$\mathbf{e}_{125}$
$\mathbf{e}_{1234}$	0	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{1234}$
$\mathbf{e}_{4235}$	0	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{45}$	0	0	0	$-\mathbf{e}_{35}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{423}$	0	$-\mathbf{e}_{435}$	$\mathbf{e}_{425}$	$\mathbf{e}_{235}$	$\mathbf{e}_{4235}$
$\mathbf{e}_{4315}$	$-\mathbf{e}_{43}$	0	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	0	$-\mathbf{e}_{45}$	0	$\mathbf{e}_{35}$	0	$-\mathbf{e}_{15}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{435}$	0	$-\mathbf{e}_{415}$	$\mathbf{e}_{315}$	$\mathbf{e}_{4315}$
$\mathbf{e}_{4125}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	0	$-\mathbf{e}_{12}$	0	0	$-\mathbf{e}_{45}$	$-\mathbf{e}_{25}$	$\mathbf{e}_{15}$	0	$-\mathbf{e}_{412}$	$-\mathbf{e}_{425}$	$\mathbf{e}_{415}$	0	$\mathbf{e}_{125}$	$\mathbf{e}_{4125}$
$\mathbf{e}_{3215}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	0	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	0	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	0	$\mathbf{e}_{3215}$
$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$

Geometric Product **a ⟑ b**

3D Conformal Geometric Algebra

a \ b	$\mathbf 1$	$\mathbf e_{1}$	$\mathbf e_{2}$	$\mathbf e_{3}$	$\mathbf e_{4}$	$\mathbf e_{5}$	$\mathbf e_{41}$	$\mathbf e_{42}$	$\mathbf e_{43}$	$\mathbf e_{23}$	$\mathbf e_{31}$	$\mathbf e_{12}$	$\mathbf e_{15}$	$\mathbf e_{25}$	$\mathbf e_{35}$	$\mathbf e_{45}$
$\mathbf 1$	$\mathbf 1$	$\mathbf e_{1}$	$\mathbf e_{2}$	$\mathbf e_{3}$	$\mathbf e_{4}$	$\mathbf e_{5}$	$\mathbf e_{41}$	$\mathbf e_{42}$	$\mathbf e_{43}$	$\mathbf e_{23}$	$\mathbf e_{31}$	$\mathbf e_{12}$	$\mathbf e_{15}$	$\mathbf e_{25}$	$\mathbf e_{35}$	$\mathbf e_{45}$
$\mathbf e_{1}$	$\mathbf e_{1}$	$\mathbf 1$	$\mathbf e_{12}$	$-\mathbf e_{31}$	$-\mathbf e_{41}$	$\mathbf e_{15}$	$-\mathbf e_{4}$	$-\mathbf e_{412}$	$\mathbf e_{431}$	$-\mathbf e_{321}$	$-\mathbf e_{3}$	$\mathbf e_{2}$	$\mathbf e_{5}$	$\mathbf e_{125}$	$-\mathbf e_{315}$	$-\mathbf e_{415}$
$\mathbf e_{2}$	$\mathbf e_{2}$	$-\mathbf e_{12}$	$\mathbf 1$	$\mathbf e_{23}$	$-\mathbf e_{42}$	$\mathbf e_{25}$	$\mathbf e_{412}$	$-\mathbf e_{4}$	$-\mathbf e_{423}$	$\mathbf e_{3}$	$-\mathbf e_{321}$	$-\mathbf e_{1}$	$-\mathbf e_{125}$	$\mathbf e_{5}$	$\mathbf e_{235}$	$-\mathbf e_{425}$
$\mathbf e_{3}$	$\mathbf e_{3}$	$\mathbf e_{31}$	$-\mathbf e_{23}$	$\mathbf 1$	$-\mathbf e_{43}$	$\mathbf e_{35}$	$-\mathbf e_{431}$	$\mathbf e_{423}$	$-\mathbf e_{4}$	$-\mathbf e_{2}$	$\mathbf e_{1}$	$-\mathbf e_{321}$	$\mathbf e_{315}$	$-\mathbf e_{235}$	$\mathbf e_{5}$	$-\mathbf e_{435}$
$\mathbf e_{4}$	$\mathbf e_{4}$	$\mathbf e_{41}$	$\mathbf e_{42}$	$\mathbf e_{43}$	0	$\mathbf e_{45} - \mathbf 1$	0	0	0	$\mathbf e_{423}$	$\mathbf e_{431}$	$\mathbf e_{412}$	$\mathbf e_{1} + \mathbf e_{415}$	$\mathbf e_{2} + \mathbf e_{425}$	$\mathbf e_{3} + \mathbf e_{435}$	$\mathbf e_{4}$
$\mathbf e_{5}$	$\mathbf e_{5}$	$-\mathbf e_{15}$	$-\mathbf e_{25}$	$-\mathbf e_{35}$	$-\mathbf 1 - \mathbf e_{45}$	0	$\mathbf e_{415} - \mathbf e_{1}$	$\mathbf e_{425} - \mathbf e_{2}$	$\mathbf e_{435} - \mathbf e_{3}$	$\mathbf e_{235}$	$\mathbf e_{315}$	$\mathbf e_{125}$	0	0	0	$-\mathbf e_{5}$
$\mathbf e_{41}$	$\mathbf e_{41}$	$\mathbf e_{4}$	$\mathbf e_{412}$	$-\mathbf e_{431}$	0	$\mathbf e_{1} + \mathbf e_{415}$	0	0	0	$-\mathbf e_{1234}$	$-\mathbf e_{43}$	$\mathbf e_{42}$	$\mathbf e_{45} - \mathbf 1$	$\mathbf e_{4125} - \mathbf e_{12}$	$\mathbf e_{31} - \mathbf e_{4315}$	$\mathbf e_{41}$
$\mathbf e_{42}$	$\mathbf e_{42}$	$-\mathbf e_{412}$	$\mathbf e_{4}$	$\mathbf e_{423}$	0	$\mathbf e_{2} + \mathbf e_{425}$	0	0	0	$\mathbf e_{43}$	$-\mathbf e_{1234}$	$-\mathbf e_{41}$	$\mathbf e_{12} - \mathbf e_{4125}$	$\mathbf e_{45} - \mathbf 1$	$\mathbf e_{4235} - \mathbf e_{23}$	$\mathbf e_{42}$
$\mathbf e_{43}$	$\mathbf e_{43}$	$\mathbf e_{431}$	$-\mathbf e_{423}$	$\mathbf e_{4}$	0	$\mathbf e_{3} + \mathbf e_{435}$	0	0	0	$-\mathbf e_{42}$	$\mathbf e_{41}$	$-\mathbf e_{1234}$	$\mathbf e_{4315} - \mathbf e_{31}$	$\mathbf e_{23} - \mathbf e_{4235}$	$\mathbf e_{45} - \mathbf 1$	$\mathbf e_{43}$
$\mathbf e_{23}$	$\mathbf e_{23}$	$-\mathbf e_{321}$	$-\mathbf e_{3}$	$\mathbf e_{2}$	$\mathbf e_{423}$	$\mathbf e_{235}$	$-\mathbf e_{1234}$	$-\mathbf e_{43}$	$\mathbf e_{42}$	$-\mathbf 1$	$-\mathbf e_{12}$	$\mathbf e_{31}$	$-\mathbf e_{3215}$	$-\mathbf e_{35}$	$\mathbf e_{25}$	$\mathbf e_{4235}$
$\mathbf e_{31}$	$\mathbf e_{31}$	$\mathbf e_{3}$	$-\mathbf e_{321}$	$-\mathbf e_{1}$	$\mathbf e_{431}$	$\mathbf e_{315}$	$\mathbf e_{43}$	$-\mathbf e_{1234}$	$-\mathbf e_{41}$	$\mathbf e_{12}$	$-\mathbf 1$	$-\mathbf e_{23}$	$\mathbf e_{35}$	$-\mathbf e_{3215}$	$-\mathbf e_{15}$	$\mathbf e_{4315}$
$\mathbf e_{12}$	$\mathbf e_{12}$	$-\mathbf e_{2}$	$\mathbf e_{1}$	$-\mathbf e_{321}$	$\mathbf e_{412}$	$\mathbf e_{125}$	$-\mathbf e_{42}$	$\mathbf e_{41}$	$-\mathbf e_{1234}$	$-\mathbf e_{31}$	$\mathbf e_{23}$	$-\mathbf 1$	$-\mathbf e_{25}$	$\mathbf e_{15}$	$-\mathbf e_{3215}$	$\mathbf e_{4125}$
$\mathbf e_{15}$	$\mathbf e_{15}$	$-\mathbf e_{5}$	$-\mathbf e_{125}$	$\mathbf e_{315}$	$\mathbf e_{415} - \mathbf e_{1}$	0	$-\mathbf 1 - \mathbf e_{45}$	$-\mathbf e_{12} - \mathbf e_{4125}$	$\mathbf e_{31} + \mathbf e_{4315}$	$-\mathbf e_{3215}$	$-\mathbf e_{35}$	$\mathbf e_{25}$	0	0	0	$-\mathbf e_{15}$
$\mathbf e_{25}$	$\mathbf e_{25}$	$\mathbf e_{125}$	$-\mathbf e_{5}$	$-\mathbf e_{235}$	$\mathbf e_{425} - \mathbf e_{2}$	0	$\mathbf e_{12} + \mathbf e_{4125}$	$-\mathbf 1 - \mathbf e_{45}$	$-\mathbf e_{23} - \mathbf e_{4235}$	$\mathbf e_{35}$	$-\mathbf e_{3215}$	$-\mathbf e_{15}$	0	0	0	$-\mathbf e_{25}$
$\mathbf e_{35}$	$\mathbf e_{35}$	$-\mathbf e_{315}$	$\mathbf e_{235}$	$-\mathbf e_{5}$	$\mathbf e_{435} - \mathbf e_{3}$	0	$-\mathbf e_{31} - \mathbf e_{4315}$	$\mathbf e_{23} + \mathbf e_{4235}$	$-\mathbf 1 - \mathbf e_{45}$	$-\mathbf e_{25}$	$\mathbf e_{15}$	$-\mathbf e_{3215}$	0	0	0	$-\mathbf e_{35}$
$\mathbf e_{45}$	$\mathbf e_{45}$	$-\mathbf e_{415}$	$-\mathbf e_{425}$	$-\mathbf e_{435}$	$-\mathbf e_{4}$	$\mathbf e_{5}$	$-\mathbf e_{41}$	$-\mathbf e_{42}$	$-\mathbf e_{43}$	$\mathbf e_{4235}$	$\mathbf e_{4315}$	$\mathbf e_{4125}$	$\mathbf e_{15}$	$\mathbf e_{25}$	$\mathbf e_{35}$	$\mathbf 1$
$\mathbf e_{423}$	$\mathbf e_{423}$	$-\mathbf e_{1234}$	$-\mathbf e_{43}$	$\mathbf e_{42}$	0	$\mathbf e_{4235} - \mathbf e_{23}$	0	0	0	$-\mathbf e_{4}$	$-\mathbf e_{412}$	$\mathbf e_{431}$	$-\mathbf e_{321} - \mathbb 1$	$-\mathbf e_{3} - \mathbf e_{435}$	$\mathbf e_{2} + \mathbf e_{425}$	$\mathbf e_{423}$
$\mathbf e_{431}$	$\mathbf e_{431}$	$\mathbf e_{43}$	$-\mathbf e_{1234}$	$-\mathbf e_{41}$	0	$\mathbf e_{4315} - \mathbf e_{31}$	0	0	0	$\mathbf e_{412}$	$-\mathbf e_{4}$	$-\mathbf e_{423}$	$\mathbf e_{3} + \mathbf e_{435}$	$-\mathbf e_{321} - \mathbb 1$	$-\mathbf e_{1} - \mathbf e_{415}$	$\mathbf e_{431}$
$\mathbf e_{412}$	$\mathbf e_{412}$	$-\mathbf e_{42}$	$\mathbf e_{41}$	$-\mathbf e_{1234}$	0	$\mathbf e_{4125} - \mathbf e_{12}$	0	0	0	$-\mathbf e_{431}$	$\mathbf e_{423}$	$-\mathbf e_{4}$	$-\mathbf e_{2} - \mathbf e_{425}$	$\mathbf e_{1} + \mathbf e_{415}$	$-\mathbf e_{321} - \mathbb 1$	$\mathbf e_{412}$
$\mathbf e_{321}$	$\mathbf e_{321}$	$-\mathbf e_{23}$	$-\mathbf e_{31}$	$-\mathbf e_{12}$	$-\mathbf e_{1234}$	$\mathbf e_{3215}$	$\mathbf e_{423}$	$\mathbf e_{431}$	$\mathbf e_{412}$	$\mathbf e_{1}$	$\mathbf e_{2}$	$\mathbf e_{3}$	$-\mathbf e_{235}$	$-\mathbf e_{315}$	$-\mathbf e_{125}$	$-\mathbb 1$
$\mathbf e_{415}$	$\mathbf e_{415}$	$-\mathbf e_{45}$	$-\mathbf e_{4125}$	$\mathbf e_{4315}$	$-\mathbf e_{41}$	$-\mathbf e_{15}$	$-\mathbf e_{4}$	$-\mathbf e_{412}$	$\mathbf e_{431}$	$-\mathbb 1$	$-\mathbf e_{435}$	$\mathbf e_{425}$	$-\mathbf e_{5}$	$-\mathbf e_{125}$	$\mathbf e_{315}$	$-\mathbf e_{1}$
$\mathbf e_{425}$	$\mathbf e_{425}$	$\mathbf e_{4125}$	$-\mathbf e_{45}$	$-\mathbf e_{4235}$	$-\mathbf e_{42}$	$-\mathbf e_{25}$	$\mathbf e_{412}$	$-\mathbf e_{4}$	$-\mathbf e_{423}$	$\mathbf e_{435}$	$-\mathbb 1$	$-\mathbf e_{415}$	$\mathbf e_{125}$	$-\mathbf e_{5}$	$-\mathbf e_{235}$	$-\mathbf e_{2}$
$\mathbf e_{435}$	$\mathbf e_{435}$	$-\mathbf e_{4315}$	$\mathbf e_{4235}$	$-\mathbf e_{45}$	$-\mathbf e_{43}$	$-\mathbf e_{35}$	$-\mathbf e_{431}$	$\mathbf e_{423}$	$-\mathbf e_{4}$	$-\mathbf e_{425}$	$\mathbf e_{415}$	$-\mathbb 1$	$-\mathbf e_{315}$	$\mathbf e_{235}$	$-\mathbf e_{5}$	$-\mathbf e_{3}$
$\mathbf e_{235}$	$\mathbf e_{235}$	$\mathbf e_{3215}$	$\mathbf e_{35}$	$-\mathbf e_{25}$	$-\mathbf e_{23} - \mathbf e_{4235}$	0	$\mathbf e_{321} - \mathbb 1$	$\mathbf e_{3} - \mathbf e_{435}$	$\mathbf e_{425} - \mathbf e_{2}$	$-\mathbf e_{5}$	$-\mathbf e_{125}$	$\mathbf e_{315}$	0	0	0	$-\mathbf e_{235}$
$\mathbf e_{315}$	$\mathbf e_{315}$	$-\mathbf e_{35}$	$\mathbf e_{3215}$	$\mathbf e_{15}$	$-\mathbf e_{31} - \mathbf e_{4315}$	0	$\mathbf e_{435} - \mathbf e_{3}$	$\mathbf e_{321} - \mathbb 1$	$\mathbf e_{1} - \mathbf e_{415}$	$\mathbf e_{125}$	$-\mathbf e_{5}$	$-\mathbf e_{235}$	0	0	0	$-\mathbf e_{315}$
$\mathbf e_{125}$	$\mathbf e_{125}$	$\mathbf e_{25}$	$-\mathbf e_{15}$	$\mathbf e_{3215}$	$-\mathbf e_{12} - \mathbf e_{4125}$	0	$\mathbf e_{2} - \mathbf e_{425}$	$\mathbf e_{415} - \mathbf e_{1}$	$\mathbf e_{321} - \mathbb 1$	$-\mathbf e_{315}$	$\mathbf e_{235}$	$-\mathbf e_{5}$	0	0	0	$-\mathbf e_{125}$
$\mathbf e_{1234}$	$\mathbf e_{1234}$	$-\mathbf e_{423}$	$-\mathbf e_{431}$	$-\mathbf e_{412}$	0	$\mathbf e_{321} + \mathbb 1$	0	0	0	$\mathbf e_{41}$	$\mathbf e_{42}$	$\mathbf e_{43}$	$\mathbf e_{23} - \mathbf e_{4235}$	$\mathbf e_{31} - \mathbf e_{4315}$	$\mathbf e_{12} - \mathbf e_{4125}$	$\mathbf e_{1234}$
$\mathbf e_{4235}$	$\mathbf e_{4235}$	$\mathbb 1$	$\mathbf e_{435}$	$-\mathbf e_{425}$	$-\mathbf e_{423}$	$\mathbf e_{235}$	$\mathbf e_{1234}$	$\mathbf e_{43}$	$-\mathbf e_{42}$	$-\mathbf e_{45}$	$-\mathbf e_{4125}$	$\mathbf e_{4315}$	$-\mathbf e_{3215}$	$-\mathbf e_{35}$	$\mathbf e_{25}$	$\mathbf e_{23}$
$\mathbf e_{4315}$	$\mathbf e_{4315}$	$-\mathbf e_{435}$	$\mathbb 1$	$\mathbf e_{415}$	$-\mathbf e_{431}$	$\mathbf e_{315}$	$-\mathbf e_{43}$	$\mathbf e_{1234}$	$\mathbf e_{41}$	$\mathbf e_{4125}$	$-\mathbf e_{45}$	$-\mathbf e_{4235}$	$\mathbf e_{35}$	$-\mathbf e_{3215}$	$-\mathbf e_{15}$	$\mathbf e_{31}$
$\mathbf e_{4125}$	$\mathbf e_{4125}$	$\mathbf e_{425}$	$-\mathbf e_{415}$	$\mathbb 1$	$-\mathbf e_{412}$	$\mathbf e_{125}$	$\mathbf e_{42}$	$-\mathbf e_{41}$	$\mathbf e_{1234}$	$-\mathbf e_{4315}$	$\mathbf e_{4235}$	$-\mathbf e_{45}$	$-\mathbf e_{25}$	$\mathbf e_{15}$	$-\mathbf e_{3215}$	$\mathbf e_{12}$
$\mathbf e_{3215}$	$\mathbf e_{3215}$	$\mathbf e_{235}$	$\mathbf e_{315}$	$\mathbf e_{125}$	$\mathbb 1 - \mathbf e_{321}$	0	$\mathbf e_{23} + \mathbf e_{4235}$	$\mathbf e_{31} + \mathbf e_{4315}$	$\mathbf e_{12} + \mathbf e_{4125}$	$\mathbf e_{15}$	$\mathbf e_{25}$	$\mathbf e_{35}$	0	0	0	$-\mathbf e_{3215}$
$\mathbb 1$	$\mathbb 1$	$\mathbf e_{4235}$	$\mathbf e_{4315}$	$\mathbf e_{4125}$	$-\mathbf e_{1234}$	$-\mathbf e_{3215}$	$\mathbf e_{423}$	$\mathbf e_{431}$	$\mathbf e_{412}$	$\mathbf e_{415}$	$\mathbf e_{425}$	$\mathbf e_{435}$	$\mathbf e_{235}$	$\mathbf e_{315}$	$\mathbf e_{125}$	$-\mathbf e_{321}$

Geometric Product **a ⟑ b**

3D Conformal Geometric Algebra

a \ b	e_{423}	e_{431}	e_{412}	e_{321}	e_{415}	e_{425}	e_{435}	e_{235}	e_{315}	e_{125}	e_{1234}	e_{4235}	e_{4315}	e_{4125}	e_{3215}	$\mathbb{1}$
1	e_{423}	e_{431}	e_{412}	e_{321}	e_{415}	e_{425}	e_{435}	e_{235}	e_{315}	e_{125}	e_{1234}	e_{4235}	e_{4315}	e_{4125}	e_{3215}	$\mathbb{1}$
e_1	e_{1234}	e_{43}	$-e_{42}$	$-e_{23}$	$-e_{45}$	$-e_{4125}$	e_{4315}	$-e_{3215}$	$-e_{35}$	e_{25}	e_{423}	$\mathbb{1}$	e_{435}	$-e_{425}$	$-e_{235}$	e_{4235}
e_2	$-e_{43}$	e_{1234}	e_{41}	$-e_{31}$	e_{4125}	$-e_{45}$	$-e_{4235}$	e_{35}	$-e_{3215}$	$-e_{15}$	e_{431}	$-e_{435}$	$\mathbb{1}$	e_{415}	$-e_{315}$	e_{4315}
e_3	e_{42}	$-e_{41}$	e_{1234}	$-e_{12}$	$-e_{4315}$	e_{4235}	$-e_{45}$	$-e_{25}$	e_{15}	$-e_{3215}$	e_{412}	e_{425}	$-e_{415}$	$\mathbb{1}$	$-e_{125}$	e_{4125}
e_4	0	0	0	e_{1234}	$-e_{41}$	$-e_{42}$	$-e_{43}$	$e_{4235} - e_{23}$	$e_{4315} - e_{31}$	$e_{4125} - e_{12}$	0	e_{423}	e_{431}	e_{412}	$e_{321} + \mathbb{1}$	$-e_{1234}$
e_5	$-e_{23} - e_{4235}$	$-e_{31} - e_{4315}$	$-e_{12} - e_{4125}$	$-e_{3215}$	$-e_{15}$	$-e_{25}$	$-e_{35}$	0	0	0	$\mathbb{1} - e_{321}$	$-e_{235}$	$-e_{315}$	$-e_{125}$	0	$-e_{3215}$
e_{41}	0	0	0	$-e_{423}$	$-e_4$	$-e_{412}$	e_{431}	$-e_{321} - \mathbb{1}$	$-e_3 - e_{435}$	$e_2 + e_{425}$	0	$-e_{1234}$	$-e_{43}$	e_{42}	$e_{23} - e_{4235}$	e_{423}
e_{42}	0	0	0	$-e_{431}$	e_{412}	$-e_4$	$-e_{423}$	$e_3 + e_{435}$	$-e_{321} - \mathbb{1}$	$-e_1 - e_{415}$	0	e_{43}	$-e_{1234}$	$-e_{41}$	$e_{31} - e_{4315}$	e_{431}
e_{43}	0	0	0	$-e_{412}$	$-e_{431}$	e_{423}	$-e_4$	$-e_2 - e_{425}$	$e_1 + e_{415}$	$-e_{321} - \mathbb{1}$	0	$-e_{42}$	e_{41}	$-e_{1234}$	$e_{12} - e_{4125}$	e_{412}
e_{23}	$-e_4$	$-e_{412}$	e_{431}	e_1	$-\mathbb{1}$	$-e_{435}$	e_{425}	$-e_5$	$-e_{125}$	e_{315}	e_{41}	$-e_{45}$	$-e_{4125}$	e_{4315}	e_{15}	e_{415}
e_{31}	e_{412}	$-e_4$	$-e_{423}$	e_2	e_{435}	$-\mathbb{1}$	$-e_{415}$	e_{125}	$-e_5$	$-e_{235}$	e_{42}	e_{4125}	$-e_{45}$	$-e_{4235}$	e_{25}	e_{425}
e_{12}	$-e_{431}$	e_{423}	$-e_4$	e_3	$-e_{425}$	e_{415}	$-\mathbb{1}$	$-e_{315}$	e_{235}	$-e_5$	e_{43}	$-e_{4315}$	e_{4235}	$-e_{45}$	e_{35}	e_{435}
e_{15}	$e_{321} - \mathbb{1}$	$e_3 - e_{435}$	$e_{425} - e_2$	e_{235}	$-e_5$	$-e_{125}$	e_{315}	0	0	0	$e_{23} + e_{4235}$	e_{3215}	e_{35}	$-e_{25}$	0	e_{235}
e_{25}	$e_{435} - e_3$	$e_{321} - \mathbb{1}$	$e_1 - e_{415}$	e_{315}	e_{125}	$-e_5$	$-e_{235}$	0	0	0	$e_{31} + e_{4315}$	$-e_{35}$	e_{3215}	e_{15}	0	e_{315}
e_{35}	$e_2 - e_{425}$	$e_{415} - e_1$	$e_{321} - \mathbb{1}$	e_{125}	$-e_{315}$	e_{235}	$-e_5$	0	0	0	$e_{12} + e_{4125}$	e_{25}	$-e_{15}$	e_{3215}	0	e_{125}
e_{45}	$-e_{423}$	$-e_{431}$	$-e_{412}$	$-\mathbb{1}$	$-e_1$	$-e_2$	$-e_3$	e_{235}	e_{315}	e_{125}	$-e_{1234}$	e_{23}	e_{31}	e_{12}	e_{3215}	$-e_{321}$
e_{423}	0	0	0	e_{41}	e_{1234}	e_{43}	$-e_{42}$	$1 - e_{45}$	$e_{12} - e_{4125}$	$e_{4315} - e_{31}$	0	$-e_4$	$-e_{412}$	e_{431}	$e_1 + e_{415}$	$-e_{41}$
e_{431}	0	0	0	e_{42}	$-e_{43}$	e_{1234}	e_{41}	$e_{4125} - e_{12}$	$1 - e_{45}$	$e_{23} - e_{4235}$	0	e_{412}	$-e_4$	$-e_{423}$	$e_2 + e_{425}$	$-e_{42}$
e_{412}	0	0	0	e_{43}	e_{42}	$-e_{41}$	e_{1234}	$e_{31} - e_{4315}$	$e_{4235} - e_{23}$	$1 - e_{45}$	0	$-e_{431}$	e_{423}	$-e_4$	$e_3 + e_{435}$	$-e_{43}$
e_{321}	$-e_{41}$	$-e_{42}$	$-e_{43}$	-1	e_{4235}	e_{4315}	e_{4125}	e_{15}	e_{25}	e_{35}	e_4	$-e_{415}$	$-e_{425}$	$-e_{435}$	$-e_5$	e_{45}
e_{415}	e_{1234}	e_{43}	$-e_{42}$	e_{4235}	1	e_{12}	$-e_{31}$	e_{3215}	e_{35}	$-e_{25}$	e_{423}	e_{321}	e_3	$-e_2$	e_{235}	$-e_{23}$
e_{425}	$-e_{43}$	e_{1234}	e_{41}	e_{4315}	$-e_{12}$	1	e_{23}	$-e_{35}$	e_{3215}	e_{15}	e_{431}	$-e_3$	e_{321}	e_1	e_{315}	$-e_{31}$
e_{435}	e_{42}	$-e_{41}$	e_{1234}	e_{4125}	e_{31}	$-e_{23}$	1	e_{25}	$-e_{15}$	e_{3215}	e_{412}	e_2	$-e_1$	e_{321}	e_{125}	$-e_{12}$
e_{235}	$1 + e_{45}$	$e_{12} + e_{4125}$	$-e_{31} - e_{4315}$	$-e_{15}$	e_{3215}	e_{35}	$-e_{25}$	0	0	0	$e_{415} - e_1$	e_5	e_{125}	$-e_{315}$	0	$-e_{15}$
e_{315}	$-e_{12} - e_{4125}$	$1 + e_{45}$	$e_{23} + e_{4235}$	$-e_{25}$	$-e_{35}$	e_{3215}	e_{15}	0	0	0	$e_{425} - e_2$	$-e_{125}$	e_5	e_{235}	0	$-e_{25}$
e_{125}	$e_{31} + e_{4315}$	$-e_{23} - e_{4235}$	$1 + e_{45}$	$-e_{35}$	e_{25}	$-e_{15}$	e_{3215}	0	0	0	$e_{435} - e_3$	e_{315}	$-e_{235}$	e_5	0	$-e_{35}$
e_{1234}	0	0	0	$-e_4$	e_{423}	e_{431}	e_{412}	$e_1 + e_{415}$	$e_2 + e_{425}$	$e_3 + e_{435}$	0	e_{41}	e_{42}	e_{43}	$1 - e_{45}$	e_4
e_{4235}	e_4	e_{412}	$-e_{431}$	$-e_{415}$	e_{321}	e_3	$-e_2$	$-e_5$	$-e_{125}$	e_{315}	$-e_{41}$	-1	$-e_{12}$	e_{31}	e_{15}	$-e_1$
e_{4315}	$-e_{412}$	e_4	e_{423}	$-e_{425}$	$-e_3$	e_{321}	e_1	e_{125}	$-e_5$	$-e_{235}$	$-e_{42}$	e_{12}	-1	$-e_{23}$	e_{25}	$-e_2$
e_{4125}	e_{431}	$-e_{423}$	e_4	$-e_{435}$	e_2	$-e_1$	e_{321}	$-e_{315}$	e_{235}	$-e_5$	$-e_{43}$	$-e_{31}$	e_{23}	-1	e_{35}	$-e_3$
e_{3215}	$e_{415} - e_1$	$e_{425} - e_2$	$e_{435} - e_3$	e_5	e_{235}	e_{315}	e_{125}	0	0	0	$1 + e_{45}$	$-e_{15}$	$-e_{25}$	$-e_{35}$	0	e_5
$\mathbb{1}$	$-e_{41}$	$-e_{42}$	$-e_{43}$	e_{45}	$-e_{23}$	$-e_{31}$	$-e_{12}$	$-e_{15}$	$-e_{25}$	$-e_{35}$	e_4	$-e_1$	$-e_2$	$-e_3$	e_5	-1

Geometric Antiproduct **a** ⟇ **b**

3D Conformal Geometric Algebra

a \ **b**	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{5}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$
$\mathbf{1}$	$-\mathbb{1}$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{4315}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{415}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{435}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{321}$
$\mathbf{e}_{1}$	$-\mathbf{e}_{4235}$	$-\mathbb{1}$	$\mathbf{e}_{435}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{235}$	$\mathbf{e}_{1234}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{45}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{4315}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{23}$
$\mathbf{e}_{2}$	$-\mathbf{e}_{4315}$	$-\mathbf{e}_{435}$	$-\mathbb{1}$	$\mathbf{e}_{415}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{315}$	$\mathbf{e}_{43}$	$\mathbf{e}_{1234}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{45}$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	$-\mathbf{e}_{31}$
$\mathbf{e}_{3}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{425}$	$-\mathbf{e}_{415}$	$-\mathbb{1}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{1234}$	$-\mathbf{e}_{4315}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{45}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	$-\mathbf{e}_{12}$
$\mathbf{e}_{4}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	$\mathbb{1}-\mathbf{e}_{321}$	0	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}+\mathbf{e}_{4235}$	$\mathbf{e}_{31}+\mathbf{e}_{4315}$	$\mathbf{e}_{12}+\mathbf{e}_{4125}$	$-\mathbf{e}_{1234}$
$\mathbf{e}_{5}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{321}+\mathbb{1}$	0	$\mathbf{e}_{23}-\mathbf{e}_{4235}$	$\mathbf{e}_{31}-\mathbf{e}_{4315}$	$\mathbf{e}_{12}-\mathbf{e}_{4125}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	0	0	0	$\mathbf{e}_{3215}$
$\mathbf{e}_{41}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	0	$\mathbf{e}_{23}+\mathbf{e}_{4235}$	0	0	0	$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbb{1}-\mathbf{e}_{321}$	$\mathbf{e}_{3}-\mathbf{e}_{435}$	$\mathbf{e}_{425}-\mathbf{e}_{2}$	$\mathbf{e}_{423}$
$\mathbf{e}_{42}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{31}+\mathbf{e}_{4315}$	0	0	0	$\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{435}-\mathbf{e}_{3}$	$\mathbb{1}-\mathbf{e}_{321}$	$\mathbf{e}_{1}-\mathbf{e}_{415}$	$\mathbf{e}_{431}$
$\mathbf{e}_{43}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	0	$\mathbf{e}_{12}+\mathbf{e}_{4125}$	0	0	0	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbf{e}_{4}$	$\mathbf{e}_{2}-\mathbf{e}_{425}$	$\mathbf{e}_{415}-\mathbf{e}_{1}$	$\mathbb{1}-\mathbf{e}_{321}$	$\mathbf{e}_{412}$
$\mathbf{e}_{23}$	$-\mathbf{e}_{415}$	$\mathbf{e}_{45}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{41}$	$\mathbf{e}_{15}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbb{1}$	$-\mathbf{e}_{435}$	$\mathbf{e}_{425}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{315}$	$\mathbf{e}_{1}$
$\mathbf{e}_{31}$	$-\mathbf{e}_{425}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{45}$	$-\mathbf{e}_{4235}$	$\mathbf{e}_{42}$	$\mathbf{e}_{25}$	$\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{435}$	$\mathbb{1}$	$-\mathbf{e}_{415}$	$\mathbf{e}_{125}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{235}$	$\mathbf{e}_{2}$
$\mathbf{e}_{12}$	$-\mathbf{e}_{435}$	$-\mathbf{e}_{4315}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{45}$	$\mathbf{e}_{43}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{425}$	$\mathbf{e}_{415}$	$\mathbb{1}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{235}$	$\mathbf{e}_{5}$	$\mathbf{e}_{3}$
$\mathbf{e}_{15}$	$-\mathbf{e}_{235}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{35}$	$\mathbf{e}_{25}$	$\mathbf{e}_{23}-\mathbf{e}_{4235}$	0	$\mathbf{e}_{321}+\mathbb{1}$	$-\mathbf{e}_{3}-\mathbf{e}_{435}$	$\mathbf{e}_{2}+\mathbf{e}_{425}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{315}$	0	0	0	$-\mathbf{e}_{235}$
$\mathbf{e}_{25}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{35}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{15}$	$\mathbf{e}_{31}-\mathbf{e}_{4315}$	0	$\mathbf{e}_{3}+\mathbf{e}_{435}$	$\mathbf{e}_{321}+\mathbb{1}$	$-\mathbf{e}_{1}-\mathbf{e}_{415}$	$\mathbf{e}_{125}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{235}$	0	0	0	$-\mathbf{e}_{315}$
$\mathbf{e}_{35}$	$-\mathbf{e}_{125}$	$-\mathbf{e}_{25}$	$\mathbf{e}_{15}$	$\mathbf{e}_{3215}$	$\mathbf{e}_{12}-\mathbf{e}_{4125}$	0	$-\mathbf{e}_{2}-\mathbf{e}_{425}$	$\mathbf{e}_{1}+\mathbf{e}_{415}$	$\mathbf{e}_{321}+\mathbb{1}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{235}$	$\mathbf{e}_{5}$	0	0	0	$-\mathbf{e}_{125}$
$\mathbf{e}_{45}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{1234}$	$-\mathbf{e}_{3215}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$-\mathbb{1}$
$\mathbf{e}_{423}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	0	$\mathbf{e}_{415}-\mathbf{e}_{1}$	0	0	0	$-\mathbf{e}_{1234}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1}-\mathbf{e}_{45}$	$\mathbf{e}_{12}+\mathbf{e}_{4125}$	$-\mathbf{e}_{31}-\mathbf{e}_{4315}$	$-\mathbf{e}_{41}$
$\mathbf{e}_{431}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	0	$\mathbf{e}_{425}-\mathbf{e}_{2}$	0	0	0	$-\mathbf{e}_{43}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}-\mathbf{e}_{4125}$	$-\mathbf{1}-\mathbf{e}_{45}$	$\mathbf{e}_{23}+\mathbf{e}_{4235}$	$-\mathbf{e}_{42}$
$\mathbf{e}_{412}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	0	$\mathbf{e}_{435}-\mathbf{e}_{3}$	0	0	0	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{31}+\mathbf{e}_{4315}$	$-\mathbf{e}_{23}-\mathbf{e}_{4235}$	$-\mathbf{1}-\mathbf{e}_{45}$	$-\mathbf{e}_{43}$
$\mathbf{e}_{321}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{4315}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$-\mathbf{1}$
$\mathbf{e}_{415}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{423}$	$\mathbf{e}_{235}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	$\mathbf{e}_{4235}$
$\mathbf{e}_{425}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	$\mathbf{e}_{431}$	$\mathbf{e}_{315}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	$\mathbf{e}_{4315}$
$\mathbf{e}_{435}$	$\mathbf{e}_{12}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{125}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{4125}$
$\mathbf{e}_{235}$	$\mathbf{e}_{15}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{315}$	$\mathbf{e}_{1}+\mathbf{e}_{415}$	0	$\mathbf{e}_{45}-\mathbf{1}$	$\mathbf{e}_{12}-\mathbf{e}_{4125}$	$\mathbf{e}_{4315}-\mathbf{e}_{31}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	0	0	0	$\mathbf{e}_{15}$
$\mathbf{e}_{315}$	$\mathbf{e}_{25}$	$\mathbf{e}_{125}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{235}$	$\mathbf{e}_{2}+\mathbf{e}_{425}$	0	$\mathbf{e}_{4125}-\mathbf{e}_{12}$	$\mathbf{e}_{45}-\mathbf{1}$	$\mathbf{e}_{23}-\mathbf{e}_{4235}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	0	0	0	$\mathbf{e}_{25}$
$\mathbf{e}_{125}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{235}$	$\mathbf{e}_{5}$	$\mathbf{e}_{3}+\mathbf{e}_{435}$	0	$\mathbf{e}_{31}-\mathbf{e}_{4315}$	$\mathbf{e}_{4235}-\mathbf{e}_{23}$	$\mathbf{e}_{45}-\mathbf{1}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	0	0	0	$\mathbf{e}_{35}$
$\mathbf{e}_{1234}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	$\mathbf{1}+\mathbf{e}_{45}$	0	0	0	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{1}-\mathbf{e}_{415}$	$\mathbf{e}_{2}-\mathbf{e}_{425}$	$\mathbf{e}_{3}-\mathbf{e}_{435}$	$\mathbf{e}_{4}$
$\mathbf{e}_{4235}$	$\mathbf{e}_{1}$	$\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{15}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{415}$
$\mathbf{e}_{4315}$	$\mathbf{e}_{2}$	$\mathbf{e}_{12}$	$\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{25}$	$\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{125}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{235}$	$-\mathbf{e}_{425}$
$\mathbf{e}_{4125}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbf{e}_{4}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{315}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{5}$	$-\mathbf{e}_{435}$
$\mathbf{e}_{3215}$	$-\mathbf{e}_{5}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$\mathbf{1}-\mathbf{e}_{45}$	0	$-\mathbf{e}_{1}-\mathbf{e}_{415}$	$-\mathbf{e}_{2}-\mathbf{e}_{425}$	$-\mathbf{e}_{3}-\mathbf{e}_{435}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	0	0	0	$-\mathbf{e}_{5}$
$\mathbb{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{5}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$\mathbf{e}_{45}$

Geometric Antiproduct $\mathbf{a} \mathbin{\vee\!\!\!\!\cdot} \mathbf{b}$

3D Conformal Geometric Algebra

a \ b	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{5}$	$\mathbf{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{415}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{41}$	$\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{15}$	$\mathbf{e}_{1}$
$\mathbf{e}_{2}$	$\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{425}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{125}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{235}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{12}$	$\mathbf{1}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{25}$	$\mathbf{e}_{2}$
$\mathbf{e}_{3}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbf{e}_{4}$	$\mathbf{e}_{435}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{315}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{5}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{1}$	$\mathbf{e}_{35}$	$\mathbf{e}_{3}$
$\mathbf{e}_{4}$	0	0	0	$\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{415} - \mathbf{e}_{1}$	$\mathbf{e}_{425} - \mathbf{e}_{2}$	$\mathbf{e}_{435} - \mathbf{e}_{3}$	0	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$\mathbf{1} + \mathbf{e}_{45}$	$\mathbf{e}_{4}$
$\mathbf{e}_{5}$	$\mathbf{e}_{1} + \mathbf{e}_{415}$	$\mathbf{e}_{2} + \mathbf{e}_{425}$	$\mathbf{e}_{3} + \mathbf{e}_{435}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	0	0	0	$\mathbf{1} - \mathbf{e}_{45}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	0	$\mathbf{e}_{5}$
$\mathbf{e}_{41}$	0	0	0	$\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{1} - \mathbf{e}_{45}$	$\mathbf{e}_{12} + \mathbf{e}_{4125}$	$-\mathbf{e}_{31} - \mathbf{e}_{4315}$	0	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{1} - \mathbf{e}_{415}$	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	0	0	0	$\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12} - \mathbf{e}_{4125}$	$-\mathbf{1} - \mathbf{e}_{45}$	$\mathbf{e}_{23} + \mathbf{e}_{4235}$	0	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{2} - \mathbf{e}_{425}$	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	0	0	0	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{31} + \mathbf{e}_{4315}$	$-\mathbf{e}_{23} - \mathbf{e}_{4235}$	$-\mathbf{1} - \mathbf{e}_{45}$	0	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3} - \mathbf{e}_{435}$	$\mathbf{e}_{43}$
$\mathbf{e}_{23}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{4235}$	$-\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{235}$	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{4315}$	$-\mathbf{e}_{12}$	$-\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	$-\mathbf{e}_{4125}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{1}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{12}$
$\mathbf{e}_{15}$	$\mathbf{e}_{45} - \mathbf{1}$	$\mathbf{e}_{12} - \mathbf{e}_{4125}$	$\mathbf{e}_{4315} - \mathbf{e}_{31}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	0	0	0	$-\mathbf{e}_{1} - \mathbf{e}_{415}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{315}$	0	$\mathbf{e}_{15}$
$\mathbf{e}_{25}$	$\mathbf{e}_{4125} - \mathbf{e}_{12}$	$\mathbf{e}_{45} - \mathbf{1}$	$\mathbf{e}_{23} - \mathbf{e}_{4235}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	0	0	0	$-\mathbf{e}_{2} - \mathbf{e}_{425}$	$\mathbf{e}_{125}$	$\mathbf{e}_{5}$	$-\mathbf{e}_{235}$	0	$\mathbf{e}_{25}$
$\mathbf{e}_{35}$	$\mathbf{e}_{31} - \mathbf{e}_{4315}$	$\mathbf{e}_{4235} - \mathbf{e}_{23}$	$\mathbf{e}_{45} - \mathbf{1}$	$-\mathbf{e}_{35}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	0	0	0	$-\mathbf{e}_{3} - \mathbf{e}_{435}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{235}$	$\mathbf{e}_{5}$	0	$\mathbf{e}_{35}$
$\mathbf{e}_{45}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$-\mathbf{1}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{25}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{415}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{435}$	$\mathbf{e}_{5}$	$\mathbf{e}_{45}$
$\mathbf{e}_{423}$	0	0	0	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321} - \mathbb{1}$	$\mathbf{e}_{435} - \mathbf{e}_{3}$	$\mathbf{e}_{2} - \mathbf{e}_{425}$	0	$\mathbf{e}_{1234}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{23} + \mathbf{e}_{4235}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	0	0	0	$\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{3} - \mathbf{e}_{435}$	$\mathbf{e}_{321} - \mathbb{1}$	$\mathbf{e}_{415} - \mathbf{e}_{1}$	0	$\mathbf{e}_{43}$	$\mathbf{e}_{1234}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{31} + \mathbf{e}_{4315}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	0	0	0	$\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{425} - \mathbf{e}_{2}$	$\mathbf{e}_{1} - \mathbf{e}_{415}$	$\mathbf{e}_{321} - \mathbb{1}$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{12} + \mathbf{e}_{4125}$	$\mathbf{e}_{412}$
$\mathbf{e}_{321}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$\mathbb{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{1234}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{3215}$	$\mathbf{e}_{321}$
$\mathbf{e}_{415}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{1}$	$-\mathbb{1}$	$\mathbf{e}_{435}$	$-\mathbf{e}_{425}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{315}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{4125}$	$-\mathbf{e}_{4315}$	$\mathbf{e}_{15}$	$\mathbf{e}_{415}$
$\mathbf{e}_{425}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{435}$	$-\mathbb{1}$	$\mathbf{e}_{415}$	$-\mathbf{e}_{125}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{235}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{4125}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{25}$	$\mathbf{e}_{425}$
$\mathbf{e}_{435}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{3}$	$\mathbf{e}_{425}$	$-\mathbf{e}_{415}$	$-\mathbb{1}$	$\mathbf{e}_{315}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{43}$	$\mathbf{e}_{4315}$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{35}$	$\mathbf{e}_{435}$
$\mathbf{e}_{235}$	$-\mathbf{e}_{321} - \mathbb{1}$	$\mathbf{e}_{3} + \mathbf{e}_{435}$	$-\mathbf{e}_{2} - \mathbf{e}_{425}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{125}$	$-\mathbf{e}_{315}$	0	0	0	$\mathbf{e}_{23} - \mathbf{e}_{4235}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{35}$	$-\mathbf{e}_{25}$	0	$\mathbf{e}_{235}$
$\mathbf{e}_{315}$	$-\mathbf{e}_{3} - \mathbf{e}_{435}$	$-\mathbf{e}_{321} - \mathbb{1}$	$\mathbf{e}_{1} + \mathbf{e}_{415}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	$-\mathbf{e}_{5}$	$\mathbf{e}_{235}$	0	0	0	$\mathbf{e}_{31} - \mathbf{e}_{4315}$	$-\mathbf{e}_{35}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	0	$\mathbf{e}_{315}$
$\mathbf{e}_{125}$	$\mathbf{e}_{2} + \mathbf{e}_{425}$	$-\mathbf{e}_{1} - \mathbf{e}_{415}$	$-\mathbf{e}_{321} - \mathbb{1}$	$-\mathbf{e}_{125}$	$\mathbf{e}_{315}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{5}$	0	0	0	$\mathbf{e}_{12} - \mathbf{e}_{4125}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{3215}$	0	$\mathbf{e}_{125}$
$\mathbf{e}_{1234}$	0	0	0	$\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{23} + \mathbf{e}_{4235}$	$\mathbf{e}_{31} + \mathbf{e}_{4315}$	$\mathbf{e}_{12} + \mathbf{e}_{4125}$	0	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321} - \mathbb{1}$	$\mathbf{e}_{1234}$
$\mathbf{e}_{4235}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{4125}$	$-\mathbf{e}_{4315}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{35}$	$\mathbf{e}_{25}$	$-\mathbf{e}_{423}$	$\mathbb{1}$	$-\mathbf{e}_{435}$	$\mathbf{e}_{425}$	$\mathbf{e}_{235}$	$\mathbf{e}_{4235}$
$\mathbf{e}_{4315}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{1234}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{4125}$	$-\mathbf{e}_{45}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{35}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{435}$	$\mathbb{1}$	$-\mathbf{e}_{415}$	$\mathbf{e}_{315}$	$\mathbf{e}_{4315}$
$\mathbf{e}_{4125}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{1234}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{4315}$	$-\mathbf{e}_{4235}$	$-\mathbf{e}_{45}$	$-\mathbf{e}_{25}$	$\mathbf{e}_{15}$	$\mathbf{e}_{3215}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{425}$	$\mathbf{e}_{415}$	$\mathbb{1}$	$\mathbf{e}_{125}$	$\mathbf{e}_{4125}$
$\mathbf{e}_{3215}$	$\mathbf{e}_{23} - \mathbf{e}_{4235}$	$\mathbf{e}_{31} - \mathbf{e}_{4315}$	$\mathbf{e}_{12} - \mathbf{e}_{4125}$	$-\mathbf{e}_{3215}$	$\mathbf{e}_{15}$	$\mathbf{e}_{25}$	$\mathbf{e}_{35}$	0	0	0	$-\mathbf{e}_{321} - \mathbb{1}$	$-\mathbf{e}_{235}$	$-\mathbf{e}_{315}$	$-\mathbf{e}_{125}$	0	$\mathbf{e}_{3215}$
$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$\mathbf{e}_{415}$	$\mathbf{e}_{425}$	$\mathbf{e}_{435}$	$\mathbf{e}_{235}$	$\mathbf{e}_{315}$	$\mathbf{e}_{125}$	$\mathbf{e}_{1234}$	$\mathbf{e}_{4235}$	$\mathbf{e}_{4315}$	$\mathbf{e}_{4125}$	$\mathbf{e}_{3215}$	$\mathbb{1}$

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

2D Conformal Exterior Algebra

a \ b	1	e_1	e_2	e_3	e_4	e_{23}	e_{31}	e_{12}	e_{41}	e_{42}	e_{43}	e_{321}	e_{423}	e_{431}	e_{412}	𝟙
1	1	e_1	e_2	e_3	e_4	e_{23}	e_{31}	e_{12}	e_{41}	e_{42}	e_{43}	e_{321}	e_{423}	e_{431}	e_{412}	𝟙
e_1	e_1	0	e_{12}	$-e_{31}$	$-e_{41}$	$-e_{321}$	0	0	0	$-e_{412}$	e_{431}	0	𝟙	0	0	0
e_2	e_2	$-e_{12}$	0	e_{23}	$-e_{42}$	0	$-e_{321}$	0	e_{412}	0	$-e_{423}$	0	0	𝟙	0	0
e_3	e_3	e_{31}	$-e_{23}$	0	$-e_{43}$	0	0	$-e_{321}$	$-e_{431}$	e_{423}	0	0	0	0	𝟙	0
e_4	e_4	e_{41}	e_{42}	e_{43}	0	e_{423}	e_{431}	e_{412}	0	0	0	𝟙	0	0	0	0
e_{23}	e_{23}	$-e_{321}$	0	0	e_{423}	0	0	0	$-$𝟙	0	0	0	0	0	0	0
e_{31}	e_{31}	0	$-e_{321}$	0	e_{431}	0	0	0	0	$-$𝟙	0	0	0	0	0	0
e_{12}	e_{12}	0	0	$-e_{321}$	e_{412}	0	0	0	0	0	$-$𝟙	0	0	0	0	0
e_{41}	e_{41}	0	e_{412}	$-e_{431}$	0	$-$𝟙	0	0	0	0	0	0	0	0	0	0
e_{42}	e_{42}	$-e_{412}$	0	e_{423}	0	0	$-$𝟙	0	0	0	0	0	0	0	0	0
e_{43}	e_{43}	e_{431}	$-e_{423}$	0	0	0	0	$-$𝟙	0	0	0	0	0	0	0	0
e_{321}	e_{321}	0	0	0	$-$𝟙	0	0	0	0	0	0	0	0	0	0	0
e_{423}	e_{423}	$-$𝟙	0	0	0	0	0	0	0	0	0	0	0	0	0	0
e_{431}	e_{431}	0	$-$𝟙	0	0	0	0	0	0	0	0	0	0	0	0	0
e_{412}	e_{412}	0	0	$-$𝟙	0	0	0	0	0	0	0	0	0	0	0	0
𝟙	𝟙	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

2D Conformal Exterior Algebra

a \ b	1	e_1	e_2	e_3	e_4	e_{23}	e_{31}	e_{12}	e_{41}	e_{42}	e_{43}	e_{321}	e_{423}	e_{431}	e_{412}	𝟙
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
e_1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	e_1
e_2	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	e_2
e_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	e_3
e_4	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	e_4
e_{23}	0	0	0	0	0	0	0	0	-1	0	0	0	0	e_3	$-e_2$	e_{23}
e_{31}	0	0	0	0	0	0	0	0	0	-1	0	0	$-e_3$	0	e_1	e_{31}
e_{12}	0	0	0	0	0	0	0	0	0	0	-1	0	e_2	$-e_1$	0	e_{12}
e_{41}	0	0	0	0	0	-1	0	0	0	0	0	e_1	$-e_4$	0	0	e_{41}
e_{42}	0	0	0	0	0	0	-1	0	0	0	0	e_2	0	$-e_4$	0	e_{42}
e_{43}	0	0	0	0	0	0	0	-1	0	0	0	e_3	0	0	$-e_4$	e_{43}
e_{321}	0	0	0	0	-1	0	0	0	e_1	e_2	e_3	0	$-e_{23}$	$-e_{31}$	$-e_{12}$	e_{321}
e_{423}	0	-1	0	0	0	0	$-e_3$	e_2	$-e_4$	0	0	e_{23}	0	$-e_{43}$	e_{42}	e_{423}
e_{431}	0	0	-1	0	0	e_3	0	$-e_1$	0	$-e_4$	0	e_{31}	e_{43}	0	$-e_{41}$	e_{431}
e_{412}	0	0	0	-1	0	$-e_2$	e_1	0	0	0	$-e_4$	e_{12}	$-e_{42}$	e_{41}	0	e_{412}
𝟙	1	e_1	e_2	e_3	e_4	e_{23}	e_{31}	e_{12}	e_{41}	e_{42}	e_{43}	e_{321}	e_{423}	e_{431}	e_{412}	𝟙

Geometric Product **a** ⟑ **b**

2D Conformal Geometric Algebra

a \ b	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{321}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbb{1}$
1	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{321}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{23}$	$\mathbb{1}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{423}$
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{41}$	$\mathbf{e}_{431}$
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	0	$-1 - \mathbf{e}_{43}$	0	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1} - \mathbf{e}_{431}$	$\mathbf{e}_{423} - \mathbf{e}_{2}$	$-\mathbf{e}_{3}$	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$\mathbb{1} - \mathbf{e}_{12}$	$-\mathbf{e}_{321}$
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43} - 1$	0	$\mathbf{e}_{2} + \mathbf{e}_{423}$	$\mathbf{e}_{431} - \mathbf{e}_{1}$	$\mathbf{e}_{412}$	0	0	$\mathbf{e}_{4}$	$\mathbf{e}_{12} + \mathbb{1}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$-\mathbf{e}_{412}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	0	$\mathbf{e}_{423} - \mathbf{e}_{2}$	0	0	$\mathbf{e}_{31}$	$\mathbf{e}_{12} - \mathbb{1}$	$-1 - \mathbf{e}_{43}$	$-\mathbf{e}_{23}$	0	$-\mathbf{e}_{3}$	$\mathbf{e}_{321}$	$\mathbf{e}_{1} + \mathbf{e}_{431}$	$\mathbf{e}_{31}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	0	$\mathbf{e}_{1} + \mathbf{e}_{431}$	0	0	$-\mathbf{e}_{23}$	$1 + \mathbf{e}_{43}$	$\mathbf{e}_{12} - \mathbb{1}$	$-\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2} - \mathbf{e}_{423}$	$-\mathbf{e}_{23}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{43}$
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$\mathbf{e}_{1} - \mathbf{e}_{431}$	0	$-\mathbf{e}_{12} - \mathbb{1}$	$1 - \mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	$\mathbf{e}_{41}$	$-\mathbf{e}_{2} - \mathbf{e}_{423}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{4}$	0	$-\mathbf{e}_{42}$
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$\mathbf{e}_{2} + \mathbf{e}_{423}$	0	$\mathbf{e}_{43} - 1$	$-\mathbf{e}_{12} - \mathbb{1}$	$-\mathbf{e}_{41}$	0	0	$\mathbf{e}_{42}$	$\mathbf{e}_{1} - \mathbf{e}_{431}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	0	$\mathbf{e}_{41}$
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	1	$\mathbf{e}_{321}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{12}$
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	0	$\mathbf{e}_{12} - \mathbb{1}$	0	0	$\mathbf{e}_{3}$	$\mathbf{e}_{423} - \mathbf{e}_{2}$	$\mathbf{e}_{1} + \mathbf{e}_{431}$	$-\mathbf{e}_{321}$	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-1 - \mathbf{e}_{43}$	$\mathbf{e}_{3}$
$\mathbf{e}_{423}$	$\mathbf{e}_{423}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{321}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{31}$	1	$\mathbf{e}_{12}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{1}$
$\mathbf{e}_{431}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{423}$	$\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{42}$	$-\mathbf{e}_{2}$
$\mathbf{e}_{412}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12} - \mathbb{1}$	0	$\mathbf{e}_{1} - \mathbf{e}_{431}$	$\mathbf{e}_{2} + \mathbf{e}_{423}$	$-\mathbf{e}_{4}$	0	0	$\mathbf{e}_{412}$	$\mathbf{e}_{43} - 1$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	0	$\mathbf{e}_{4}$
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{4}$	-1

Geometric Antiproduct **a** ⟇ **b**

2D Conformal Geometric Algebra

a \ b	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{321}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbb{1}$
1	$-\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{4}$	1
$\mathbf{e}_{1}$	$\mathbf{e}_{423}$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{31}$	1	$-\mathbf{e}_{12}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{1}$
$\mathbf{e}_{2}$	$\mathbf{e}_{431}$	$\mathbf{e}_{43}$	$\mathbb{1}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{423}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{1}$	$\mathbf{e}_{23}$	$\mathbf{e}_{12}$	1	$-\mathbf{e}_{42}$	$\mathbf{e}_{2}$
$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	0	$-\mathbf{e}_{12} - \mathbb{1}$	0	0	$-\mathbf{e}_{3}$	$\mathbf{e}_{2} + \mathbf{e}_{423}$	$\mathbf{e}_{431} - \mathbf{e}_{1}$	$-\mathbf{e}_{321}$	0	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$1 - \mathbf{e}_{43}$	$\mathbf{e}_{3}$
$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{12} - \mathbb{1}$	0	$-\mathbf{e}_{1} - \mathbf{e}_{431}$	$\mathbf{e}_{423} - \mathbf{e}_{2}$	$\mathbf{e}_{4}$	0	0	$\mathbf{e}_{412}$	$1 + \mathbf{e}_{43}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	0	$\mathbf{e}_{4}$
$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{3}$	$\mathbf{e}_{321}$	0	$\mathbf{e}_{1} - \mathbf{e}_{431}$	0	0	$-\mathbf{e}_{23}$	$\mathbf{e}_{43} - 1$	$\mathbf{e}_{12} + \mathbb{1}$	$-\mathbf{e}_{31}$	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{2} - \mathbf{e}_{423}$	$\mathbf{e}_{23}$
$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{3}$	0	$\mathbf{e}_{2} + \mathbf{e}_{423}$	0	0	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12} - \mathbb{1}$	$\mathbf{e}_{43} - 1$	$\mathbf{e}_{23}$	0	$-\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{1} - \mathbf{e}_{431}$	$\mathbf{e}_{31}$
$\mathbf{e}_{12}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	-1	$\mathbf{e}_{321}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{12}$
$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423} - \mathbf{e}_{2}$	0	$-1 - \mathbf{e}_{43}$	$\mathbf{e}_{12} - \mathbb{1}$	$\mathbf{e}_{41}$	0	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{1} + \mathbf{e}_{431}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	0	$\mathbf{e}_{41}$
$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{412}$	$\mathbf{e}_{1} + \mathbf{e}_{431}$	0	$\mathbb{1} - \mathbf{e}_{12}$	$-1 - \mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	$\mathbf{e}_{41}$	$\mathbf{e}_{2} - \mathbf{e}_{423}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	0	$\mathbf{e}_{42}$
$\mathbf{e}_{43}$	$\mathbf{e}_{12}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{3}$	$\mathbf{e}_{431}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{43}$
$\mathbf{e}_{321}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	0	$\mathbf{e}_{43} - 1$	0	0	$-\mathbf{e}_{321}$	$\mathbf{e}_{1} - \mathbf{e}_{431}$	$\mathbf{e}_{2} + \mathbf{e}_{423}$	$\mathbf{e}_{3}$	0	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12} - \mathbb{1}$	$\mathbf{e}_{321}$
$\mathbf{e}_{423}$	$-\mathbf{e}_{1}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{412}$	$-\mathbf{e}_{431}$	$\mathbf{e}_{23}$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{423}$
$\mathbf{e}_{431}$	$-\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{3}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{423}$	$\mathbf{e}_{31}$	$\mathbf{e}_{43}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{431}$
$\mathbf{e}_{412}$	$\mathbf{e}_{4}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-1 - \mathbf{e}_{43}$	0	$\mathbf{e}_{423} - \mathbf{e}_{2}$	$\mathbf{e}_{1} + \mathbf{e}_{431}$	$\mathbf{e}_{412}$	0	0	$-\mathbf{e}_{4}$	$\mathbf{e}_{12} - \mathbb{1}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	0	$\mathbf{e}_{412}$
$\mathbb{1}$	1	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbf{e}_{321}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbb{1}$

Appendix B

Geometric Properties

This appendix contains data tables filled with the properties of the various types of flat and round geometries discussed throughout the book. Information about flat geometries in n dimensions pertains to their representations in the $(n+1)$-dimensional rigid algebra, and information about round geometries in n dimensions pertains to their representations in the $(n+2)$-dimensional conformal algebra.

Point (3D)

3D Rigid Algebras

$$\mathbf{p} = \underbrace{p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\,\mathbf{e}_4}_{\text{Weight}}$$

Degrees of Freedom	$\operatorname{DOF}(3,0) = 3$	Attitude	$\operatorname{att}(\mathbf{p}) = p_w\mathbf{1}$
Right Complement	$\overline{\mathbf{p}} = p_x\,\mathbf{e}_{423} + p_y\,\mathbf{e}_{431} + p_z\,\mathbf{e}_{412} + p_w\,\mathbf{e}_{321}$		
Bulk Dual	$\mathbf{p}^{\bigstar} = p_x\,\mathbf{e}_{423} + p_y\,\mathbf{e}_{431} + p_z\,\mathbf{e}_{412}$	Bulk Norm	$\lVert\mathbf{p}\rVert_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$
Weight Dual	$\mathbf{p}^{\star} = p_w\,\mathbf{e}_{321}$	Weight Norm	$\lVert\mathbf{p}\rVert_\circ = \lvert p_w\rvert\,\mathbb{1}$
Antisupport	$\operatorname{asp}(\mathbf{p}) = -p_x p_w\,\mathbf{e}_{423} - p_y p_w\,\mathbf{e}_{431} - p_z p_w\,\mathbf{e}_{412} + \left(p_x^2 + p_y^2 + p_z^2\right)\mathbf{e}_{321}$		

Line (3D)

3D Rigid Algebras

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

Degrees of Freedom	$\operatorname{DOF}(3,1) = 4$	Constraints	$\boldsymbol{l}_{\mathrm{v}} \cdot \boldsymbol{l}_{\mathrm{m}} = 0$
Attitude	$\operatorname{att}(\boldsymbol{l}) = l_{vx}\,\mathbf{e}_1 + l_{vy}\,\mathbf{e}_2 + l_{vz}\,\mathbf{e}_3$		
Right Complement	$\overline{\boldsymbol{l}} = -l_{mx}\,\mathbf{e}_{41} - l_{my}\,\mathbf{e}_{42} - l_{mz}\,\mathbf{e}_{43} - l_{vx}\,\mathbf{e}_{23} - l_{vy}\,\mathbf{e}_{31} - l_{vz}\,\mathbf{e}_{12}$		
Bulk Dual	$\boldsymbol{l}^{\bigstar} = -l_{mx}\,\mathbf{e}_{41} - l_{my}\,\mathbf{e}_{42} - l_{mz}\,\mathbf{e}_{43}$	Bulk Norm	$\lVert\boldsymbol{l}\rVert_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$
Weight Dual	$\boldsymbol{l}^{\star} = -l_{vx}\,\mathbf{e}_{23} - l_{vy}\,\mathbf{e}_{31} - l_{vz}\,\mathbf{e}_{12}$	Weight Norm	$\lVert\boldsymbol{l}\rVert_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$
Support	$\operatorname{sup}(\boldsymbol{l}) = (l_{vy}l_{mz} - l_{vz}l_{my})\,\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\,\mathbf{e}_2 + (l_{vx}l_{my} - l_{vy}l_{mx})\,\mathbf{e}_3 + \boldsymbol{l}_{\mathrm{v}}^2\,\mathbf{e}_4$		
Antisupport	$\operatorname{asp}(\boldsymbol{l}) = (l_{vz}l_{my} - l_{vy}l_{mz})\,\mathbf{e}_{423} + (l_{vx}l_{mz} - l_{vz}l_{mx})\,\mathbf{e}_{431} + (l_{vy}l_{mx} - l_{vx}l_{my})\,\mathbf{e}_{412} + \boldsymbol{l}_{\mathrm{m}}^2\,\mathbf{e}_{321}$		

Plane (3D)

3D Rigid Algebras

$$\mathbf{g} = \underbrace{g_x\,\mathbf{e}_{423} + g_y\,\mathbf{e}_{431} + g_z\,\mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w\,\mathbf{e}_{321}}_{\text{Position}}$$

Degrees of Freedom	$\operatorname{DOF}(3,2) = 3$	Attitude	$\operatorname{att}(\mathbf{g}) = g_x\,\mathbf{e}_{23} + g_y\,\mathbf{e}_{31} + g_z\,\mathbf{e}_{12}$
Right Complement	$\overline{\mathbf{g}} = -g_x\,\mathbf{e}_1 - g_y\,\mathbf{e}_2 - g_z\,\mathbf{e}_3 - g_w\,\mathbf{e}_4$		
Bulk Dual	$\mathbf{g}^{\bigstar} = -g_w\,\mathbf{e}_4$	Bulk Norm	$\lVert\mathbf{g}\rVert_\bullet = \lvert g_w\rvert\,\mathbf{1}$
Weight Dual	$\mathbf{g}^{\star} = -g_x\,\mathbf{e}_1 - g_y\,\mathbf{e}_2 - g_z\,\mathbf{e}_3$	Weight Norm	$\lVert\mathbf{g}\rVert_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$
Support	$\operatorname{sup}(\mathbf{g}) = -g_x g_w\,\mathbf{e}_1 - g_y g_w\,\mathbf{e}_2 - g_w g_z\,\mathbf{e}_3 + \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{e}_4$		

Round Point (3D)

3D Conformal Algebras

$$\mathbf{a} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + \mathbf{e}_4 + \frac{\mathbf{p}^2 + r^2}{2}\mathbf{e}_5$$

$\mathbf{p}$ = center position, r = radius

$$\mathbf{a} = \underbrace{a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3 + a_w\mathbf{e}_4}_{\text{Carrier Point}} + \underbrace{a_u\mathbf{e}_5}_{\text{Infinity}}$$

(when $a_x = a_y = a_z = a_w = 0$)

Center	$\operatorname{cen}(\mathbf{a}) = a_xa_w\mathbf{e}_1 + a_ya_w\mathbf{e}_2 + a_za_w\mathbf{e}_3 + a_w^2\mathbf{e}_4 + a_wa_u\mathbf{e}_5$		
Container	$\operatorname{con}(\mathbf{a}) = -a_w^2\mathbf{e}_{1234} + a_xa_w\mathbf{e}_{4235} + a_ya_w\mathbf{e}_{4315} + a_za_w\mathbf{e}_{4125} + \left(a_wa_u - a_x^2 - a_y^2 - a_z^2\right)\mathbf{e}_{3215}$		
Partner	$\operatorname{par}(\mathbf{a}) = a_xa_w^2\mathbf{e}_1 + a_ya_w^2\mathbf{e}_2 + a_za_w^2\mathbf{e}_3 + a_w^3\mathbf{e}_4 + \left(a_x^2 + a_y^2 + a_z^2 - a_wa_u\right)a_w\mathbf{e}_5$		
Carrier	$\operatorname{car}(\mathbf{a}) = a_x\mathbf{e}_{15} + a_y\mathbf{e}_{25} + a_z\mathbf{e}_{35} + a_w\mathbf{e}_{45}$ (flat point)		
Cocarrier	$\operatorname{ccr}(\mathbf{a}) = a_w\mathbb{1}$ (full space)		
Attitude	$\operatorname{att}(\mathbf{a}) = a_w\mathbf{1}$		
Dual	$\mathbf{a}^\star = -a_w\mathbf{e}_{1234} + a_x\mathbf{e}_{4235} + a_y\mathbf{e}_{4315} + a_z\mathbf{e}_{4125} - a_u\mathbf{e}_{3215}$		
Degrees of Freedom	$\operatorname{DOF}(4,0) = 4$		
Center Norm	$\lVert\mathbf{a}\rVert_\odot = \sqrt{a_x^2 + a_y^2 + a_z^2}$	Radius Norm	$\lVert\mathbf{a}\rVert_\circledcirc = \sqrt{2a_wa_u - a_x^2 - a_y^2 - a_z^2}$
Weight Norm	$\lVert\mathbf{a}\rVert_\circ = \lvert a_w\rvert$		

Sphere (3D)

3D Conformal Algebras

$$\mathbf{s} = p_x\mathbf{e}_{4235} + p_y\mathbf{e}_{4315} + p_z\mathbf{e}_{4125} - \mathbf{e}_{1234} - \frac{\mathbf{p}^2 - r^2}{2}\mathbf{e}_{3215}$$

$\mathbf{p}$ = center position, r = radius

$$\mathbf{s} = \underbrace{s_u\mathbf{e}_{1234}}_{\text{Carrier Space}} + \underbrace{s_x\mathbf{e}_{4235} + s_y\mathbf{e}_{4315} + s_z\mathbf{e}_{4125} + s_w\mathbf{e}_{3215}}_{\text{Flat Plane}}$$

(when $s_u = 0$)

Center	$\operatorname{cen}(\mathbf{s}) = -s_xs_u\mathbf{e}_1 - s_ys_u\mathbf{e}_2 - s_zs_u\mathbf{e}_3 + s_u^2\mathbf{e}_4 + \left(s_x^2 + s_y^2 + s_z^2 - s_ws_u\right)\mathbf{e}_5$		
Container	$\operatorname{con}(\mathbf{s}) = -s_u^2\mathbf{e}_{1234} - s_xs_u\mathbf{e}_{4235} - s_ys_u\mathbf{e}_{4315} - s_zs_u\mathbf{e}_{4125} - s_ws_u\mathbf{e}_{3215}$		
Partner	$\operatorname{par}(\mathbf{s}) = s_u^3\mathbf{e}_{1234} + s_xs_u^2\mathbf{e}_{4235} + s_ys_u^2\mathbf{e}_{4315} + s_zs_u^2\mathbf{e}_{4125} + \left(s_x^2 + s_y^2 + s_z^2 - s_ws_u\right)s_u\mathbf{e}_{3215}$		
Carrier	$\operatorname{car}(\mathbf{s}) = s_u\mathbb{1}$ (full space)		
Cocarrier	$\operatorname{ccr}(\mathbf{s}) = s_x\mathbf{e}_{15} + s_y\mathbf{e}_{25} + s_z\mathbf{e}_{35} - s_u\mathbf{e}_{45}$ (flat point)		
Attitude	$\operatorname{att}(\mathbf{s}) = s_u\mathbf{e}_{321} + s_x\mathbf{e}_{235} + s_y\mathbf{e}_{315} + s_z\mathbf{e}_{125}$		
Dual	$\mathbf{s}^\star = -s_x\mathbf{e}_1 - s_y\mathbf{e}_2 - s_z\mathbf{e}_3 + s_u\mathbf{e}_4 + s_w\mathbf{e}_5$		
Degrees of Freedom	$\operatorname{DOF}(4,3) = 4$		
Center Norm	$\lVert\mathbf{s}\rVert_\odot = \sqrt{s_x^2 + s_y^2 + s_z^2}$	Radius Norm	$\lVert\mathbf{s}\rVert_\circledcirc = \sqrt{s_x^2 + s_y^2 + s_z^2 - 2s_ws_u}$
Weight Norm	$\lVert\mathbf{s}\rVert_\circ = \lvert s_u\rvert$		

Dipole (3D)

3D Conformal Algebras

$$\mathbf{d} = n_x\mathbf{e}_{41} + n_y\mathbf{e}_{42} + n_z\mathbf{e}_{43} + (p_yn_z - p_zn_y)\mathbf{e}_{23} + (p_zn_x - p_xn_z)\mathbf{e}_{31} + (p_xn_y - p_yn_x)\mathbf{e}_{12}$$

$$+(\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + \mathbf{e}_{45}) - \frac{\mathbf{p}^2 + r^2}{2}(n_x\mathbf{e}_{15} + n_y\mathbf{e}_{25} + n_z\mathbf{e}_{35})$$

$\mathbf{p}$ = center position, $\mathbf{n}$ = line direction, r = radius

Cocarrier Normal

Cocarrier Position

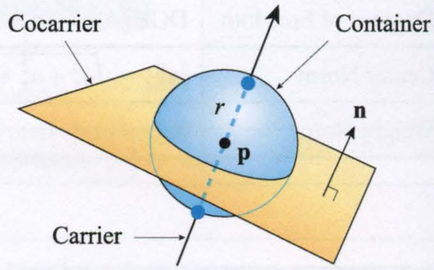
$$\mathbf{d} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12} + d_{px}\mathbf{e}_{15} + d_{py}\mathbf{e}_{25} + d_{pz}\mathbf{e}_{35} + d_{pw}\mathbf{e}_{45}$$

Carrier Line

Flat Point

(when $d_{vz} = d_{vy} = d_{vz} = d_{mx} = d_{my} = d_{mz} = 0$)

Center	$\begin{aligned}\operatorname{cen}(\mathbf{d}) = {} & (d_{vy}d_{mz} - d_{vz}d_{my} + d_{vx}d_{pw})\mathbf{e}_1 \\ & + (d_{vz}d_{mx} - d_{vx}d_{mz} + d_{vy}d_{pw})\mathbf{e}_2 \\ & + (d_{vx}d_{my} - d_{vy}d_{mx} + d_{vz}d_{pw})\mathbf{e}_3 \\ & + (d_{vx}^2 + d_{vy}^2 + d_{vz}^2)\mathbf{e}_4 \\ & + (d_{pw}^2 - d_{vx}d_{px} - d_{vy}d_{py} - d_{vz}d_{pz})\mathbf{e}_5\end{aligned}$
Container	$\begin{aligned}\operatorname{con}(\mathbf{d}) = {} & -(d_{vx}^2 + d_{vy}^2 + d_{vz}^2)\mathbf{e}_{1234} \\ & + (d_{vy}d_{mz} - d_{vz}d_{my} + d_{vx}d_{pw})\mathbf{e}_{4235} \\ & + (d_{vz}d_{mx} - d_{vx}d_{mz} + d_{vy}d_{pw})\mathbf{e}_{4315} \\ & + (d_{vx}d_{my} - d_{vy}d_{mx} + d_{vz}d_{pw})\mathbf{e}_{4125} \\ & - (d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{vx}d_{px} + d_{vy}d_{py} + d_{vz}d_{pz})\mathbf{e}_{3215}\end{aligned}$



Partner	$\begin{aligned}\operatorname{par}(\mathbf{d}) = {} & (d_{vx}^2 + d_{vy}^2 + d_{vz}^2)(d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12} + d_{pw}\mathbf{e}_{45}) \\ & + (d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - d_{vx}d_{px} - d_{vy}d_{py} - d_{vz}d_{pz})(d_{vx}\mathbf{e}_{15} + d_{vy}\mathbf{e}_{25} + d_{vz}\mathbf{e}_{35}) \\ & + (d_{mz}d_{vy} - d_{my}d_{vz})d_{pw}\mathbf{e}_{15} + (d_{mx}d_{vz} - d_{mz}d_{vx})d_{pw}\mathbf{e}_{25} + (d_{my}d_{vx} - d_{mx}d_{vy})d_{pw}\mathbf{e}_{35}\end{aligned}$
Carrier	$\operatorname{car}(\mathbf{d}) = d_{vx}\mathbf{e}_{415} + d_{vy}\mathbf{e}_{425} + d_{vz}\mathbf{e}_{435} + d_{mx}\mathbf{e}_{235} + d_{my}\mathbf{e}_{315} + d_{mz}\mathbf{e}_{125}$ (flat line)
Cocarrier	$\operatorname{ccr}(\mathbf{d}) = d_{vx}\mathbf{e}_{4235} + d_{vy}\mathbf{e}_{4315} + d_{vz}\mathbf{e}_{4125} - d_{pw}\mathbf{e}_{3215}$ (flat plane)
Attitude	$\operatorname{att}(\mathbf{d}) = d_{vx}\mathbf{e}_1 + d_{vy}\mathbf{e}_2 + d_{vz}\mathbf{e}_3 + d_{pw}\mathbf{e}_5$
Dual	$\mathbf{d}^\star = -d_{vx}\mathbf{e}_{423} - d_{vy}\mathbf{e}_{431} - d_{vz}\mathbf{e}_{412} + d_{pw}\mathbf{e}_{321} - d_{mx}\mathbf{e}_{415} - d_{my}\mathbf{e}_{425} - d_{mz}\mathbf{e}_{435} - d_{px}\mathbf{e}_{235} - d_{py}\mathbf{e}_{315} - d_{pz}\mathbf{e}_{125}$
Constraints	$\mathbf{d}_{pxyz} \times \mathbf{d}_\mathbf{v} - d_{pw}\mathbf{d}_\mathbf{m} = 0,\ \ \mathbf{d}_\mathbf{v} \cdot \mathbf{d}_\mathbf{m} = 0,\ \ \mathbf{d}_{pxyz} \cdot \mathbf{d}_\mathbf{m} = 0$
DOF	$\operatorname{DOF}(4,1) = 6$
Center Norm	$\lVert\mathbf{d}\rVert_\odot = \sqrt{d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{pw}^2}$
Radius Norm	$\lVert\mathbf{d}\rVert_\oslash = \sqrt{d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - 2(d_{px}d_{vx} + d_{py}d_{vy} + d_{pz}d_{vz})}$
Weight Norm	$\lVert\mathbf{d}\rVert_\circ = \sqrt{d_{vx}^2 + d_{vy}^2 + d_{vz}^2}$

Circle (3D)

3D Conformal Algebras

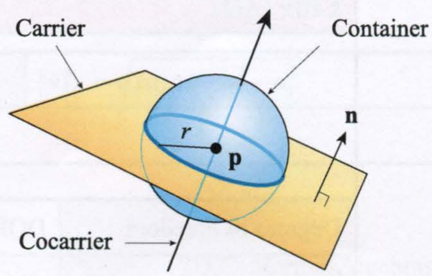
$$\mathbf{c} = n_x\mathbf{e}_{423} + n_y\mathbf{e}_{431} + n_z\mathbf{e}_{412} + (p_y n_z - p_z n_y)\mathbf{e}_{415} + (p_z n_x - p_x n_z)\mathbf{e}_{425} + (p_x n_y - p_y n_x)\mathbf{e}_{435}$$
$$+ (\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{235} + p_y\mathbf{e}_{315} + p_z\mathbf{e}_{125} - \mathbf{e}_{321}) - \frac{\mathbf{p}^2 - r^2}{2}(n_x\mathbf{e}_{235} + n_y\mathbf{e}_{315} + n_z\mathbf{e}_{125})$$

$\mathbf{p}$ = center position, $\mathbf{n}$ = plane normal, r = radius

$$\mathbf{c} = \underbrace{\overbrace{c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412}}^{\text{Cocarrier Direction}} + c_{gw}\mathbf{e}_{321}}_{\text{Carrier Plane}} + \underbrace{\overbrace{c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435}}^{\text{Cocarrier Moment}} + c_{mx}\mathbf{e}_{235} + c_{my}\mathbf{e}_{315} + c_{mz}\mathbf{e}_{125}}_{\text{Flat Line}}$$

Flat Line (when $c_{gx} = c_{gy} = c_{gz} = c_{gw} = 0$)

Center	$\operatorname{cen}(\mathbf{c}) = (c_{gy}c_{vz} - c_{gz}c_{vy} - c_{gx}c_{gw})\mathbf{e}_1 + (c_{gz}c_{vx} - c_{gx}c_{vz} - c_{gy}c_{gw})\mathbf{e}_2 + (c_{gx}c_{vy} - c_{gy}c_{vx} - c_{gz}c_{gw})\mathbf{e}_3 + (c_{gx}^2 + c_{gy}^2 + c_{gz}^2)\mathbf{e}_4 + (c_{vx}^2 + c_{vy}^2 + c_{vz}^2 + c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz})\mathbf{e}_5$
Container	$\operatorname{con}(\mathbf{c}) = -(c_{gx}^2 + c_{gy}^2 + c_{gz}^2)\mathbf{e}_{1234} + (c_{gy}c_{vz} - c_{gz}c_{vy} - c_{gx}c_{gw})\mathbf{e}_{4235} + (c_{gz}c_{vx} - c_{gx}c_{vz} - c_{gy}c_{gw})\mathbf{e}_{4315} + (c_{gx}c_{vy} - c_{gy}c_{vx} - c_{gz}c_{gw})\mathbf{e}_{4125} + (c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz} - c_{gw}^2)\mathbf{e}_{3215}$



Partner	$\operatorname{par}(\mathbf{c}) = (c_{gx}^2 + c_{gy}^2 + c_{gz}^2)(c_{gx}\mathbf{e}_{423} + c_{gy}\mathbf{e}_{431} + c_{gz}\mathbf{e}_{412} + c_{gw}\mathbf{e}_{321} + c_{vx}\mathbf{e}_{415} + c_{vy}\mathbf{e}_{425} + c_{vz}\mathbf{e}_{435}) + (c_{gw}^2 - c_{vx}^2 - c_{vy}^2 - c_{vz}^2 - c_{gx}c_{mx} - c_{gy}c_{my} - c_{gz}c_{mz})(c_{gx}\mathbf{e}_{235} + c_{gy}\mathbf{e}_{315} + c_{gz}\mathbf{e}_{125}) + (c_{vy}c_{gz} - c_{vz}c_{gy})c_{gw}\mathbf{e}_{235} + (c_{vz}c_{gx} - c_{vx}c_{gz})c_{gw}\mathbf{e}_{315} + (c_{vx}c_{gy} - c_{vy}c_{gx})c_{gw}\mathbf{e}_{125}$
Carrier	$\operatorname{car}(\mathbf{c}) = c_{gx}\mathbf{e}_{4235} + c_{gy}\mathbf{e}_{4315} + c_{gz}\mathbf{e}_{4125} + c_{gw}\mathbf{e}_{3215}$ (flat plane)
Cocarrier	$\operatorname{ccr}(\mathbf{c}) = -c_{gx}\mathbf{e}_{415} - c_{gy}\mathbf{e}_{425} - c_{gz}\mathbf{e}_{435} - c_{vx}\mathbf{e}_{235} - c_{vy}\mathbf{e}_{315} - c_{vz}\mathbf{e}_{125}$ (flat line)
Attitude	$\operatorname{att}(\mathbf{c}) = c_{gx}\mathbf{e}_{23} + c_{gy}\mathbf{e}_{31} + c_{gz}\mathbf{e}_{12} + c_{vx}\mathbf{e}_{15} + c_{vy}\mathbf{e}_{25} + c_{vz}\mathbf{e}_{35}$
Dual	$\mathbf{c}^\star = c_{gx}\mathbf{e}_{41} + c_{gy}\mathbf{e}_{42} + c_{gz}\mathbf{e}_{43} + c_{vx}\mathbf{e}_{23} + c_{vy}\mathbf{e}_{31} + c_{vz}\mathbf{e}_{12} + c_{mx}\mathbf{e}_{15} + c_{my}\mathbf{e}_{25} + c_{mz}\mathbf{e}_{35} - c_{gw}\mathbf{e}_{45}$
Constraints	$\mathbf{c}_{gxyz} \times \mathbf{c}_\mathbf{m} - c_{gw}\mathbf{c}_\mathbf{v} = 0,\ \ \mathbf{c}_\mathbf{v}\cdot\mathbf{c}_\mathbf{m} = 0,\ \ \mathbf{c}_{gxyz}\cdot\mathbf{c}_\mathbf{v} = 0$
DOF	$\operatorname{DOF}(4,2) = 6$
Center Norm	$\lVert\mathbf{c}\rVert_\odot = \sqrt{c_{gw}^2 + c_{vx}^2 + c_{vy}^2 + c_{vz}^2}$
Radius Norm	$\lVert\mathbf{c}\rVert_\oslash = \sqrt{c_{vx}^2 + c_{vy}^2 + c_{vz}^2 - c_{gw}^2 + 2(c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz})}$
Weight Norm	$\lVert\mathbf{c}\rVert_\circ = \sqrt{c_{gx}^2 + c_{gy}^2 + c_{gz}^2}$

Point (2D)

2D Rigid Algebras

$$\mathbf{p} = \underbrace{p_x \mathbf{e}_1 + p_y \mathbf{e}_2}_{\text{Position}} + \underbrace{p_z \mathbf{e}_3}_{\text{Weight}}$$

Degrees of Freedom	$\operatorname{DOF}(2,0) = 2$		
Attitude	$\operatorname{att}(\mathbf{p}) = p_z \mathbf{1}$		
Complement	$\overline{\mathbf{p}} = -p_x \mathbf{e}_{23} - p_y \mathbf{e}_{31} - p_z \mathbf{e}_{12}$		
Bulk Dual	$\mathbf{p}^{\bigstar} = -p_x \mathbf{e}_{23} - p_y \mathbf{e}_{31}$	Bulk Norm	$\lVert\mathbf{p}\rVert_{\bullet} = \mathbf{1}\sqrt{p_x^2 + p_y^2}$
Weight Dual	$\mathbf{p}^{\text{☆}} = -p_z \mathbf{e}_{12}$	Weight Norm	$\lVert\mathbf{p}\rVert_{\circ} = \lvert p_z \rvert \mathbb{1}$
Antisupport	$\operatorname{asp}(\mathbf{p}) = -p_x p_z \mathbf{e}_{23} - p_y p_z \mathbf{e}_{31} + \left(p_x^2 + p_y^2\right) \mathbf{e}_{12}$		

Line (2D)

2D Rigid Algebras

$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31}}_{\text{Normal}} + \underbrace{g_z \mathbf{e}_{12}}_{\text{Position}}$$

Degrees of Freedom	$\operatorname{DOF}(2,1) = 2$		
Attitude	$\operatorname{att}(\mathbf{g}) = g_y \mathbf{e}_1 - g_x \mathbf{e}_2$		
Complement	$\overline{\mathbf{g}} = -g_x \mathbf{e}_1 - g_y \mathbf{e}_2 - g_z \mathbf{e}_3$		
Bulk Dual	$\mathbf{g}^{\bigstar} = -g_z \mathbf{e}_3$	Bulk Norm	$\lVert\mathbf{g}\rVert_{\bullet} = \lvert g_z \rvert \mathbf{1}$
Weight Dual	$\mathbf{g}^{\text{☆}} = -g_x \mathbf{e}_1 - g_y \mathbf{e}_2$	Weight Norm	$\lVert\mathbf{g}\rVert_{\circ} = \mathbb{1}\sqrt{g_x^2 + g_y^2}$
Support	$\operatorname{sup}(\mathbf{g}) = -g_x g_z \mathbf{e}_1 - g_y g_z \mathbf{e}_2 + \left(g_x^2 + g_y^2\right) \mathbf{e}_3$		

Round Point (2D)

2D Conformal Algebras

$$\mathbf{a} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + \mathbf{e}_3 + \frac{\mathbf{p}^2 + r^2}{2}\mathbf{e}_4$$ $\mathbf{p}$ = center position, r = radius	$$\mathbf{a} = \underbrace{a_x\mathbf{e}_1 + a_y\mathbf{e}_2 + a_z\mathbf{e}_3}_{\text{Carrier Point}} + \underbrace{a_w\mathbf{e}_4}_{\text{Infinity}}$$ $(\text{when } a_x = a_y = a_z = 0)$

Center	$\operatorname{cen}(\mathbf{a}) = a_x a_z\mathbf{e}_1 + a_y a_z\mathbf{e}_2 + a_z^2\mathbf{e}_3 + a_z a_w\mathbf{e}_4$		
Container	$\operatorname{con}(\mathbf{a}) = -a_z^2\mathbf{e}_{321} + a_x a_z\mathbf{e}_{423} + a_y a_z\mathbf{e}_{431} + \left(a_z a_w - a_x^2 - a_y^2\right)\mathbf{e}_{412}$		
Partner	$\operatorname{par}(\mathbf{a}) = a_x a_z^2\mathbf{e}_1 + a_y a_z^2\mathbf{e}_2 + a_z^3\mathbf{e}_3 + \left(a_x^2 + a_y^2 - a_z a_w\right)a_z\mathbf{e}_5$		
Carrier	$\operatorname{car}(\mathbf{a}) = a_x\mathbf{e}_{41} + a_y\mathbf{e}_{42} + a_z\mathbf{e}_{43}$	(flat point)	
Cocarrier	$\operatorname{ccr}(\mathbf{a}) = a_z\mathbb{1}$	(full plane)	
Attitude	$\operatorname{att}(\mathbf{a}) = a_z\mathbf{1}$		
Dual	$\mathbf{a}^\star = -a_z\mathbf{e}_{321} + a_x\mathbf{e}_{423} + a_y\mathbf{e}_{431} - a_w\mathbf{e}_{412}$		
Degrees of Freedom	$\operatorname{DOF}(3,0) = 3$		
Center Norm	$\lVert\mathbf{a}\rVert_\odot = \sqrt{a_x^2 + a_y^2}$	Radius Norm	$\lVert\mathbf{a}\rVert_\oslash = \sqrt{2a_z a_w - a_x^2 - a_y^2}$
Weight Norm	$\lVert\mathbf{a}\rVert_\bigcirc = \lvert a_z\rvert$		

Circle (2D)

2D Conformal Algebras

$$\mathbf{c} = p_x\mathbf{e}_{423} + p_y\mathbf{e}_{431} - \mathbf{e}_{321} - \frac{\mathbf{p}^2 - r^2}{2}\mathbf{e}_{412}$$ $\mathbf{p}$ = center position, r = radius	$$\mathbf{c} = \underbrace{c_w\mathbf{e}_{321}}_{\text{Carrier Plane}} + \underbrace{c_x\mathbf{e}_{423} + c_y\mathbf{e}_{431} + c_z\mathbf{e}_{412}}_{\text{Flat Line}}$$ $(\text{when } c_w = 0)$

Center	$\operatorname{cen}(\mathbf{c}) = -c_x c_w\mathbf{e}_1 - c_y c_w\mathbf{e}_2 + c_w^2\mathbf{e}_3 + \left(c_x^2 + c_y^2 - c_z c_w\right)\mathbf{e}_4$		
Container	$\operatorname{con}(\mathbf{c}) = -c_w^2\mathbf{e}_{321} - c_x c_w\mathbf{e}_{423} - c_y c_w\mathbf{e}_{431} - c_z c_w\mathbf{e}_{412}$		
Partner	$\operatorname{par}(\mathbf{c}) = c_w^3\mathbf{e}_{321} + c_x c_w^2\mathbf{e}_{423} + c_y c_w^2\mathbf{e}_{431} + \left(c_x^2 + c_y^2 - c_z c_w\right)c_w\mathbf{e}_{412}$		
Carrier	$\operatorname{car}(\mathbf{c}) = c_w\mathbb{1}$	(full plane)	
Cocarrier	$\operatorname{ccr}(\mathbf{c}) = -c_x\mathbf{e}_{41} - c_y\mathbf{e}_{42} + c_w\mathbf{e}_{43}$	(flat point)	
Attitude	$\operatorname{att}(\mathbf{c}) = -c_w\mathbf{e}_{12} - c_y\mathbf{e}_{41} + c_x\mathbf{e}_{42}$		
Dual	$\mathbf{c}^\star = c_x\mathbf{e}_1 + c_y\mathbf{e}_2 - c_w\mathbf{e}_3 - c_z\mathbf{e}_4$		
Degrees of Freedom	$\operatorname{DOF}(3,2) = 3$		
Center Norm	$\lVert\mathbf{c}\rVert_\odot = \sqrt{c_x^2 + c_y^2}$	Radius Norm	$\lVert\mathbf{c}\rVert_\oslash = \sqrt{c_x^2 + c_y^2 - 2c_z c_w}$
Weight Norm	$\lVert\mathbf{c}\rVert_\bigcirc = \lvert c_w\rvert$		

Dipole (2D)

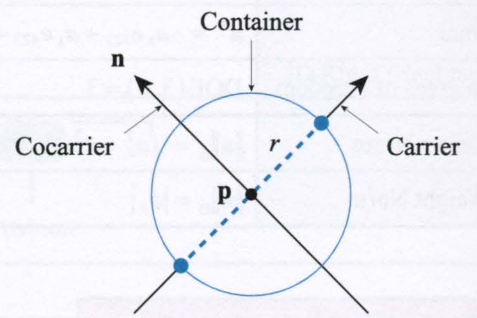
2D Conformal Algebras

$$\mathbf{d} = n_x\mathbf{e}_{23} + n_y\mathbf{e}_{31} - (\mathbf{p}\cdot\mathbf{n})\,\mathbf{e}_{12} + \frac{\mathbf{p}^2 + r^2}{2}(n_y\mathbf{e}_{41} - n_x\mathbf{e}_{42}) - (p_x n_y - p_y n_x)(p_x\mathbf{e}_{41} + p_y\mathbf{e}_{42} + \mathbf{e}_{43})$$

$\mathbf{p}$ = center position, $\mathbf{n}$ = line normal, r = radius

$$\mathbf{d} = \underbrace{\overbrace{d_{gx}\mathbf{e}_{23} + d_{gy}\mathbf{e}_{31} + d_{gz}\mathbf{e}_{12}}^{\text{Cocarrier Normal}}}_{\text{Carrier Line}} + \underbrace{\overbrace{d_{px}\mathbf{e}_{41} + d_{py}\mathbf{e}_{42} + d_{pz}\mathbf{e}_{43}}^{\text{Cocarrier Position}}}_{\text{Flat Point}}$$

(when $d_{gx} = d_{gy} = d_{gz} = 0$)



Center	$\mathrm{cen}(\mathbf{d}) = -(d_{gx}d_{gz} + d_{gy}d_{pz})\,\mathbf{e}_1 + (d_{gx}d_{pz} - d_{gy}d_{gz})\,\mathbf{e}_2 + (d_{gx}^2 + d_{gy}^2)\,\mathbf{e}_3 + (d_{pz}^2 - d_{gx}d_{py} + d_{gy}d_{px})\,\mathbf{e}_4$
Container	$\mathrm{con}(\mathbf{d}) = -(d_{gx}^2 + d_{gy}^2)\,\mathbf{e}_{321} - (d_{gx}d_{gz} + d_{gy}d_{pz})\,\mathbf{e}_{423} + (d_{gx}d_{pz} - d_{gy}d_{gz})\,\mathbf{e}_{431} + (d_{gy}d_{px} - d_{gx}d_{py} - d_{gz}^2)\,\mathbf{e}_{412}$
Partner	$\mathrm{par}(\mathbf{d}) = (d_{gx}^2 + d_{gy}^2)(d_{gx}\mathbf{e}_{23} + d_{gy}\mathbf{e}_{31} + d_{gz}\mathbf{e}_{12} + d_{pz}\mathbf{e}_{43}) + (d_{gz}^2 - d_{pz}^2 + d_{gx}d_{py} - d_{gy}d_{px})(d_{gy}\mathbf{e}_{41} - d_{gx}\mathbf{e}_{42}) - d_{gz}d_{pz}(d_{gx}\mathbf{e}_{41} + d_{gy}\mathbf{e}_{42})$
Carrier	$\mathrm{car}(\mathbf{d}) = d_{gx}\mathbf{e}_{423} + d_{gy}\mathbf{e}_{431} + d_{gz}\mathbf{e}_{412}$ (flat line)
Cocarrier	$\mathrm{ccr}(\mathbf{d}) = -d_{gy}\mathbf{e}_{423} + d_{gx}\mathbf{e}_{431} - d_{pz}\mathbf{e}_{412}$ (flat line)
Attitude	$\mathrm{att}(\mathbf{d}) = d_{gy}\mathbf{e}_1 - d_{gx}\mathbf{e}_2 - d_{pz}\mathbf{e}_4$
Dual	$\mathbf{d}^{\star} = d_{gy}\mathbf{e}_{23} - d_{gx}\mathbf{e}_{31} - d_{pz}\mathbf{e}_{12} - d_{py}\mathbf{e}_{41} + d_{px}\mathbf{e}_{42} - d_{gz}\mathbf{e}_{43}$
Constraints	$\mathbf{d_g}\cdot\mathbf{d_p} = 0$
Degrees of Freedom	$\mathrm{DOF}(3,1) = 4$
Center Norm	$\lVert\mathbf{d}\rVert_{\odot} = \sqrt{d_{gz}^2 + d_{pz}^2}$
Radius Norm	$\lVert\mathbf{d}\rVert_{\oslash} = \sqrt{d_{pz}^2 - d_{gz}^2 - 2(d_{gx}d_{py} - d_{gy}d_{px})}$
Weight Norm	$\lVert\mathbf{d}\rVert_{\circ} = \sqrt{d_{gx}^2 + d_{gy}^2}$

Appendix C

Notation Reference

This appendix summarizes the mathematical notation pertaining to geometric algebra as used throughout this book. Each entry in the tables that follow includes a brief description identifying the purpose of the notation and a reference to the location where the notation is introduced. Most entries also include the code point corresponding to the Unicode character used by the notation, which is provided to encourage a consistent appearance in materials written on the subject.

Notation	Unicode	Description	Reference
$\mathbf{1}$	U+01D7CF	Scalar unit	40
$\mathbb{1}$	U+01D7D9	Antiscalar unit / Volume element	40
$\boldsymbol{\mathfrak{g}}$	U+01D58C	Metric tensor	66
$\mathbf{G}$	U+01D406	Metric / Metric exomorphism	67
$\mathbb{G}$	U+01D53E	Antimetric / Metric antiexomorphism	68

Table C.1. Notation for foundational elements.

Notation	Unicode	Description	Reference
$u_{\mathbf{1}}$	U+01D7CF	Scalar part	122
$u_{\mathbb{1}}$	U+01D7D9	Antiscalar part	122
$\langle \mathbf{u} \rangle_k$	—	Grade selection	120
$\mathbf{u}_{\bullet}$	U+25CF	Bulk / Round bulk	69
$\mathbf{u}_{\circ}$	U+25CB	Weight / Round weight	69
$\mathbf{u}_{\blacksquare}$	U+25A0	Flat bulk	197
$\mathbf{u}_{\square}$	U+25A1	Flat weight	197

Table C.2. Notation for component extraction.

Notation	Unicode	Description	Reference
$\overline{\mathbf{u}}$	U+0304	Right complement	45
$\underline{\mathbf{u}}$	U+0331	Left complement	45
$\mathbf{u}^{\bigstar}$	U+2605	Dual / Bulk dual	81
$\mathbf{u}^{\text{☆}}$	U+2606	Antidual / Weight dual	83
$\tilde{\mathbf{u}}$	U+0303	Reverse	130
$\underset{\sim}{\mathbf{u}}$	U+0330	Antireverse	131

Table C.3. Notation for various unary operations.

Notation	Unicode	Description	Reference
$\mathbf{a} \wedge \mathbf{b}$	U+2227	Wedge product / Exterior product	33
$\mathbf{a} \vee \mathbf{b}$	U+2228	Antiwedge product / Exterior antiproduct	46
$\mathbf{a}$ ⟑ $\mathbf{b}$	U+27D1	Geometric product	117
$\mathbf{a}$ ⟇ $\mathbf{b}$	U+27C7	Geometric antiproduct	118
$\mathbf{a} \bullet \mathbf{b}$	U+2022	Dot product / Inner product	71
$\mathbf{a} \circ \mathbf{b}$	U+2218	Antidot product / Inner antiproduct	72

Table C.4. Notation for various products.

Notation	Unicode	Description	Reference
$\left\Vert\mathbf{u}\right\Vert_{\bullet}$	U+25CF	Norm / Bulk norm	76
$\left\Vert\mathbf{u}\right\Vert_{\circ}$	U+25CB	Antinorm / Weight norm	76
$\left\Vert\mathbf{u}\right\Vert$	—	Geometric norm	77
$\hat{\mathbf{u}}$	U+0302	Unitization	76
$\left\Vert\mathbf{u}\right\Vert_{\odot}$	U+2299	Center norm	197
$\left\Vert\mathbf{u}\right\Vert_{\oslash}$	not defined	Radius norm	198

Table C.5. Notation for norms and related operations.

Notation	Description	Reference
$\operatorname{gr}(\mathbf{u})$	Grade	42
$\operatorname{ag}(\mathbf{u})$	Antigrade	42
$\operatorname{att}(\mathbf{u})$	Attitude	71
$\operatorname{sup}(\mathbf{u})$	Support	100
$\operatorname{asp}(\mathbf{u})$	Antisupport	104
$\operatorname{car}(\mathbf{u})$	Carrier	193
$\operatorname{ccr}(\mathbf{u})$	Cocarrier	193
$\operatorname{cen}(\mathbf{u})$	Center	193
$\operatorname{con}(\mathbf{u})$	Container	194
$\operatorname{par}(\mathbf{u})$	Partner	195

Table C.6. Notation for various properties of an object.

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Made in United States
Orlando, FL
13 March 2024

44733134R00164

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projectivegeometricalgebra.org

ISBN 9798985358254

9 798985 358254

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